

ON STABILITY OF THE THIRD ORDER PARTIAL DELAY  
DIFFERENTIAL EQUATION WITH NONLOCAL  
BOUNDARY CONDITIONS

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**Abstract:** In the present paper, the stability of the initial value problem for the third order partial delay differential equation with nonlocal boundary condition is studied. The first of accuracy absolute stable difference scheme for solution of this problem is presented. Stability estimates for solution of this difference scheme are proved. Numerical results are provided.

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**Key Words:** time delay; third order partial differential equations; stability; difference scheme; nonlocal condition

## 1. Introduction

Classical and nonclassical problems for third order partial differential equations have been studied widely in the literature (for instance, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]).

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Time delay is one of the most common phenomena occurring in many engineering applications. In control theory, the process of sampled-data control is a typical example where time delay happens in the transmission from measurement to controller.

Theory and applications of delay linear and nonlinear third order ordinary and partial differential and difference equations with the delay term were widely investigated (see, e.g., [15], [16], [17], [18], [19], [20], [21], [22] and the references given therein).

Our goal in this paper is to investigate the initial value problem for third order partial delay differential and difference equations with nonlocal boundary condition. The paper is organized as follows. Section 3 is Introduction. In Section 2, a theorem on stability of the initial value problem for the third order partial delay differential equation with nonlocal boundary condition is established. In Section 3, the first order of accuracy difference scheme for solution of this problem is studied. Stability estimates for solution of this difference scheme are proved. In Section 4, numerical results are provided. Finally, Section 5 is conclusion.

## 2. Stability of differential problem

In  $[0, \infty) \times (0, l)$ , the initial boundary value problem for the third order partial differential equation with time delay and nonlocal boundary conditions

$$\left\{ \begin{array}{l} \frac{\partial^3 u(t, x)}{\partial t^3} - (a(x)u_{tx}(t, x))_x + \delta u(t, x) \\ = -b((-a(x)u_x(t - w, x))_x + \delta u(t, x)) \\ + f(t, x), \quad 0 < t < \infty, (0, l), \\ u(t, x) = g(t, x), \quad -w \leq t \leq 0, x \in [0, l], \\ u(t, 0) = u(t, l), u_x(t, 0) = u_x(t, l), \quad 0 \leq t < \infty \end{array} \right. \quad (1)$$

is considered. Throughout this paper, we will assume that  $a(x) \geq \underline{a} > 0$ ,  $x \in (0, l)$  and  $a(l) = a(0)$ .

We consider the Hilbert space  $L_2[0, l]$  of the all square integrable functions defined on  $[0, l]$ , equipped with the norm

$$\|f\|_{L_2[0, l]} = \left( \int_0^l |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Under compatibility conditions problem, (1) has a unique solution  $u(t, x)$  for the smooth functions  $a(x)$ ,  $x \in (0, \ell)$ ,  $g(t, x)$ ,  
 $-w \leq t \leq 0$ ,  $x \in [0, l]$ ,  $f(t, x)$ ,  $0 < t < \infty$ ,  $x \in (0, l)$ , and  $\delta > 0, b \in R^1$ .

Let us present a theorem on stability of problem (1).

**Theorem 1.** *For the solutions of problem (1), we have following stability estimates:*

$$\begin{aligned} & \max_{0 \leq t \leq nw} \|v_{tt}(t, \cdot)\|_{W_2^1[0, l]}, \max_{0 \leq t \leq nw} \|v_t(t, \cdot)\|_{W_2^2[0, l]}, \max_{0 \leq t \leq nw} \|v(t, \cdot)\|_{W_2^3[0, l]} \\ & \leq M_2 \left[ (2 + |b|w)^n a_0 + \int_{j=1}^n (2 + |b|w)^{n-j} \int_{(j-1)w}^{jw} \|f(s, \cdot)\|_{W_2^1[0, l]} ds \right], \\ & a_0 = \max \left\{ \max_{-w \leq t \leq 0} \|g_{tt}(t, \cdot)\|_{W_2^1[0, l]}, \max_{-w \leq t \leq 0} \|g(t, \cdot)\|_{W_2^3[0, l]} \right\}, \end{aligned}$$

where  $M_2$  does not depend on  $g(t, x)$  and  $f(t, x)$ . Here,  $W_2^k[0, l]$ ,  $k = 1, 2, 3$  are Sobolev spaces of all square integrable functions  $\psi(x)$  defined on  $[0, l]$  equipped with the norm

$$\|\psi\|_{W_2^k[0, l]} = \left( \int_0^l \sum_{j=0}^k \left( \underbrace{\psi x \cdots x}_{j \text{ time}}(x) \right)^2 dx \right)^{\frac{1}{2}}.$$

*Proof.* This allows us to reduce the problem (1) to the initial value problem

$$\begin{cases} \frac{d^3 v(t)}{dt^3} + A \frac{dv(t)}{dt} = bAv(t-w) + f(t), & 0 < t < \infty, \\ v(t) = g(t), & -w \leq t \leq 0 \end{cases} \quad (2)$$

in a Hilbert space  $H = L_2[0, l]$  with a self-adjoint positive definite operator  $A$  defined by formula

$$Au(x) = -(a(x)u_x(x))_x + \delta u(x) \quad (3)$$

with domain

$$\begin{aligned} D(A) = \{ & u(x) : u(x), u_x(x), (a(x)u_x)_x \in L_2[0, l], u(l) = u(0), \\ & u_x(l) = u_x(0) \}. \end{aligned}$$

The proof of Theorem 1 is based on the self-adjointness and positive definiteness of the space operator  $A$  defined by formula (3) [25] and the following theorem on stability of the solution of the abstract problem (2).  $\square$

**Theorem 2.** ([23]) *For the solution of problem (2) the following estimate holds:*

$$\begin{aligned} & \max_{0 \leq t \leq nw} \left\| A^{\frac{1}{2}} \frac{d^2 v(t)}{dt^2} \right\|_H, \max_{0 \leq t \leq nw} \left\| A \frac{dv(t)}{dt} \right\|_H, \frac{1}{2} \max_{0 \leq t \leq nw} \left\| A^{\frac{3}{2}} v(t) \right\|_H \\ & \leq (2 + |b|w)^n a_0 + \int_0^{nw} \left\| A^{\frac{1}{2}} f(s) \right\|_H ds, \quad n = 1, 2, \dots, \end{aligned} \quad (4)$$

where

$$\begin{aligned} a_0 = \max \left\{ \max_{-w \leq t \leq 0} \left\| A^{\frac{1}{2}} \frac{d^2 g(t)}{dt^2} \right\|_H, \max_{-w \leq t \leq 0} \left\| A \frac{dg(t)}{dt} \right\|_H, \right. \\ \left. \max_{-w \leq t \leq 0} \left\| A^{\frac{3}{2}} g(t) \right\|_H \right\}. \end{aligned}$$

### 3. Stability of the difference scheme

Now, we study the stable difference scheme for the approximate solution of problem (1). The discretization of problem (1) is carried out in two steps.

In the first step, the spatial discretization is carried out. We define the grid space

$$[0, \ell]_h = \{x = x_n \mid x_n = nh, \ 0 \leq n \leq M, \ Mh = \ell\}.$$

We introduce the Hilbert space  $L_{2h} = L_2([0, \ell]_h)$  of the grid functions  $\varphi^h(x) = \{\varphi^n\}_0^M$  defined on  $[0, \ell]_h$ , equipped with the norm

$$\|\varphi^h\|_{L_{2h}} = \left( \sum_{x \in [0, \ell]_h} |\varphi^h(x)|^2 h \right)^{1/2}.$$

To the differential operator  $A$  defined by the formula (3), we assign the difference operator  $A_h^x$  by the formula

$$A_h^x \varphi^h(x) = \left\{ - (a(x) \varphi_{\bar{x}}^n)_x + \delta \varphi^n(x) \right\}_{-M+1}^{M-1}, \quad (5)$$

acting in the space of grid functions  $\varphi^h(x) = \{\varphi^n\}_0^M$  and satisfying the conditions  $\varphi^0 = \varphi^M, \varphi^1 - \varphi^0 = \varphi^M - \varphi^{M-1}$ . Here

$$\varphi_{\bar{x}}^n = \frac{\varphi^n - \varphi^{n-1}}{h}, \quad 1 \leq n \leq M, \quad \varphi_x^n = \frac{\varphi^{n+1} - \varphi^n}{h}, \quad 0 \leq n \leq M-1.$$

It is well-known that  $A_h^x$ , defined by (5), is a self-adjoint positive definite operator in  $L_{2h}$ . With the help of  $A_h^x$ , the first discretization step results in the following problem

$$\begin{cases} \frac{\partial^3 u^h(t, x)}{\partial t^3} + A_h^x u^h(t, x) = -b A_h^x u^h(t - w, x) \\ + f^h(t, x), \quad x \in [0, \ell]_h, \quad 0 < t < \infty, \\ u^h(t, x) = g^h(t, x), \quad -w \leq t \leq 0, \quad x \in [0, \ell]_h, \quad -w < t < 0. \end{cases} \quad (6)$$

In the second step, we replace the problem (6) with the following first order of accuracy difference scheme

$$\begin{cases} \frac{u_{k+2}^h(x) - 3u_{k+1}^h(x) + 3u_k^h(x) - u_{k-1}^h(x)}{\tau^3} + A_h^x \frac{u_{k+2}^h(x) - u_{k+1}^h(x)}{\tau} \\ = b A_h^x u_{k-N}^h(x) + f_k^h(x), f_k^h(x) \\ = f^h(t_k, x), k \geq 1, \quad x \in [0, \ell]_h, \\ u_k^h(x) = g^h(t_k, x), -N \leq k \leq 0, \\ (I_h + \tau^2 A_h^x) \frac{u_1^h(x) - u_0^h(x)}{\tau} = g_t^h(0, x), \\ (I_h + \tau^2 A_h^x) \frac{u_2^h(x) - 2u_1^h(x) + u_0^h(x)}{\tau^2} = g_{tt}^h(0, x), x \in [0, \ell]_h, \\ (I_h + \tau^2 A_h^x) \frac{u_{mN+1}^h(x) - u_{mN}^h(x)}{\tau} = \frac{u_{mN}^h(x) - u_{mN-1}^h(x)}{\tau}, x \in [0, \ell]_h, \\ (I_h + \tau^2 A_h^x) \frac{u_{mN+2}^h(x) - 2u_{mN+1}^h(x) + u_{mN}^h(x)}{\tau^2} \\ = \frac{u_{mN}^h(x) - 2u_{mN-1}^h(x) + u_{mN-2}^h(x)}{\tau^2}, x \in [0, \ell]_h, m = 1, 2, \dots, \end{cases} \quad (7)$$

where  $\tau = 1/N$  and  $t_k = k\tau$ ,  $-N \leq k < \infty$ .

**Theorem 3.** *Let  $\tau$  and  $h$  be sufficiently small numbers. For the solution of difference scheme (7) the following estimates*

$$\begin{aligned} \max_{0 \leq k \leq (m+1)N-2} \left\| \frac{u_{k+2}^h - 2u_{k+1}^h + u_k^h}{\tau^2} \right\|_{W_{2h}^1}, \max_{1 \leq k \leq (m+1)N} \left\| \frac{u_k^h - u_{k-1}^h}{\tau} \right\|_{W_{2h}^2}, \\ \max_{0 \leq k \leq (m+1)N} \|u_k^h\|_{W_{2h}^3} \leq C_1 \left[ (2 + \tau|b|(N-2))^m b_0^h \right] \end{aligned}$$

$$+ \sum_{j=1}^m (2 + \tau|b|(N-2))^{m-j} \tau \sum_{s=(j-1)N+1}^{jN} \|f(t_s)\|_{W_{2h}^1} \Big], m = 0, 1, \dots,$$

$$b_0^h = \max \left\{ \max_{-N \leq k \leq 0} \|g_{tt}^h(t_k)\|_{W_{2h}^1}, \max_{-N \leq k \leq 0} \|g_t^h(t_k)\|_{W_{2h}^2}, \right. \\ \left. \max_{-N \leq k \leq 0} \|g^h(t_k)\|_{W_{2h}^3} \right\}$$

hold, where  $C_1$  does not depend on  $\tau, h, g^h(t_k)$ , and  $f_k^h(x)$ . Here,  $W_{2h}^k$ ,  $k = 1, 2, 3$  are spaces of all mesh functions  $\psi^h(x)$  defined on  $[0, l]_h$  equipped with the norm

$$\|\psi^h\|_{W_{2h}^k} = \left( \sum_{x \in [0, l]_h} \sum_{j=0}^k \left( \underbrace{\psi_x^h \cdots x}_{j \text{ time}}(x) \right)^2 h^k \right)^{\frac{1}{2}}.$$

*Proof.* The difference scheme (7) can be written in the abstract form

$$\left\{ \begin{array}{l} \frac{u_{k+2}^h - 3u_{k+1}^h + 3u_k^h - u_{k-1}^h}{\tau^3} + A_h \frac{u_{k+2}^h - u_{k+1}^h}{\tau} \\ = bA_h u_{k-N}^h + f_k^h, k \geq 1, \\ u_k^h = g_k^h, -N \leq k \leq 0, \\ (I_h + \tau^2 A_h) \frac{u_1^h - u_0^h}{\tau} = g_t^h(0), (I_h + \tau^2 A_h) \frac{u_2^h - 2u_1^h + u_0^h}{\tau^2} = g_{tt}^h(0), \\ (I_h + \tau^2 A_h) \frac{u_{mN+2}^h - 2u_{mN+1}^h + u_{mN}^h}{\tau^2} = \frac{u_{mN}^h - 2u_{mN-1}^h + u_{mN-2}^h}{\tau^2}, \\ (I_h + \tau^2 A_h) \frac{u_{mN+1}^h - u_{mN}^h}{\tau} = \frac{u_{mN}^h - u_{mN-1}^h}{\tau}, m = 1, 2, \dots \end{array} \right. \quad (8)$$

in a Hilbert space  $L_{2h}$  with self-adjoint positive definite operator  $A_h = A_h^x$  by formula (5).

Here,  $g_k^h = g_k^h(x)$ ,  $f_k^h = f_k^h(x)$  and  $u_k^h = u_k^h(x)$  are known and unknown abstract mesh functions defined on  $[0, l]_h$  with the values in  $H = L_{2h}$ . Therefore, the proof of Theorem 2 is based on the self-adjointness and positive definiteness of the space operator  $A_h$  (5) [26] and the following theorem on stability of the solution of the difference scheme (8).  $\square$

**Theorem 4.** ([27]) *For the solution of difference scheme (8) the following estimate holds:*

$$\begin{aligned} & \frac{1}{2} \max_{0 \leq k \leq (m+1)N-2} \left\| A_h^{\frac{1}{2}} \frac{u_{k+2}^h - 2u_{k+1}^h + u_k^h}{\tau^2} \right\|_H, \max_{1 \leq k \leq (m+1)N} \left\| A_h \frac{u_k^h - u_{k-1}^h}{\tau} \right\|_H, \\ & \max_{0 \leq k \leq (m+1)N} \|A_h^{\frac{3}{2}} u_k^h\|_H \leq C_1 \left[ (2 + \tau|b|(N-2))^m b_0^h \right. \\ & \left. + \sum_{j=1}^m (2 + \tau|b|(N-2))^{m-j} \tau \sum_{s=(j-1)N+1}^{jN} \|A_H^{\frac{1}{2}} f(t_s)\|_H \right], \quad m = 0, 1, \dots, \end{aligned}$$

where

$$\begin{aligned} b_0 = \max \{ & \max_{-N \leq k \leq 0} \|A_h^{\frac{1}{2}} g''(t_k)\|_H, \max_{-N \leq k \leq 0} \|A_h g_t^h(t_k)\|_H, \\ & \max_{-N \leq k \leq 0} \|A_h^{\frac{3}{2}} g^h(t_k)\|_H \}. \end{aligned}$$

#### 4. Numerical results

When the analytical methods do not work properly, the numerical methods for obtaining the approximate solutions of partial differential equations play an important role in applied mathematics. In this section, we will use the first order of accuracy difference scheme to approximate the solution of a simple test problem

$$\begin{cases} \frac{\partial^3 u(t, x)}{\partial t^3} - \frac{\partial^3 u(t, x)}{\partial t \partial x^2} + 16 \frac{\partial u(t, x)}{\partial t} \\ = -0.1 \left( -\frac{\partial^2 u(t-1, x)}{\partial x^2} + 16u(t-1, x) \right) \\ - 48e^{-2t} \sin 2x + 2e^{-2(t-1)} \sin 2x, \\ 0 < t < \infty, \quad 0 < x < \pi, \\ u(t, x) = e^{-2t} \sin 2x, \quad -1 \leq t \leq 0, \quad 0 \leq x \leq \pi, \\ u(t, 0) = u(t, \pi), \quad u_x(t, 0) = u_x(t, \pi), \quad 0 \leq t < \infty. \end{cases} \quad (9)$$

The exact solution of problem (9) is  $u(t, x) = e^{-2t} \sin 2x, 0 \leq x \leq \pi, -1 \leq t < \infty$ . For the approximate solutions of the problem (9), using the set of grid points

$$[-1, \infty)_\tau \times [0, \pi]_h = \{(t_k, x_n) : t_k = k\tau, -N \leq k, N\tau = 1,$$

$$x_n = nh, \quad -M \leq n \leq M, \quad Mh = \pi\},$$

we get the first order of accuracy in  $t$  difference scheme

$$\left\{ \begin{aligned} & \frac{u_n^{k+2} - 3u_n^{k+1} + 3u_n^k - u_n^{k-1}}{\tau^3} - \frac{u_{n+1}^{k+2} - u_{n+1}^{k+1} - 2(u_n^{k+2} - u_n^{k+1}) + u_{n-1}^{k+2} - u_{n-1}^{k+1}}{\tau h^2} \\ & + 16 \frac{u_n^{k+2} - u_n^{k+1}}{\tau} = -(0.1) \left( -\frac{u_{n+1}^{k-N} - 2u_n^{k-N} + u_{n-1}^{k-N}}{h^2} + 16u_n^{k-N} \right) \\ & - 48e^{-2t_k} \sin 2x_n + 2e^{-2(t_k-N)} \sin 2x_n, \\ & t_k = k\tau, \quad mN + 1 \leq k \leq (m+1)N - 2, \\ & m = 0, 1, \dots, \quad 1 \leq n \leq M - 1, \\ & N\tau = 1, \quad x_n = nh, \quad 1 \leq n \leq M - 1, \quad Mh = \pi, \\ & u_n^0 = \sin(2nh), \\ & \frac{u_n^1 - u_n^0}{\tau} + \tau \left( -\frac{u_{n+1}^1 - 2u_n^1 + u_{n-1}^1}{h^2} + 16u_n^1 \right) + \\ & \tau \left( \frac{u_{n+1}^0 - 2u_n^0 + u_{n-1}^0}{h^2} - 16u_n^0 \right) = -2 \sin(2nh), \\ & \frac{u_n^2 - 2u_n^1 + u_n^0}{\tau^2} + \left( -\frac{u_{n+1}^2 - 2u_n^2 + u_{n-1}^2}{h^2} + 16u_n^2 \right) \\ & + 2 \left( \frac{u_{n+1}^1 - 2u_n^1 + u_{n-1}^1}{h^2} - 16u_n^1 \right) \\ & + \left( -\frac{u_{n+1}^0 - 2u_n^0 + u_{n-1}^0}{h^2} + 16u_n^0 \right) = 4 \sin(2nh), \quad 0 \leq n \leq M, \\ & \frac{u_n^{mN+1} - u_n^{mN}}{\tau} + \tau \left( -\frac{u_{n+1}^{mN+1} - 2u_n^{mN+1} + u_{n-1}^{mN+1}}{h^2} + 16u_n^{mN+1} \right) \\ & + \tau \left( \frac{u_{n+1}^{mN} - 2u_n^{mN} + u_{n-1}^{mN}}{h^2} - 16u_n^{mN} \right) = \frac{u_n^{mN} - u_n^{mN-1}}{\tau}, \\ & \frac{u_n^{mN+2} - 2u_n^{mN+1} + u_n^{mN}}{\tau^2} \\ & + \left( -\frac{u_{n+1}^{mN+2} - 2u_n^{mN+2} + u_{n-1}^{mN+2}}{h^2} + 16u_n^{mN+2} \right) \\ & + 2 \left( \frac{u_{n+1}^{mN+1} - 2u_n^{mN+1} + u_{n-1}^{mN+1}}{h^2} - 16u_n^{mN+1} \right) \\ & + \left( -\frac{u_{n+1}^{mN} - 2u_n^{mN} + u_{n-1}^{mN}}{h^2} + 16u_n^{mN} \right) \\ & = \frac{u_n^{mN} - 2u_n^{mN-1} + u_n^{mN-2}}{\tau^2}, \quad 0 \leq n \leq M, \quad m = 1, 2, \dots, \\ & u_0^k = u_M^k, \quad u_1^k - u_0^k = u_M^k - u_{M-1}^k, \\ & mN \leq k \leq (m+1)N, \quad m = 1, 2, \dots \end{aligned} \right. \quad (10)$$



We can write (10) in the matrix form

$$\left\{ \begin{array}{l} BU^{k+2} + CU^{k+1} + DU^k + EU^{k-1} = \varphi(U^{k-N}), \\ k = 1, 2, 3, \dots, U^0 = \begin{bmatrix} 0 \\ \sin(2h) \\ \vdots \\ \sin(2(M-1)h) \\ 0 \end{bmatrix}, \\ U^1 = (1 - 2\tau)U^0, U^2 = U^1 - (1 - 4\tau^2)U^0 \\ U^{mN+1} = F^{-1}HU^{mN} - F^{-1}U^{mN-1}, \\ U^{mN+2} = 2U^{mN+1} + F^{-1}PU^{mN} - 2F^{-1}U^{mN-1} \\ + F^{-1}U^{mN-2}, m = 1, 2, \dots, \end{array} \right. \quad (11)$$

where  $B, C, D, E, F, H$  and  $P$  are  $(M+1) \times (M+1)$  matrices,  $\varphi(U^{k-N})$ ,  $U^0$ ,  $U^1$  and  $U^r, r = k, k \pm 1, k + 2$  are  $(M+1) \times 1$  column vectors defined by

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & -1 \\ a & b & a & & \cdot & 0 & 0 & 0 \\ 0 & a & b & a & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & b & a & 0 \\ 0 & 0 & 0 & 0 & \cdot & a & b & a \\ 1 & -1 & 0 & 0 & \cdot & 0 & -1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ -a & c & -a & & \cdot & 0 & 0 & 0 \\ 0 & -a & c & -a & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & c & -a & 0 \\ 0 & 0 & 0 & 0 & \cdot & -a & c & -a \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$DE(g) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & g & 0 & & \cdot & 0 & 0 & 0 \\ 0 & 0 & g & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & g & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & g & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix},$$

$$FH(z) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & -1 \\ -s & z & -s & & \cdot & 0 & 0 & 0 \\ 0 & -s & z & -s & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & z & -s & 0 \\ 0 & 0 & 0 & 0 & \cdot & -s & z & -s \\ 1 & -1 & 0 & 0 & \cdot & 0 & -1 & 1 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & -1 \\ s & p & s & & \cdot & 0 & 0 & 0 \\ 0 & s & p & s & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & p & s & 0 \\ 0 & 0 & 0 & 0 & \cdot & s & p & s \\ 1 & -1 & 0 & 0 & \cdot & 0 & -1 & 1 \end{bmatrix},$$

$$DE(g) = \begin{cases} D, & g = d, \\ E, & g = e, \end{cases}$$

$$FH(z) = \begin{cases} F, & z = q, \\ H, & z = t, \end{cases}$$

$$\varphi(U^{k-N}) = \begin{bmatrix} 0 \\ \varphi_1^k \\ \vdots \\ \varphi_{M-1}^k \\ 0 \end{bmatrix}, \quad U^r = \begin{bmatrix} U_0^r \\ U_1^r \\ \vdots \\ U_{M-1}^r \\ U_M^r \end{bmatrix}, \quad r = k, k \pm 1, k + 2,$$

where

$$\begin{aligned} \varphi_n^k = -(0.1) & \left( -\frac{u_{n+1}^{k-N} - 2u_n^{k-N} + u_{n-1}^{k-N}}{h^2} + 16u_n^{k-N} \right) \\ & - 48e^{-2t_k} \sin 2x_n + 2e^{-2(t_k-N)} \sin 2x_n, \\ t_k = k\tau, \quad mN + 1 \leq k \leq (m+1)N - 2, \\ m = 0, 1, \dots, \quad 1 \leq n \leq M - 1. \end{aligned}$$

Here, we denote  $a = -\frac{1}{\tau h^2}$ ,  $b = \frac{1}{\tau^3} + \frac{2}{\tau h^2} + \frac{16}{\tau}$ ,  $c = -\frac{3}{\tau^3} - \frac{2}{\tau h^2} - \frac{16}{\tau}$ ,  $d = \frac{3}{\tau^3}$ ,  $e = -\frac{1}{\tau^3}$ ,  $t = 2 + \frac{2\tau^2}{h^2} + 16\tau^2$ ,  $p = -\frac{2\tau^2}{h^2} - 16\tau^2$ ,

$$q = 1 + \frac{2\tau^2}{h^2} + 16\tau^2, \quad s = \frac{\tau^2}{h^2}.$$

The numerical solutions are recorded for different values of  $N$  and  $M$ , and  $u_n^k$  represents the numerical solution of this difference scheme at  $u(t_k, x_n)$ . Table 1 is constructed for  $N = M = 40, 80, 160$  in  $t \in [0, 1]$ ,  $t \in [1, 2]$ ,  $t \in [2, 3]$  respectively and the errors are computed by

$$mE_M^N = \max_{mN+1 \leq k \leq (m+1)N, 0 \leq n \leq M} |u(t_k, x_n) - u_n^k|.$$

**Table 1.** Errors of difference scheme (10).

(N, M)	N = M = 40	N = M = 80	N = M = 160
$t \in [0, 1]$	0.0790	0.0400	0.0200
$t \in [1, 2]$	0.0823	0.0411	0.0205
$t \in [2, 3]$	0.0789	0.0402	0.0202

If  $N$  and  $M$  are doubled, the values of the errors are decreases by a factor of approximately 1/2 for the first order difference scheme (10). The errors presented in this table indicates the accuracy of difference scheme.

## 5. Conclusion

In the present paper, the stability of the initial boundary value problem for the third order partial delay differential equation with nonlocal conditions is investigated. The first order of accuracy difference scheme for solution of this problem is presented. Stability estimates for solution of this difference scheme are proved. Numerical results are provided. Some statements of the present paper were published, without proof, in [23], [24].

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