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WAVELET METHODS FOR SOLVING THIRD ORDER ODEs

Inderdeep Singh[1], Manbir Kaur[2 §]

Department of Physical Sciences
Sant Baba Bhag Singh University
Jalandhar - 144030, INDIA

Abstract: In this research, we propose a numerical technique based on Hermite wavelets for solving third order ordinary differential equations, which have several applications in science and engineering. A comparative study has been presented to illustrate the accuracy of the proposed scheme. For this purpose, we have utilized Haar wavelets based approximation technique. Considerable amount of research work has been carried out to find the numerical solution of such equations, where computer symbolic systems facilitate the computational work.

AMS Subject Classification: 65N99
Key Words: Haar wavelets; Hermite wavelets; third order ODEs; operational matrices; function approximation

1. Introduction

Third order differential equations are arising in several applications of engineering and science. Many analytical, semi-analytical and numerical techniques have been developed for solving these differential equations such as finite difference method (FDM), Adomian decomposition method (ADM), Homotopy perturbation method (HPM), Homotopy analysis method (HAM), Runge-Kutta method (RKM) of different orders, Euler's method and Taylor's series methods. Nowadays, wavelets based approximate techniques attract the interest of scientists and researchers as wavelets based techniques are simpler, less time consuming and show very high accuracy. Basis functions of Haar wavelets have

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[§]Correspondence author

been used for solving differential and integral equations in [2], [3], [6], [12], [13] and [14], whereas Hermite wavelets have been used in [4], [5], [9], [10] and [11]. Some third order ordinary differential equations have been discussed in [1], [7] and [8].

2. Wavelets and their properties

2.1. Haar wavelets

Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. Haar wavelet is a sequence of rescaled square shaped functions which together forms a wavelet family or basis. The Haar wavelet function $h_i(x)$ is defined in the interval $[\alpha, \gamma]$ as

$$h_i(x) = \begin{cases} 1, & \alpha \leq x < \beta, \\ -1, & \beta \leq x < \gamma, \\ 0, & \text{elsewhere}, \end{cases} \tag{1}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, $m = 2^j$ and $j = 0, 1, 2, ..., J$. J denotes the level of resolution. The integer $k = 0, 1, 2, ..., m-1$ is the translation parameter. The index i is calculated as: $i = m + k + 1$. The minimal value of $i = 2$. The maximal value of i is 2^{j+1}.

The collocation points are calculated as:

$$x_l = \frac{l - 0.5}{2M}, \qquad l = 1, 2, 3, ..., 2M, \;\; M = 2^J. \tag{2}$$

The operational matrix P, which is $2M \times 2M$, is calculated as below:

$$P_{1,i}(x) = \int_0^x h_i(x)dx, \tag{3}$$

and

$$P_{n+1,i}(x) = \int_0^x P_{n,i}(x)dx, \quad n = 1, 2, 3, ... \tag{4}$$

2.2. Hermite wavelets

Wavelets consist of a family of functions from dilation and translation of a single function known as mother wavelet. The continuous variation of dilation parameter α and translation parameter β, form a family of continuous wavelets as:

$$\psi_{\alpha,\beta}(x) = |\ \alpha\ |^{-\frac{1}{2}} \psi\Big(\frac{x-\alpha}{\beta}\Big), \quad \alpha, \beta \in R, \quad \alpha \neq 0. \tag{5}$$

If the dilation and translation parameters are restricted to discrete values by setting $\alpha = {\alpha_0}^{-k}$, $\beta = n\beta_0{\alpha_0}^{-k}$, $\alpha_0 > 1$, $\beta_0 > 0$, we obtain the following family of discrete wavelets:

$$\psi_{k,n}(x) = |\ \alpha\ |^{-\frac{1}{2}} \psi({\alpha_0}^k x - n\beta_0), \quad \alpha, \beta \in R, \quad \alpha \neq 0, \tag{6}$$

where $\psi_{k,n}$, form a wavelet basis for $L^2(R)$. For special case, if $\alpha_0 = 2$ and $\beta_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as:

$$\psi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} H_m(2^k m - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where $m = 0, 1, 2, 3, ..., M - 1$ and $n = 1, 2, 3, ..., 2^{k-1}$ and k is assumed any positive integer. Also, H_m are Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1 - x^2}$ on the real line R and satisfies the following recurrence relation

$$H_{m+2}(x) = 2xH_{m+1}(x) - 2(m + 1)H_m(x), \tag{8}$$

where $m = 0, 1, 2, ...,$ $H_0(x) = 1$ and $H_1(x) = 2x$.

3. Function Approximation

3.1. Haar wavelets

Consider any square integrable function $y(x)$ can be expanded in terms of infinite series of Haar basis functions as:

$$y(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{9}$$

where a_i are constants of this infinite series, known as Haar wavelet coefficients. For numerical approximation the above infinite series is truncated upto finite terms as:

$$y(x) = \sum_{i=1}^{2M} a_i h_i(x) = A^T h(x), \tag{10}$$

where A and $h(x)$ are $2M \times 1$ matrices and are given by

$$A^T = [a_1, a_2, ..., a_{2M}], \tag{11}$$

and

$$h(x) = [h_1(x), h_2(x), ..., h_{2M}(x)]^T. \tag{12}$$

3.2. Hermite wavelets

Consider any square integrable function $u(x)$ can be expanded in terms of infinite series of Hermite basis functions as:

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \psi_{n,m}(x), \tag{13}$$

where $C_{n,m}$ are constants of this infinite series, known as Hermite wavelet coefficients. For numerical approximation the above infinite series is truncated upto finite terms as:

$$u(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \psi_{n,m}(x) = C^T \Psi(x), \tag{14}$$

where C and Ψ are $2^{k-1}M \times 1$ matrices and are given by

$$C^T = [C_{1,0}, ..., C_{1,M-1}, ..., C_{2^{k-1},0}, ..., C_{2^{k-1},M-1}] \tag{15}$$

and

$$\Psi = [\psi_{1,0}, ..., \psi_{1,M-1}, ..., \psi_{2^{k-1},0}, ..., \psi_{2^{k-1},M-1}]^T. \tag{16}$$

4. Proposed methods for solving third order ODEs

Consider the following third order ODE

$$ay'''(x) + by''(x) + cy'(x) + dy(x) = F, \tag{17}$$

with initial conditions $y(0) = \alpha$, $y'(0) = \beta$ and $y''(0) = \gamma$, where α, β, γ are any constants and a, b, c, d and F are either functions of x or constants.

4.1. Haar wavelet collocation method(HWCM)

Consider the wavelet approximation

$$y'''(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{18}$$

Integrating (18), thrice with respect to x, from 0 to x, we get

$$y''(x) = y''(0) + \sum_{i=1}^{2M} a_i P_{1,i}(x), \tag{19}$$

$$y'(x) = y'(0) + xy''(0) + \sum_{i=1}^{2M} a_i P_{2,i}(x), \tag{20}$$

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2} y''(0) + \sum_{i=1}^{2M} a_i P_{3,i}(x). \tag{21}$$

Substituting y''', y'', y' and y in (17), and applying initial conditions, we get

$$\sum_{i=1}^{2M} \Big[ah_i(x) + bP_{1,i}(x) + cP_{2,i}(x) + dP_{3,i}(x)\Big] = G(x), \tag{22}$$

where $G(x)$ contains remaining terms including $y''(0), y'(0)$ and $y(0)$. From (22), we get Haar wavelet coefficients. The Haar wavelet solution $y(x)$ is obtained by substituting the values of wavelet coefficients into (21).

4.2. Hermite wavelet collocation method (HeWCM)

Consider the wavelet approximation

$$y'''(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T \Psi(x), \tag{23}$$

Integrating (23) with respect to x, from 0 to x, we get

$$y''(x) = y''(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \int_0^x \psi_{n,m}(x)dx, \tag{24}$$

$$y''(x) = y''(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} P\psi_{n,m}(x), \tag{25}$$

where

$$P\psi_{n,m}(x) = \int_0^x \psi_{n,m}(x)dx. \tag{26}$$

Integrating (25) with respect to x, we obtain

$$y'(x) = y'(0) + xy''(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \int_0^x P\psi_{n,m}(x)dx, \tag{27}$$

$$y'(x) = y'(0) + xy''(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} Q\psi_{n,m}(x), \tag{28}$$

where

$$Q\psi_{n,m}(x) = \int_0^x P\psi_{n,m}(x)dx. \tag{29}$$

Again, integrating (28) with respect to x, we obtain

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \int_0^x Q\psi_{n,m}(x)dx, \tag{30}$$

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} R\psi_{n,m}(x), \tag{31}$$

where

$$R\psi_{n,m}(x) = \int_0^x Q\psi_{n,m}(x)dx. \tag{32}$$

Substituting the values of y''', y'', y' and y in (17), and applying initial conditions, we get

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \Big[a\psi_{n,m}(x) + bP\psi_{n,m}(x) + cQ\psi_{n,m}(x) + dR\psi_{n,m}(x) \Big] = G_1(x), \tag{33}$$

where $G_1(x)$ contains all the remaining terms. From (33), we get Hermite wavelet coefficients. The Hermite wavelet solution $y(x)$ is obtained by substituting the values of wavelet coefficients into (31).

5. Convergence Analysis

5.1. Convergence of Haar wavelet collocation method

If $y_e(x)$ is the exact solution and $y_a(x)$ is the approximate solution of the differential equation (1). Let E_m be the corresponding error function and is defined as:

$$E_m = |y_e(x) - y_a(x)| \simeq \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k} h_{2^j+k}(x), \; x \in [0,1]. \tag{34}$$

Theorem. *Suppose that $y(x)$ satisfies a Lipschitz's condition on $[0,1]$, there exists constant $S > 0$ (dependent on both $y(x)$ and interval), such that*

$$|y(x_1) - y(x_2)| \leq S|x_1 - x_2|, \quad \forall x_1, x_2 \in [0,1]. \tag{35}$$

Then, the Haar wavelet method will be convergent in the sense that E_m goes to zero as m goes to infinity. The order of convergence is:

$$\|E_m\|_2 \simeq O(\frac{1}{m}). \tag{36}$$

Proof. Since

$$\|E_m\|_2{}^2 = \int_0^1 (\sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k} h_{2^j+k}(x))^2 dx. \tag{37}$$

After splitting the summation, from (37), we obtain

$$\begin{aligned} \|E_m\|_2{}^2 = & \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k}^2 \int_0^1 h_{2^j+k}^2(x) dx \\ & + \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{p=J+1}^{\infty} \sum_{q=0, q\neq k}^{2^p-1} a_{2^j+k} a_{2^p+q} \int_0^1 h_{2^j+k}(x) h_{2^p+q}(x) dx. \end{aligned} \tag{38}$$

Using orthogonality condition, we obtain

$$\|E_m\|_2{}^2 = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k}^2 (\frac{1}{2^j}). \tag{39}$$

We know that

$$a_{2^j+k} = 2^j \int_0^1 y(x) h_i(x) dx. \tag{40}$$

By the mean value theorem and equation (40), there exist

$$x_{11}^{jk} \in [\frac{k}{2^j}, \frac{k+0.5}{2^j}], \quad x_{22}^{jk} \in [\frac{k+0.5}{2^j}, \frac{k+1}{2^j}], \tag{41}$$

such that

$$a_{2^j+k} = 2^j[(\frac{k+0.5}{2^j} - \frac{k}{2^j})y(x_{11}^{jk}) - (\frac{k+1}{2^j} - \frac{k+0.5}{2^j})y(x_{22}^{jk})]. \tag{42}$$

After simplification, from (42), we obtain

$$a_{2^j+k} = \frac{1}{2}[y(x_{11}^{jk}) - y(x_{22}^{jk})]. \tag{43}$$

From (35) and (43), we obtain

$$a_{2^j+k} \leq \frac{1}{2}S(x_{11}^{jk} - x_{22}^{jk}) \leq (\frac{1}{2})S(\frac{1}{2^j}) \simeq S(\frac{1}{2^{j+1}}). \tag{44}$$

From (39) and (44), we obtain

$$\|E_m\|_2{}^2 = \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} S^2\frac{1}{2^{2j+2}}(\frac{1}{2^j}). \tag{45}$$

After simplification, from (45), we obtain

$$\|E_m\|_2{}^2 = \frac{S^2}{4}\sum_{j=J+1}^{\infty} 2^j(\frac{1}{2^{3j}}). \tag{46}$$

After summation of series, from (46), we obtain

$$\|E_m\|_2{}^2 = \frac{S^2}{3}(\frac{1}{2^{J+1}})^2. \tag{47}$$

But, $m = 2^{J+1}$, we obtain,

$$\|E_m\|_2 \simeq O(\frac{1}{m}). \tag{48}$$

Thus, when m goes to infinity, then, E_m tends to zero.

5.2. Convergence of Hermite wavelet collocation method [11]

Theorem. *Consider a function $u(x)$, which is continuous and bounded in $H^2[0,1)$, then the Hermite wavelet expansion of $u(x)$ converge to it.*

Proof. The Hermite wavelet coefficients of a continuous function $u(x)$ are defined as:

$$C_{n,m} = \int_0^1 u(x)\psi_{n,m}(x)dx. \tag{49}$$

Substituting the value of $\psi_{n,m}(x)$, from (7), we obtain

$$C_{n,m} = \int_\Omega u(x)\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}H_m(2^kx - 2n + 1)dx, \tag{50}$$

where $\Omega = \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}$. Substituting $2^kx - 2n + 1 = s$, we obtain

$$C_{n,m} = \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\int_{-1}^1 u(\frac{s-1+2n}{2^k})H_m(s)2^{-k}ds, \tag{51}$$

$$C_{n,m} = \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} \int_{-1}^{1} u(\frac{s-1+2n}{2^k}) H_m(s) ds. \tag{52}$$

Using GMVT for integrals, we obtain

$$C_{n,m} = \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} u(\frac{z-1+2n}{2^k}) \int_{-1}^{1} H_m(s) ds, \quad \text{for some} \quad z \in (-1, 1). \tag{53}$$

Set $\int_{-1}^{1} H_m(s)ds = K$, we obtain

$$|C_{n,m}| = |\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}|.|u(\frac{z-1+2n}{2^k})|.|K|. \tag{54}$$

As u is bounded, therefore, $\sum_{n,m=0}^{\infty} C_{n,m}$ converges absolutely. Hence the Hermite series expansion of u is convergent.

6. Numerical Observations

In this section, some numerical illustrations have been presented to obtain the accuracy and efficiency of these numerical schemes. Numerical results and accuracy of the proposed schemes have been obtained by using $k = 1, M = 8$ for Hermite wavelets collocation method (HeWCM) and $2M = 8$ for Haar wavelets collocation method (HWCM).

Example 1. Consider the third order linear ODE

$$y'''(x) = 3 \sin x, \tag{55}$$

with initial condition $y(0) = 1$, $y'(0) = 0$ and $y''(0) = -2$. The exact solution of the equation is

$$y(x) = 3 \cos x + \frac{x^2}{2} - 2. \tag{56}$$

Table 1 represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of Example 1. Table 2 represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method.

Example 2. Consider the third order ODE

$$y'''(x) = -y(x), \tag{57}$$

with initial condition $y(0) = 1$, $y'(0) = -1$ and $y''(0) = 1$. The exact solution of the equation is

$$y(x) = e^{-x}. \tag{58}$$

x	Exact solution	Hermite wavelet solution	Haar wavelet solution
1/16	0.9960956571	0.9960956570	0.9961013744
3/16	0.9649980643	0.9649980642	0.9650647394
5/16	0.9035319691	0.9035319691	0.9037200851
7/16	0.8131441752	0.8131441752	0.8135132729
9/16	0.6959766226	0.6959766226	0.6965848480
11/16	0.5548329634	0.5548329633	0.5557366367
13/16	0.3931348116	0.3931348115	0.3943880248
15/16	0.2148683502	0.2148683501	0.2165225995

Table 1: Numerical solutions of Example 1.

x	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/6	$5.7173e-006$	$3.7725e-013$
3/16	$6.6675e-005$	$4.6695e-012$
5/16	$1.8812e-004$	$1.3043e-011$
7/16	$3.6910e-004$	$2.5534e-011$
9/16	$6.0823e-004$	$4.2077e-011$
11/16	$9.0367e-004$	$6.2671e-011$
13/16	$1.2532e-003$	$8.7210e-011$
15/16	$1.6542e-003$	$1.1565e-010$

Table 2: Comparison of absolute errors of Example 1.

x	Exact solution	Hermite wavelet solution	Haar wavelet solution
1/16	0.9394130628	0.9394130628	0.9394149001
3/16	0.8290291181	0.8290291181	0.8290505437
5/16	0.7316156289	0.7316156289	0.7316752258
7/16	0.6456485264	0.6456485264	0.6457638052
9/16	0.5697828247	0.5697828247	0.5699702485
11/16	0.5028315779	0.5028315780	0.5031065403
13/16	0.4437473100	0.4437473101	0.4441240699
15/16	0.3916056266	0.3916056267	0.3920972039

Table 3: Numerical solutions of Example 2.

x	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/6	$1.8373e-006$	$1.8874e-013$
3/16	$2.1426e-005$	$2.3351e-012$
5/16	$5.9597e-005$	$6.5320e-012$
7/16	$1.1528e-004$	$1.2779e-011$
9/16	$1.8742e-004$	$2.1016e-011$
11/16	$2.7496e-004$	$3.1186e-011$
13/16	$3.7676e-004$	$4.3160e-011$
15/16	$4.9158e-004$	$5.6794e-011$

Table 4: Comparison of absolute errors of Example 2.

Table 3 represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of Example 2. Table 4 represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method.

Example 3. Consider the following third order ODE

$$y'''(x) + 4y'(x) = x, \tag{59}$$

with initial condition $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 1$. The exact solution of the equation is

$$y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{x^2}{8}. \tag{60}$$

y	Exact solution	Hermite wavelet solution	Haar wavelet solution
1/6	0.0019512186	0.0019512187	0.0019455547
3/16	0.0174243521	0.0174243531	0.0173590571
5/16	0.0476514463	0.0476514492	0.0474724339
7/16	0.0912388703	0.0912388758	0.0909026849
9/16	0.1462051843	0.1462051930	0.1456832055
11/16	0.2101043360	0.2101043482	0.2093857463
13/16	0.2801777440	0.2801777600	0.2792710024
15/16	0.3535258136	0.3535258333	0.3524584928

Table 5: Numerical solutions of Example 3.

y	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/6	$5.6639e-006$	$8.6019e-011$
3/16	$6.5295e-005$	$1.0541e-009$
5/16	$1.7901e-004$	$2.8850e-009$
7/16	$3.3619e-004$	$5.4749e-009$
9/16	$5.2198e-004$	$8.6529e-009$
11/16	$7.1859e-004$	$1.2230e-008$
13/16	$9.0674e-004$	$1.5975e-008$
15/16	$1.0673e-003$	$1.9663e-008$

Table 6: Comparison of absolute errors of Example 3.

Table 5 represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of Example 3. Table 6 represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method.

Example 4. Consider the following third order ODE

$$y'''(x) = e^{-x}, \quad 0 \leq x \leq 1 \tag{61}$$

with initial condition $y(0) = 3$, $y'(0) = 1$ and $y''(0) = 5$. The exact solution of the equation is

$$y(x) = 2 + 2x^2 + e^x. \tag{62}$$

Figure 1 and Figure 2 show the absolute errors obtained by Hermite wavelets and Haar wavelets based collocation methods respectively for Example 4.

Example 5. Consider the following third order ODE

$$y'''(x) = y''(x) - y'(x) + y(x) + e^{-x}, \quad 0 \leq x \leq 1 \tag{63}$$

with initial condition $y(0) = 1$, $y'(0) = 1$ and $y''(0) = 0$. The exact solution of the equation is

$$y(x) = \frac{1}{2}xe^x + \cos x + \frac{1}{2}\sin x. \tag{64}$$

Figure 3 and Figure 4 show the absolute errors obtained by Hermite wavelets and Haar wavelets based collocation methods respectively for Example 5.

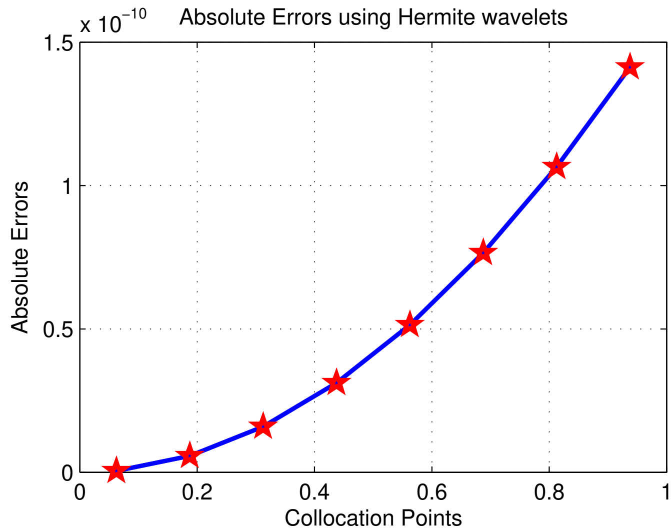

Figure 1: Absolute errors using Hermite wavelets for Example 4

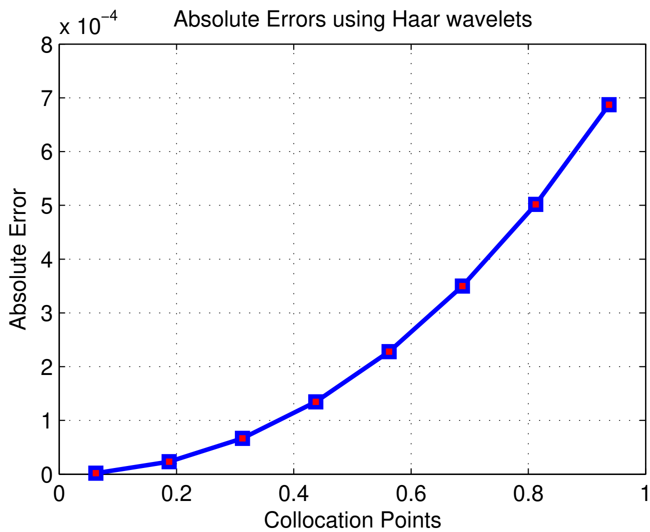

Figure 2: Absolute errors using Haar wavelets for Example 4

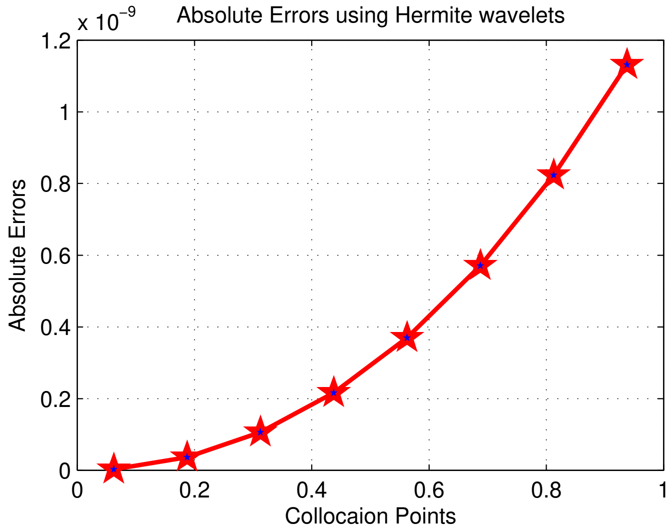

Figure 3: Absolute errors using Hermite wavelets for Example 5

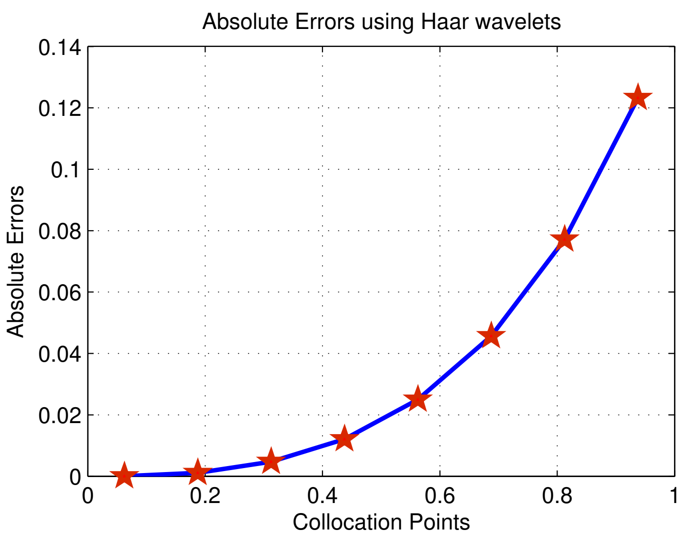

Figure 4: Absolute errors using Haar wavelets for Example 5

Conclusion

From above discussion, it is concluded that the Hermite wavelet based collocation method is much better in comparison to Haar wavelet based collocation method for solving third order ordinary differential equations arising in science and engineering. To obtain more accurate results, the number of collocation points may be increased.

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