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GENERALIZATION OF SOME TYPES OF DIFFERENCE SEQUENCE SPACE BY $\mathcal{I}$-CONVERGENCE

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Abstract: The purpose of this paper is to use the notions of $\mathcal{I}$-convergence of sequences and a Musielak-Orlicz function, as well as a sequence of modulus functions, to study certain difference sequence spaces, which are generalizations of the spaces investigated in [13]. Among other results, we investigated some algebraic and topological properties of these spaces. Moreover, we establish some inclusion relationships between these spaces under certain conditions on the mathematical tools that define them.

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1. Introduction

Sequence spaces have been a topic of great importance in the development of functional analysis from its origins to the present. The spaces ℓ_∞, c, c_0, ℓ_1 and ℓ_p were the first spaces that obtained relevance for being used as cases particular of Banach spaces. These sequence spaces have many applications in various branches of functional analysis, among which we can highlight the

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theory of functions, the theory of locally convex spaces, matrix transformations, and the theory of summability invariably (see [1], [2], [3], [11], [14], [15], [16]).

In 1971, Lindenstrauss and Tzafriri [9] used the notion of Orlicz function to construct the sequence spaces

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\},$$

where ω is the space of all complex sequences, and M is an Orlicz function. The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. In their work, Lindenstrauss and Tzafriri showed that each Orlicz sequence space ℓ_M is isomorphic to ℓ_p for some $p \geq 1$, answering positively a general conjecture that each infinite-dimensional Banach space contains a closed subspace isomorphic to c_0 or some ℓ_p, for a class of spaces. Later, the sequence spaces defined by using Orlicz functions continued to be the center of attention for many investigations, such as the work of Mursaleen et al. [12], in which the difference sequence spaces $\mathcal{C}_0(\Delta, M)$, $\mathcal{C}(\Delta, L)$ and $\ell_\infty(\Delta, M)$ were introduced, where $\Delta x_k = x_k - x_{k+1}$, $k = 1, 2, \cdots$. The sequence spaces described above are norm spaces with a suitable norm $\|\cdot\|_\Delta$ and also with this norm $\ell_\infty(\Delta, M)$ is a Banach space [12]. Following this line of research, in 2015, Raj and Kiliçman [13] used a seminormed space (X, q), where X is a complex linear space and q the seminorm, a Musielak-Orlicz function $\mathcal{M} = (M_k)$ (which is a sequence of Orlicz functions), a bounded sequence $p = (p_k)$ of nonnegative real numbers, and a sequence $u = (u_k)$ of positive real numbers, to study the difference sequence spaces $w_0(\mathcal{M}, \Delta_m^n, p, q, u)$, $w(\mathcal{M}, \Delta_m^n, p, q, u)$ and $w_\infty(\mathcal{M}, \Delta_m^n, p, q, u)$, showing that these spaces are linear and also that the space $w_\infty(\mathcal{M}, \Delta_m^n,$
$p, q, u)$ is paranormed. Similarly, in [13] also other spaces of sequences were studied, which were defined by replacing the Musielak-Orlicz function with a sequence of modulus functions in the spaces described above.

The notion of an ideal on a nonempty set was originally introduced in 1930, by Kuratowski [8] in the classic text “Topologie I”. Since then many mathematicians have based their research on generalizing concepts and properties of general topology and mathematical analysis using the notion of ideal due to Kuratowski. In particular, in 2000, Kostyrko et al. [6] used an ideal on the set $\mathbb{N}$ of the natural numbers to introduce the concept of $\mathcal{I}$-convergence, as a generalization of statistical convergence. In the last two decades, there have been some studies on sequence spaces using the notion of $\mathcal{I}$-convergence, as can be seen in [18] and [19]. The purpose of this paper is to use the notions of $\mathcal{I}$-convergence of sequences and a Musielak-Orlicz function, as well as a sequence of modulus functions, to study certain difference sequence spaces, which are generalizations of the spaces investigated in [13]. Among other results, we investigated some algebraic and topological properties of these spaces. In addition, we establish some inclusion relationships between these spaces under certain conditions on the mathematical tools that define them. This study provides new results that could be useful to address problems that arise in many investigations with interest in sequence spaces and matrix transformations in the context of summability theory.

2. Preliminaries

Let X be a linear space. A function $f : X \to \mathbb{R}$ is called *convex*, if the following inequality is hold:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in X$ and $\lambda \in [0, 1]$.

Theorem 1. [7] *If f is a convex function and $f(0) = 0$, then $f(\lambda x) \leq \lambda f(x)$, for all $\lambda \in [0, 1]$.*

An *Orlicz function* is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex, with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. According to [7], we say that an Orlicz function M satisfies Δ_2-*condition*, if for each $x \in [0, \infty)$, there exists a constant $K > 0$ such that $M(2x) \leq K\, M(x)$.

Theorem 2. [7] *An Orliz function M satisfies Δ_2-condition if and only if for each $x \in [0, \infty)$ and each $l > 1$ there exists a constant $R = R(l) > 0$ such that $M(lx) \leq R\, M(x) \leq R\, l\, M(x)$.*

Observe that the function $M(x) = x^p$ with $x \in [0, \infty)$ and $1 \leq p < \infty$, is an Orlicz function which satisfies Δ_2-condition, because $M(2x) = (2x)^p = 2^p x^p =$

$2^p M(x)$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function.*

Definition 3. A *modulus function* is a function $f : [0, \infty) \to [0, \infty)$ which satisfies the following properties:

(1) $f(x) = 0$ if only if $x = 0$;

(2) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$;

(3) f is increasing;

(4) f is continuous from right at 0.

From Definition 3, it follows that f must be continuous on $[0, \infty)$. A modulus function may be bounded or unbounded. For example, $f(x) = \dfrac{x}{x+1}$ is a modulus function bounded, while $f(x) = x^p$, with $0 < p < 1$, is a modulus function unbounded. If f is a modulus function, then $f(nx) \leq nf(x)$ for each $n \in \mathbb{N}$.

Theorem 4. [5] *Let f be a modulus function and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f(x) \leq 2f(1) \cdot \dfrac{x}{\delta}$.*

The following result is known as *Maddox's inequality*, [10]. Let $p = (p_k)$ be a bounded sequence such that $0 < p_k \leq \sup p_k = H < \infty$. Then,

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\},$$

where $D = \max\{1, 2^{H-1}\}$. This inequality plays an important role in the investigation of properties of many sequence spaces and in particular, those that we will deal with in this work.

The notion of $\mathcal{I}$-convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [6] using an ideal of subsets of $\mathbb{N}$. Next, we present the notion of $\mathcal{I}$-convergence of sequences, but before we recall the notion of an ideal of subsets of a given set, due to Kuratowski [8]. An *ideal* $\mathcal{I}$ on a set X is a nonempty collection of subsets of X which satisfies the following properties:

(1) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$;

(2) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

Let $\mathcal{I}$ be an ideal on X. According to [6], we say that $\mathcal{I}$ is *non trivial* if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$, while $\mathcal{I}$ is *admissible* if $\mathcal{I}$ is non trivial and $\{x\} \in \mathcal{I}$ for each $x \in X$. Given a metric space (X, d) and a non trivial ideal $\mathcal{I}$ of subsets of $\mathbb{N}$, we say that a sequence (x_k) of elements in X is *$\mathcal{I}$-convergent* to $L \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{k \in \mathbb{N} : d(x_k, L) \geq \varepsilon\}$ belongs to $\mathcal{I}$, see [6]. In this case, the element L is called the $\mathcal{I}$-limit of the sequence $x = (x_k)$ and is denote by $L = \mathcal{I}\text{-}\lim_{k\to\infty} x_k$. If $\mathcal{I}$ is an admissible ideal then usual convergence in X implies $\mathcal{I}$-convergence in X; also, if $\mathcal{I}$ does not contain any infinite set, both concepts coincide.

The notion the difference sequence spaces was introduced by Kızmaz [4], when he studied the space $Z(\Delta) = \{x = (x_k) \in w : \Delta x \in Z\}$, where $Z = \ell_\infty, c, c_0$, $\Delta x = x_k - x_{k+1}$ for all $k \in \mathbb{N}$, and w is the space of all real or complex sequences. This notion was extended by Triparthy et al. [17] by introducing a generalized difference operators as follows. If m, n are non-negative integers, then we have sequence spaces $Z(\Delta_m^n) = \{x = (x_k) \in w : \Delta_m^n x \in Z\}$, for $Z = \ell_\infty, c, c_0$, where $\Delta_m^n x = (\Delta_m^n x_k) = \left(\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m}\right)$ and $\Delta_m^0 x_k = x_k$ for each $k \in \mathbb{N}$. The difference $\Delta_m^n x_k$ has the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

In the sequel, we assume that X is a complex linear space with zero element $\mathbf{0}$ and let q be a seminorm on X. We consider sequences $x = (x_k)$, where $(x_k) \subset X$, with the usual coordinatewise operations: $\alpha x = (\alpha x_k)$ and $x + y = (x_k + y_k)$ for each $\alpha \in \mathbb{C}$. For a scalar sequence $\lambda = (\lambda_k)$ and $x = (x_k)$, we write $\lambda x = (\lambda_k x_k)$. Let $p = (p_k)$ be any bounded sequence of non-negative real numbers, $u = (u_k)$ be a sequence of positive real numbers, and $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. According to [13], we have the following difference sequence spaces defined by using a Musielak-Orlicz function:

$$w(\mathcal{M}, \Delta_m^n, p, q, u) = \Bigg\{ x = (x_k) : \frac{1}{n} \sum_{k=1}^{n} \vartheta_k(L, \rho) \to 0 \text{ as } n \to \infty,$$

$$\text{for some } L \in X,\ \rho > 0 \Bigg\},$$

$$w_0(\mathcal{M}, \Delta_m^n, p, q, u) = \Bigg\{ x = (x_k) : \frac{1}{n} \sum_{k=1}^{n} \vartheta_k(\rho) \to 0 \text{ as } n \to \infty,$$

$$\text{for some } \rho > 0 \Bigg\},$$

$$w_\infty(\mathcal{M},\Delta_m^n,p,q,u)=\left\{x=(x_k):\sup\frac{1}{n}\sum_{k=1}^{n}\vartheta_k(\rho)<\infty,\right.$$

$$\left.\text{for some }\ \rho>0\right\},$$

where

$$\vartheta_k(L,\rho)=\left[M_k\left(\frac{q(u_k\Delta_m^n x_k-L)}{\rho}\right)\right]^{p_k}$$

and

$$\vartheta_k(\rho)=\left[M_k\left(\frac{q(u_k\Delta_m^n x_k)}{\rho}\right)\right]^{p_k}.$$

Similarly, in [13] also some difference sequence spaces were introduced using sequences of modulus functions, which we present below. Let $F=(f_k)$ be a sequence of modulus functions, $p=(p_k)$ be any bounded sequence of non-negative real numbers and $u=(u_k)$ be a sequence of positive real numbers. Let (X,q) be a seminormed space by q. We define the following sequence spaces:

$$w(F,\Delta_m^n,p,q,u)=\left\{x=(x_k):\frac{1}{n}\sum_{k=1}^{n}\theta_k(L,\rho)\to 0\ \text{ as }\ n\to\infty,\right.$$

$$\left.\text{for some }\ L\in X,\ \rho>0\right\},$$

$$w_0(F,\Delta_m^n,p,q,u)=\left\{x=(x_k):\frac{1}{n}\sum_{k=1}^{n}\theta_k(\rho)\to 0\ \text{ as }\ n\to\infty,\right.$$

$$\left.\text{for some }\ \rho>0\right\},$$

$$w_\infty(F,\Delta_m^n,p,q,u)=\left\{x=(x_k):\sup\frac{1}{n}\sum_{k=1}^{n}\theta_k(\rho)<\infty,\right.$$

$$\left.\text{for some }\ \rho>0\right\},$$

where

$$\theta_k(L,\rho)=\left[f_k\left(\frac{q(u_k\Delta_m^n x_k-L)}{\rho}\right)\right]^{p_k}$$

and

$$\theta_k(\rho)=\left[f_k\left(\frac{q(u_k\Delta_m^n x_k)}{\rho}\right)\right]^{p_k}.$$

Motivated by the fact that the $\mathcal{I}$-convergence of sequences is a natural generalization of the usual convergence, next, we use the notion of $\mathcal{I}$-convergence to construct and study new difference sequences spaces which generalize the spaces introduced by Raj and Kiliçman [13].

3. Generalized difference sequence spaces defined by $\mathcal{I}$-convergence and a Musielak-Orlicz function

In this section, we consider a sequence of complex numbers $u = (u_k)$ such that $u_k \neq 0$ for each k, a bounded sequence of non-negative real numbers $p = (p_k)$, a sequence of seminorms $q = (q_k)$ on X and a Musielak-Orlicz function $\mathcal{M} = (M_k)$. For each $x = (x_k) \subset X$, we define the following sequence spaces:

$$\begin{aligned} w^{\mathcal{I}}&(\mathcal{M}, \Delta_m^n, p, q, u) \\ &= \Bigg\{ x : \forall \varepsilon > 0, \Bigg\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \Omega_k(L, \rho) \geq \varepsilon \Bigg\} \in \mathcal{I}, \\ &\qquad \text{for some } \ L \in X, \ \rho > 0 \Bigg\}, \end{aligned}$$

$$\begin{aligned} w_0^{\mathcal{I}}&(\mathcal{M}, \Delta_m^n, p, q, u) \\ &= \Bigg\{ x : \forall \varepsilon > 0, \Bigg\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \Omega_k(\rho) \geq \varepsilon \Bigg\} \in \mathcal{I}, \\ &\qquad \text{for some } \ \rho > 0 \Bigg\}, \end{aligned}$$

$$\begin{aligned} w_\infty^{\mathcal{I}}&(\mathcal{M}, \Delta_m^n, p, q, u) \\ &= \Bigg\{ x : \exists K > 0 \text{ such that } \Bigg\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \Omega_k(\rho) \geq K \Bigg\} \in \mathcal{I}, \\ &\qquad \text{for some } \ \rho > 0 \Bigg\}, \end{aligned}$$

where

$$\Omega_k(L, \rho) = \left[M_k \left(\frac{q_k(u_k \Delta_m^n x_k - L)}{\rho} \right) \right]^{p_k}$$

and

$$\Omega_k(\rho) = \left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho}\right)\right]^{p_k}.$$

Also, we define the spaces:

$$w_\infty(\mathcal{M},\Delta_m^n,p,q,u) = \left\{x : \sup_n \frac{1}{n}\sum_{k=1}^n \Omega_k(\rho) < \infty, \text{ for some } \rho > 0\right\},$$

$$\eta^{\mathcal{I}}(\mathcal{M},\Delta_m^n,p,q,u) = w^{\mathcal{I}}(\mathcal{M},\Delta_m^n,p,q,u) \cap w_\infty(\mathcal{M},\Delta_m^n,p,q,u) \text{ and}$$
$$\eta_0^{\mathcal{I}}(\mathcal{M},\Delta_m^n,p,q,u) = w_0^{\mathcal{I}}(\mathcal{M},\Delta_m^n,p,q,u) \cap w_\infty(\mathcal{M},\Delta_m^n,p,q,u).$$

Remark 5. It is important to highlight the presence of $\mathcal{I}$-convergence in the definition of some of the above spaces. For example,

$$w^{\mathcal{I}}(\mathcal{M},\Delta_m^n,p,q,u) = \left\{x : \mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \Omega_k(L,\rho) = 0,\right.$$
$$\left.\text{for some } L \in X,\ \rho > 0\right\}.$$

Theorem 6. *$w^{\mathcal{I}}(\mathcal{M},\Delta_m^n,p,q,u)$ is a linear space over the field of complex numbers $\mathbb{C}$.*

Proof. Assume that $x = (x_k)$, $y = (y_k) \in w^{\mathcal{I}}(\mathcal{M},\Delta_m^n,p,q,u)$ and $\alpha,\beta \in \mathbb{C}$. Let $\varepsilon > 0$ given. We show that there exist $\rho > 0$ and $L \in X$ such that

$$A = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^n \left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k + \beta y_k) - L)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in \mathcal{I}.$$

Since $x, y \in w^{\mathcal{I}}(\mathcal{M},\Delta_m^n,p,q,u)$, there exist two positive numbers ρ_1 and ρ_2 such that

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^n \left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k - L_1)}{\rho_1}\right)\right]^{p_k} \geq \frac{\varepsilon}{2D}\right\} \in \mathcal{I}$$

for some $L_1 \in X$ and

$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^n \left[M_k\left(\frac{q_k(u_k\Delta_m^n y_k - L_2)}{\rho_2}\right)\right]^{p_k} \geq \frac{\varepsilon}{2D}\right\} \in \mathcal{I}$$

for some $L_2 \in X$, where $D = \max\{1, 2^{H-1}\}$ and $H = \sup_k p_k \geq p_k > 0$. Let $\rho = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$ and $L = \alpha L_1 + \beta L_2$. Then, using the triangular inequality for q_k and the fact that M_k is non-decreasing, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k+\beta y_k)-L)}{\rho}\right)\right]^{p_k}\\
&= \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(\alpha(u_k\Delta_m^n x_k-L_1)+\beta(u_k\Delta_m^n y_k-L_2))}{\rho}\right)\right]^{p_k}\\
&\leq \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k-L_1)}{2\rho_1}+\frac{q_k(u_k\Delta_m^n y_k-L_2)}{2\rho_2}\right)\right]^{p_k}.
\end{aligned}$$

By the convexity of M_k, it follows that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k+\beta y_k)-L)}{\rho}\right)\right]^{p_k}\\
&\leq \frac{1}{n}\sum_{k=1}^{n}\left[\frac{1}{2}M_k\left(\frac{q_k(u_k\Delta_m^n x_k-L_1)}{\rho_1}\right)\right.\\
&+ \left.\frac{1}{2}M_k\left(\frac{q_k(u_k\Delta_m^n y_k-L_2)}{\rho_2}\right)\right]^{p_k}\\
&\leq \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{p_k}}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k-L_1)}{\rho_1}\right)\right.\\
&+ \left.M_k\left(\frac{q_k(u_k\Delta_m^n y_k-L_2)}{\rho_2}\right)\right]^{p_k}.
\end{aligned}$$

Applying Maddox's inequality, we obtain that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k+\beta y_k)-L)}{\rho}\right)\right]^{p_k}\\
&\leq \frac{1}{n}\sum_{k=1}^{n}\frac{D}{2^{p_k}}\left\{\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k-L_1)}{\rho_1}\right)\right]^{p_k}\right.\\
&+ \left.\left[M_k\left(\frac{q_k(u_k\Delta_m^n y_k-L_2)}{\rho_2}\right)\right]^{p_k}\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{D}{n}\left\{\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k - L_1)}{\rho_1}\right)\right]^{p_k}\right.\\
&+ \left.\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n y_k - L_2)}{\rho_2}\right)\right]^{p_k}\right\}.
\end{aligned}$$

If $n \in A_1^c \cap A_2^c$, then of this last inequality, we conclude that

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k + \beta y_k) - L)}{\rho}\right)\right]^{p_k}\\
&\leq D\left\{\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k - L_1)}{\rho_1}\right)\right]^{p_k}\right.\\
&+ \left.\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n y_k - L_2)}{\rho_2}\right)\right]^{p_k}\right\} < D\left\{\frac{\varepsilon}{2D} + \frac{\varepsilon}{2D}\right\} = \varepsilon,
\end{aligned}$$

which implies that $n \in A^c$. Therefore, $(A_1 \cup A_2)^c = A_1^c \cap A_2^c \subset A^c$ and so, $A \subset A_1 \cup A_2 \in \mathcal{I}$, it follows that $A \in \mathcal{I}$. This shows that $\alpha x + \beta y \in w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and hence, $w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ is a linear space. □

Corollary 7. *$w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ is a linear space over the field of complex numbers $\mathbb{C}$.*

Theorem 8. *$w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ is a linear space over the field of complex numbers $\mathbb{C}$.*

Proof. Assume that $x = (x_k)$, $y = (y_k) \in w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and $\alpha, \beta \in \mathbb{C}$. We show that there exist two positive numbers K and ρ such that

$$A = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k + \beta y_k))}{\rho}\right)\right]^{p_k} \geq K\right\} \in \mathcal{I}.$$

By hypothesis, there exist four positive numbers K_1, K_2, ρ_1 and ρ_2 such that

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho_1}\right)\right]^{p_k} \geq \frac{K_1}{2D}\right\} \in \mathcal{I}$$

and

$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n y_k)}{\rho_2}\right)\right]^{p_k} \geq \frac{K_2}{2D}\right\} \in \mathcal{I},$$

where $D = \max\{1, 2^{H-1}\}$ and $H = \sup_k p_k \geq p_k > 0$. Let $K = \frac{K_1+K_2}{2}$ and $\rho = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Then, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k+\beta y_k))}{\rho}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(\alpha(u_k\Delta_m^n x_k))}{\rho}+\frac{q(\beta(u_k\Delta_m^n y_k))}{\rho}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{2\rho_1}+\frac{q_k(u_k\Delta_m^n y_k)}{2\rho_2}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{p_k}}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho_1}\right)+M_k\left(\frac{q_k(u_k\Delta_m^n y_k)}{\rho_2}\right)\right]^{p_k}.$$

By Maddox's inequality, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k+\beta y_k))}{\rho}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\frac{D}{2^{p_k}}\left\{\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho_1}\right)\right]^{p_k} + \left[M_k\left(\frac{q_k(u_k\Delta_m^n y_k)}{\rho_2}\right)\right]^{p_k}\right\}$$

$$\leq \frac{D}{n}\left\{\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho_1}\right)\right]^{p_k} + \sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n y_k)}{\rho_2}\right)\right]^{p_k}\right\}.$$

If $n \in A_1^c \cap A_2^c$, then of this last inequality, we get that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n(\alpha x_k+\beta y_k))}{\rho}\right)\right]^{p_k}$$

$$\leq D\left\{\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho_1}\right)\right]^{p_k}\right.$$

$$+ \quad \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n y_k)}{\rho_2}\right)\right]^{p_k}\Bigg\} < D\left\{\frac{K_1}{2D}+\frac{K_2}{2D}\right\} = K,$$

which implies that $n \in A^c$. Thus, $(A_1 \cup A_2)^c = A_1^c \cap A_2^c \subset A^c$ and hence, $A \subset A_1 \cup A_2 \in \mathcal{I}$, it follows that $A \in \mathcal{I}$. This shows that $\alpha x + \beta y \in w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and hence, $w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ is a linear space over the field of complex numbers $\mathbb{C}$. ☐

Corollary 9. *$\eta^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and $\eta_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Theorem 10. *$\eta^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and $\eta_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ are paranormed spaces with paranorm defined by*

$$\varphi(x) = \inf\left\{\rho^{\frac{p_k}{H}} : \sup_n \left\{\frac{1}{n}\sum_{k=1}^{n}\Omega_k(\rho)\right\}^{\frac{1}{H}} \le 1,\ \rho > 0\right\},$$

where $H = \max\{1, \sup_k p_k\}$.

Proof. The proof is similar to that of [13, Theorem 2.2]. ☐

Theorem 11. *Let $\mathcal{M} = (M_k)$ and $\mathcal{M}' = (M_k')$ be two Musielak-Orlicz functions which satisfies the Δ_2-condition. The following statements hold:*

(1) *If $p = (p_k)$ is a bounded sequence of non-negative real numbers with $H = \sup p_k < \infty$, then $Z(\mathcal{M}, \Delta_m^n, p, q, u) \subseteq Z(\mathcal{M}' \circ \mathcal{M}, \Delta_m^n, p, q, u)$ for $Z = w_0^{\mathcal{I}}$, $w^{\mathcal{I}}$, $w_\infty^{\mathcal{I}}$, $\eta^{\mathcal{I}}$, $\eta_0^{\mathcal{I}}$.*

(2) *$Z(\mathcal{M}, \Delta_m^n, p, q, u) \cap Z(\mathcal{M}', \Delta_m^n, p, q, u) \subseteq Z(\mathcal{M} + \mathcal{M}', \Delta_m^n, p, q, u)$ for $Z = w_0^{\mathcal{I}}$, $w^{\mathcal{I}}$, $w_\infty^{\mathcal{I}}$, $\eta^{\mathcal{I}}$, $\eta_0^{\mathcal{I}}$.*

Proof. (1) Let $x = (x_k) \in w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$. There exists $\rho > 0$ such that

$$\mathcal{I}\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k\left(u_k\Delta_m^n x_k\right)}{\rho}\right)\right]^{p_k} = 0.$$

Let $\varepsilon > 0$ given. Since each M_k is continuous from right at 0, there exists $0 < \delta < 1$ such that $0 \le t \le \delta$ implies that $M_k(t) < \varepsilon$. Let $y_k = M_k\left(\dfrac{q_k\left(u_k\Delta_m^n x_k\right)}{\rho}\right)$

for each $k \in \mathbb{N}$. We define the sets $N_1 = \{k \in \{1, \dots, n\} : y_k \leq \delta\}$ and $N_2 = \{k \in \{1, \dots, n\} : y_k > \delta\}$. Observe that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k'(y_k)\right]^{p_k} = \frac{1}{n}\sum_{k\in N_1}\left[M_k'(y_k)\right]^{p_k} + \frac{1}{n}\sum_{k\in N_2}\left[M_k'(y_k)\right]^{p_k}.$$

If $k \in N_1$, then $0 \leq y_k \leq 1$. By Theorem 1, we have $M_k'(y_k) \leq y_k \cdot M_k'(1)$ and so,

$$\begin{aligned}\frac{1}{n}\sum_{k\in N_1}\left[M_k'(y_k)\right]^{p_k} &\leq \frac{1}{n}\sum_{k\in N_1}\left[M_k'(1)\cdot y_k\right]^{p_k}\\ &\leq \max\left\{[M_k'(1)]^H\right\}\cdot\frac{1}{n}\sum_{k\in N_1}[y_k]^{p_k}.\end{aligned}$$

For $k \in N_2$, we have $y_k < \dfrac{y_k}{\delta} < 1 + \dfrac{y_k}{\delta}$. Since each M_k' is non-decreasing and satisfies Δ_2-condition for each $k \in \mathbb{N}$, by Theorem 2, there exists a constant $R > 0$ such that

$$\begin{aligned}M_k'(y_k) &< M_k'\left(1+\frac{y_k}{\delta}\right) = M_k'\left(\left(1+\frac{y_k}{\delta}\right)\cdot 1\right)\\ &\leq R\cdot M_k'(1) < R\cdot M_k'(1)\cdot\frac{y_k}{\delta}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{n}\sum_{k\in N_2}\left[M_k'(y_k)\right]^{p_k} &\leq \frac{1}{n}\sum_{k\in N_2}\left[R\cdot M_k'(1)\cdot\frac{y_k}{\delta}\right]^{p_k}\\ &\leq \left(\frac{R}{\delta}\right)^H\cdot\max\left\{[M_k'(1)]^H\right\}\cdot\frac{1}{n}\sum_{k\in N_2}[y_k]^{p_k}.\end{aligned}$$

Consequently, from of above results, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k'(y_k)\right]^{p_k} \leq \left[1+\left(\frac{R}{\delta}\right)^H\right]\cdot\max\left\{[M_k'(1)]^H\right\}\cdot\frac{1}{n}\sum_{k=1}^{n}[y_k]^{p_k}.$$

Since $\mathcal{I}\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[y_k]^{p_k} = 0$, we conclude that

$$\begin{aligned}&\mathcal{I}\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[M_k'\left(M_k\left(\frac{q_k\left(u_k\Delta_m^n x_k\right)}{\rho}\right)\right)\right]^{p_k}\\ &\qquad = \mathcal{I}\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[M_k'(y_k)\right]^{p_k} = 0.\end{aligned}$$

Thus, $x = (x_k) \in w_0^{\mathcal{I}}(\mathcal{M}' \circ \mathcal{M}, \Delta_m^n, p, q, u)$ and hence,

$$w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u) \subseteq w_0^{\mathcal{I}}(\mathcal{M}' \circ \mathcal{M}, \Delta_m^n, p, q, u).$$

(2) Let $x = (x_k) \in w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u) \cap w_0^{\mathcal{I}}(\mathcal{M}', \Delta_m^n, p, q, u)$. There exist two positive numbers ρ_1 and ρ_2 such that

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{q_k\left(u_k \Delta_m^n x_k\right)}{\rho_1} \right) \right]^{p_k} = 0$$

and

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[M_k' \left(\frac{q_k\left(u_k \Delta_m^n x_k\right)}{\rho_2} \right) \right]^{p_k} = 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$ and $M_k'' = M_k + M_k'$. By Maddox's inequality, we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} \left[M_k'' \left(\frac{q_k\left(u_k \Delta_m^n x_k\right)}{\rho} \right) \right]^{p_k} &\leq D \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{q_k\left(u_k \Delta_m^n x_k\right)}{\rho_1} \right) \right]^{p_k} \right. \\ &+ \left. \frac{1}{n} \sum_{k=1}^{n} \left[M_k' \left(\frac{q_k\left(u_k \Delta_m^n x_k\right)}{\rho_2} \right) \right]^{p_k} \right\}. \end{aligned}$$

From this inequality, we conclude that

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[\left(M_k + M_k'\right) \left(\frac{q_k\left(u_k \Delta_m^n x_k\right)}{\rho} \right) \right]^{p_k} = 0$$

and hence, $x = (x_k) \in w_0^{\mathcal{I}}(\mathcal{M} + \mathcal{M}', \Delta_m^n, p, q, u)$. Thus,

$$w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u) \cap w_0^{\mathcal{I}}(\mathcal{M}', \Delta_m^n, p, q, u) \subseteq w_0^{\mathcal{I}}(\mathcal{M} + \mathcal{M}', \Delta_m^n, p, q, u).$$

□

Theorem 12. *If $n, m \geq 1$, then the following inclusions hold:*

(1) $w^{\mathcal{I}}(\mathcal{M}, \Delta_m^{n-1}, p, q, u) \subseteq w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$.

(2) $w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^{n-1}, p, q, u) \subseteq w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$.

(3) $\eta^{\mathcal{I}}(\mathcal{M}, \Delta_m^{n-1}, p, q, u) \subseteq \eta^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$.

(4) $\eta_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^{n-1}, p, q, u) \subseteq \eta_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$.

(5) $w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^{n-1}, p, q, u) \subseteq w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$.

Proof. We will only prove (1). The proof of (2) is an immediate consequence of (1) and, the proofs of (3)-(5) are similar to the proof of (1). Suppose that $x = (x_k) \in w^{\mathcal{I}}(\mathcal{M}, \Delta_m^{n-1}, p, q, u)$. Let $\varepsilon > 0$ given. We will show that there exist $\rho > 0$ and $L \in X$ such that

$$A = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{q_k(u_k \Delta_m^n x_k - L)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in \mathcal{I}.$$

Since $x = (x_k) \in w^{\mathcal{I}}(\mathcal{M}, \Delta_m^{n-1}, p, q, u)$, there exist $\rho_1 > 0$ and $L_1 \in X$ such that

$$B = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{q_k(u_k \Delta_m^{n-1} x_k - L_1)}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \in \mathcal{I},$$

where $D = \max\{1, 2^{H-1}\}$ y $H = \sup_k p_k \geq p_k > 0$. Let $\rho = 2\rho_1$ and $L = 2L_1$. Put $y_k = M_k \left(\frac{q_k(u_k \Delta_m^n x_k - L)}{\rho} \right)$ for each $k \in \mathbb{N}$. Since M_k is non-decreasing and convex for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{1}{n} \sum_{k=1}^{n} [y_k]^{p_k} \\
&\leq \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{q_k(u_k \Delta_m^{n-1} x_k - L_1)}{2\rho_1} + \frac{q_k(u_k \Delta_m^{n-1} x_{k+m} - L_1)}{2\rho_1} \right) \right]^{p_k} \\
&\leq \frac{1}{n} \sum_{k=1}^{n} \left[\frac{1}{2} M_k \left(\frac{q_k(u_k \Delta_m^{n-1} x_k - L_1)}{\rho_1} \right) \right. \\
&+ \left. \frac{1}{2} M_k \left(\frac{q_k(u_k \Delta_m^{n-1} x_{k+m} - L_1)}{\rho_1} \right) \right]^{p_k} \\
&\leq \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{q_k(u_k \Delta_m^{n-1} x_k - L_1)}{\rho_1} \right) \right. \\
&+ \left. M_k \left(\frac{q_k(u_k \Delta_m^{n-1} x_{k+m} - L_1)}{\rho_1} \right) \right]^{p_k}.
\end{aligned}$$

By Maddox's inequality, it follows that

$$\frac{1}{n} \sum_{k=1}^{n} [y_k]^{p_k} \leq D \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[M_k \left(\frac{q_k(u_k \Delta_m^{n-1} x_k - L_1)}{\rho_1} \right) \right]^{p_k} \right.$$

$$+ \quad \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^{n-1}x_{k+m}-L_1)}{\rho_1}\right)\right]^{p_k}\Bigg\}.$$

If $n \in B^c$, then from above inequality, we get

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^{n}x_{k}-L)}{\rho}\right)\right]^{p_k} \quad \leq \quad D\left\{\frac{\varepsilon}{2D}+\frac{\varepsilon}{2D}\right\}=\varepsilon,$$

which implies that $n \in A^c$. Consequently, $A \subset B \in \mathcal{I}$ and hence, $A \in \mathcal{I}$. This shows that $x=(x_k) \in w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and so, $w^{\mathcal{I}}(\mathcal{M}, \Delta_m^{n-1}, p, q, u) \subseteq w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$. □

Theorem 13. *If $0 < p_k \leq r_k$ for each $k \in \mathbb{N}$ and $\frac{r_k}{p_k}$ is bounded, then*

$$Z(\mathcal{M}, \Delta_m^n, r, q, u) \subset Z(\mathcal{M}, \Delta_m^n, p, q, u) \text{ for each } Z = w^{\mathcal{I}}, w_0^{\mathcal{I}}.$$

Proof. Let $x=(x_k) \in w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, r, q, u)$. Given $\epsilon > 0$ there exist $\rho > 0$ and $L \in X$ such that

$$A=\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^{n}x_{k}-L)}{\rho}\right)\right]^{r_k} \geq \varepsilon\right\} \in \mathcal{I}.$$

That is,

$$\mathcal{I}\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^{n}x_{k}-L)}{\rho}\right)\right]^{r_k}=0.$$

Let $t_k = \left[M_k\left(\frac{q_k(u_k\Delta_m^{n}x_{k}-L)}{\rho}\right)\right]^{r_k}$ and $\lambda_k = \frac{p_k}{r_k}$, for each $k \in \mathbb{N}$. Since $0 < p_k \leq r_k$, we have $0 < \lambda_k \leq 1$ for each $k \in \mathbb{N}$. We choose $0 < \lambda < \lambda_k$ for each $k \in \mathbb{N}$. We define the sequences (φ_k) and (ψ_k) as follows:

$$\varphi_k = \begin{cases} 0, & \text{si } t_k < 1 \\ t_k, & \text{si } t_k \geq 1 \end{cases} \qquad \text{and} \qquad \psi_k = \begin{cases} t_k, & \text{si } t_k < 1 \\ 0, & \text{si } t_k \geq 1. \end{cases}$$

Obviously, $t_k = \varphi_k + \psi_k$ and $t_k^{\lambda_k} = \varphi_k^{\lambda_k} + \psi_k^{\lambda_k}$, for each $k \in \mathbb{N}$. According to this, $\varphi_k^{\lambda_k} \leq \varphi_k \leq t_k$ and $\psi_k^{\lambda_k} \leq \psi_k^{\lambda}$. By Hölder's inequality, it follows that

$$\begin{aligned} &\frac{1}{n}\sum_{k=1}^{n} t_k^{\lambda_k} \\ &= \frac{1}{n}\sum_{k=1}^{n}\left(\varphi_k^{\lambda_k}+\psi_k^{\lambda_k}\right) \leq \frac{1}{n}\sum_{k=1}^{n}\left(t_k+\psi_k^{\lambda}\right) \end{aligned}$$

$$= \frac{1}{n}\sum_{k=1}^{n} t_k + \sum_{k=1}^{n}\left(\frac{1}{n}\psi_k\right)^{\lambda}\left(\frac{1}{n}\right)^{1-\lambda}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n} t_k + \left(\sum_{k=1}^{n}\frac{1}{n}\psi_k\right)^{\lambda}\left(\sum_{k=1}^{n}\frac{1}{n}\right)^{1-\lambda} \leq \frac{1}{n}\sum_{k=1}^{n} t_k + \left(\frac{1}{n}\sum_{k=1}^{n} t_k\right)^{\lambda}.$$

From the above inequality, we have

$$0 \leq \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k - L)}{\rho}\right)\right]^{p_k}$$

$$= \frac{1}{n}\sum_{k=1}^{n} t_k^{\lambda_k} \leq \frac{1}{n}\sum_{k=1}^{n} t_k + \left(\sum_{k=1}^{n} t_k\right)^{\lambda}.$$

Since $\mathcal{I}\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} t_k = 0$, we conclude that

$$\mathcal{I}\text{-}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k - L)}{\rho}\right)\right]^{p_k} = 0.$$

This shows that $x = (x_k) \in w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and therefore,

$$w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, r, q, u) \subset w^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u).$$ □

Theorem 14. *The sequence spaces $w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and $w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ are solid and hence monotone.*

Proof. Let $x = (x_k) \in w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ and let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for each $k \in \mathbb{N}$. There exist two positive numbers ρ and K such that

$$A = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho}\right)\right]^{p_k} \geq K\right\} \in \mathcal{I}$$

Put

$$B = \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n \alpha_k x_k)}{\rho}\right)\right]^{p_k} \geq K\right\}.$$

If $n \notin A$, then

$$\frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n\alpha_k x_k)}{\rho}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}|\alpha_k|^{p_k}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho}\right)\right]^{p_k}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho}\right)\right]^{p_k} < K,$$

which implies that $n \notin B$. Thus, $B \subset A \in \mathcal{I}$ and hence, $B \in \mathcal{I}$. This shows that $(\alpha_k x_k) \in w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ for all sequence of scalars (α_k) with $|\alpha_k| \leq 1$ for each $k \in \mathbb{N}$, whenever $(x_k) \in w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$. Therefore, $w_\infty^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ is solid.

The proof for $w_0^{\mathcal{I}}(\mathcal{M}, \Delta_m^n, p, q, u)$ is similar. □

4. Generalized difference sequence spaces defined by $\mathcal{I}$-convergence and a sequence of modulus functions

In this section, we consider a sequence of modulus functions $F = (f_k)$, a bounded sequence of non-negative real numbers $p = (p_k)$ and a sequence of complex numbers $u = (u_k)$ with $u_k \neq 0$. Le $q = (q_k)$ be a sequence of norms on X. Now, we define the following spaces:

$$w^{\mathcal{I}}(F, \Delta_m^n, p, q, u) = \left\{x : \forall\varepsilon > 0, \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Theta_k(L, \rho) \geq \varepsilon\right\} \in \mathcal{I}, \text{ for some } L \in X \text{ and some } \rho > 0\right\},$$

$$w_0^{\mathcal{I}}(F, \Delta_m^n, p, q, u) = \left\{x : \forall\varepsilon > 0, \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Theta_k(\rho) \geq \varepsilon\right\} \in \mathcal{I}, \text{ for some } \rho > 0\right\},$$

$$w_\infty^{\mathcal{I}}(F,\Delta_m^n,p,q,u)$$
$$=\left\{x : \exists K>0 \text{ such that } \left\{n\in\mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Theta_k(\rho)\geq K\right\}\in\mathcal{I}, \right.$$
$$\left. \text{for some } \rho>0\right\}.$$

where
$$\Theta_k(L,\rho)=\left[f_k\left(\frac{q_k(u_k\Delta_m^n x_k-L)}{\rho}\right)\right]^{p_k}$$
and
$$\Theta_k(\rho)=\left[f_k\left(\frac{q_k(u_k\Delta_m^n x_k)}{\rho}\right)\right]^{p_k}.$$

Also, we define the following spaces:

$$w_\infty(F,\Delta_m^n,p,q,u)=\left\{x : \sup_n\frac{1}{n}\sum_{k=1}^{n}\Theta_k(\rho)<\infty, \text{ for some } \rho>0\right\},$$
$$\eta^{\mathcal{I}}(F,\Delta_m^n,p,q,u)=w^{\mathcal{I}}(F,\Delta_m^n,p,q,u)\cap w_\infty(F,\Delta_m^n,p,q,u) \text{ and}$$
$$\eta_0^{\mathcal{I}}(F,\Delta_m^n,p,q,u)=w_0^{\mathcal{I}}(F,\Delta_m^n,p,q,u)\cap w_\infty(F,\Delta_m^n,p,q,u).$$

The proofs of Theorems 6, 8 and 10 holds along the same lines for the following two theorems and so we omit them.

Theorem 15. *The sequence spaces* $w^{\mathcal{I}}(F,\Delta_m^n,p,q,u)$, $w_0^{\mathcal{I}}(F,\Delta_m^n,$ $p,q,u)$, $w_\infty^{\mathcal{I}}(F,\Delta_m^n,p,q,u)$, $\eta^{\mathcal{I}}(F,\Delta_m^n,p,q,u)$ *and* $\eta_0^{\mathcal{I}}(F,\Delta_m^n,p,q,u)$ *are linear spaces over the complex field* $\mathbb{C}$.

Theorem 16. $\eta^{\mathcal{I}}(F,\Delta_m^n,p,q,u)$ *and* $\eta_0^{\mathcal{I}}(F,\Delta_m^n,p,q,u)$ *are paranormed spaces with paranorm defined by*

$$\varphi(x)=\inf\left\{\rho^{\frac{p_k}{H}} : \sup_n\left\{\frac{1}{n}\sum_{k=1}^{n}\Theta_k(\rho)\right\}^{\frac{1}{H}}\leq 1,\ \rho>0\right\},$$

where $H=\max\{1,\sup_k p_k\}$.

Theorem 17. *Let* $F=(f_k)$ *and* $G=(g_k)$ *be two sequences of modulus functions. For any two seminorms sequence* $q=(q_k)$ *and* $r=(r_k)$*, the following properties are satisfied:*

(1) $Z(F, \Delta_m^n, q, u) \subset Z(G \circ F, \Delta_m^n, q, u)$ *for* $Z = w_0^{\mathcal{I}}$, $w^{\mathcal{I}}$, $w_\infty^{\mathcal{I}}$, $\eta^{\mathcal{I}}$, $\eta_0^{\mathcal{I}}$.

(2) $Z(F, \Delta_m^n, p, q, u) \cap Z(F, \Delta_m^n, p, r, u) \subseteq Z(F, \Delta_m^n, p, q + r, u)$ *for* $Z = w_0^{\mathcal{I}}$, $w^{\mathcal{I}}$, $w_\infty^{\mathcal{I}}$, $\eta^{\mathcal{I}}$, $\eta_0^{\mathcal{I}}$.

(3) $Z(F, \Delta_m^n, p, q, u) \cap Z(G, \Delta_m^n, p, q, u) \subseteq Z(F + G, \Delta_m^n, p, q, u)$ *for* $Z = w_0^{\mathcal{I}}$, $w^{\mathcal{I}}$, $w_\infty^{\mathcal{I}}$, $\eta^{\mathcal{I}}$, $\eta_0^{\mathcal{I}}$.

Proof. (1) We will only show the inclusion $w_0^{\mathcal{I}}(F, \Delta_m^n, q, u) \subset w_0^{\mathcal{I}}(F \circ G, \Delta_m^n, q, u)$. The other inclusions are showed analogously. Let $(x_k) \in w_0^{\mathcal{I}}(F, \Delta_m^n, q, u)$. Given $\varepsilon > 0$, we choose $0 < \delta < 1$ such that $g_k(t) < \frac{\varepsilon}{2}$ for each $0 \le t \le \delta$ and each $k \in \mathbb{N}$. By hypothesis,

$$A = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{q_k(u_k \Delta_m^n x_k)}{\rho_1} \right) \right] \ge \frac{\varepsilon}{2R} \right\} \in \mathcal{I},$$

where $R = 2\delta^{-1} g_k(1)$. Put $y_k = f_k \left(\frac{q_k(u_k \Delta_m^n x_k)}{\rho} \right)$ for each $k \in \mathbb{N}$, and let

$$B = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} [g_k(y_k)] \ge \varepsilon \right\}.$$

We will show that $B \subset A$. Suppose that $n \notin A$ and we consider the sets $N_1 = \{k \in \{1, \ldots, n\} : y_k \le \delta\}$ and $N_2 = \{k \in \{1, \ldots, n\} : y_k > \delta\}$. Then,

$$\frac{1}{n} \sum_{k=1}^{n} [g_k(y_k)] = \frac{1}{n} \sum_{k \in N_1} [g_k(y_k)] + \frac{1}{n} \sum_{k \in N_2} [g_k(y_k)].$$

If $k \in N_1$, then $0 \le y_k \le \delta$ and, by the continuity of g_k, it follows that

$$\sum_{k \in N_1} [g_k(y_k)] \le \sum_{k=1}^{n} [g_k(y_k)] < \frac{n\varepsilon}{2}.$$

For $k \in N_2$, we obtain by Theorem 4, the inequality $g_k(y_k) \le 2 g_k(1) \cdot \frac{y_k}{\delta}$. Thus,

$$\sum_{k \in N_2} [g_k(y_k)] \le 2\delta^{-1} g_k(1) \sum_{k \in N_2} y_k \le 2\delta^{-1} g_k(1) \sum_{k=1}^{n} y_k.$$

From the results previously deduced, it follows that

$$\frac{1}{n} \sum_{k=1}^{n} [g_k(y_k)] < \frac{1}{n} \cdot \left(\frac{n\varepsilon}{2} \right) + 2\delta^{-1} g_k(1) \cdot \frac{1}{n} \sum_{k=1}^{n} y_k < \frac{\varepsilon}{2} + R \cdot \frac{\varepsilon}{2R} = \varepsilon,$$

which implies that $n \notin B$. Hence, $B \subset A \in \mathcal{I}$ and so, $B \in \mathcal{I}$. This shows that $x = (x_k) \in w_0^{\mathcal{I}}(F \circ G, \Delta_m^n, q, u)$.

(2) Let $x = (x_k) \in w_0^{\mathcal{I}}(F, \Delta_m^n, p, q, u) \cap w_0^{\mathcal{I}}(F, \Delta_m^n, p, r, u)$. Then, there exist two positive numbers ρ_1 and ρ_2 such that

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho_1} \right) \right]^{p_k} = 0$$

and

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{r_k \left(u_k \Delta_m^n x_k \right)}{\rho_2} \right) \right]^{p_k} = 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. Then,

$$\begin{aligned}
&\frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{(q_k + r_k) \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} \\
&\quad = \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} + \frac{r_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} \\
&\quad \leq \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) + f_k \left(\frac{r_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} \\
&\quad \leq \frac{D}{n} \sum_{k=1}^{n} \left\{ \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} + \left[f_k \left(\frac{r_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} \right\} \\
&\quad \leq D \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho_1} \right) \right]^{p_k} \right. \\
&\quad + \left. \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{r_k \left(u_k \Delta_m^n x_k \right)}{\rho_2} \right) \right]^{p_k} \right\}
\end{aligned}$$

From this last inequality, we conclude that

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{(q_k + r_k) \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} = 0$$

and hence, $x = (x_k) \in w_0^{\mathcal{I}}(F, \Delta_m^n, p, q+r, u)$. This shows that $w_0^{\mathcal{I}}(F, \Delta_m^n, p, q, u) \cap w_0^{\mathcal{I}}(F, \Delta_m^n, p, r, u) \subseteq w_0^{\mathcal{I}}(F, \Delta_m^n, p, q+r, u)$.

(3) Let $x = (x_k) \in w_0^{\mathcal{I}}(F, \Delta_m^n, p, q, u) \cap w_0^{\mathcal{I}}(G, \Delta_m^n, p, q, u)$. Then, there exist two positive numbers ρ_1 and ρ_2 such that

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho_1} \right) \right]^{p_k} = 0$$

and

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[g_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho_2} \right) \right]^{p_k} = 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. Then, applying Maddox's inequality, we obtain that

$$\begin{aligned}
&\frac{1}{n} \sum_{k=1}^{n} \left[(f_k + g_k) \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} \\
&= \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) + g_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} \\
&\le \frac{D}{n} \sum_{k=1}^{n} \left\{ \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} + \left[g_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} \right\} \\
&\le D \left\{ \frac{1}{n} \sum_{k=1}^{n} \left[f_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho_1} \right) \right]^{p_k} \right. \\
&+ \left. \frac{1}{n} \sum_{k=1}^{n} \left[g_k \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho_2} \right) \right]^{p_k} \right\}.
\end{aligned}$$

From this inequality, we conclude that

$$\mathcal{I}\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[(f_k + g_k) \left(\frac{q_k \left(u_k \Delta_m^n x_k \right)}{\rho} \right) \right]^{p_k} = 0$$

and therefore, $x = (x_k) \in w_0^{\mathcal{I}}(F + G, \Delta_m^n, p, q, u)$. Thus,

$$w_0^{\mathcal{I}}(F, \Delta_m^n, p, q, u) \cap w_0^{\mathcal{I}}(G, \Delta_m^n, p, q, u) \subseteq w_0^{\mathcal{I}}(F + G, \Delta_m^n, p, q, u).$$

□

Corollary 18. *Let $G = (g_k)$ be a sequence of modulus functions. Then,*

$$Z(\Delta_m^n, q, u) \subset Z(G, \Delta_m^n, q, u) \textit{ for } Z = w_0^{\mathcal{I}},\, w^{\mathcal{I}},\, w_\infty^{\mathcal{I}},\, \eta^{\mathcal{I}},\, \eta_0^{\mathcal{I}}.$$

References

[1] J. Banaś, M. Mursaleen, *Sequences Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Delhi (2014).

[2] J. A. Fridy, On statistical convergence, *Analysis*, **5** (1985), 301-313.

[3] P. K. Kamthan, M. Gupta, *Sequences Spaces and Series*, Marcel Dekker, New York (1981).

[4] H. Kızmaz, On certain sequence spaces, *Canad. Math. Bull.*, **24** (1981), 169-176.

[5] S. Pehlivan, B. Fisher, On some sequence spaces, *Indian J. Pure Appl. Math.*, **25** (10) (1994), 1067-1071.

[6] P. Kostyrko, T. Salát, W. Wilezynski, $\mathcal{I}$-convergence, *Real Anal. Exchange*, **26** (2000), 669-686.

[7] Krasnosel'skiĭ, Ya. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, P. Noordhoff Ltd., Groningen (1961).

[8] K. Kuratowski, *Topologie I*, Monografie Matematyczne Tom 3, PWN-Polish Scientific Publishers, Warszawa (1933).

[9] J. Lindenstrauss, L. Tzafriri, On Orlicz sequences spaces, *Israel J. Math.*, **10** (1971), 379-390.

[10] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, London (1970).

[11] M. Mursaleen, *Applied Summability Methods*, Springer, Cham (2014).

[12] M. Mursaleen, M. A. Khan, Qamaruddin, Difference sequence spaces defined by Orlicz function, *Demonstratio Math.*, **32**, No 1 (1999), 145-150.

[13] K. Raj, A. Kiliçman, On certain generalized paranormed spaces, *J. Inequal. Appl.*, **2015** (2015), Art. 37, 12 pp.; DOI 10.1186/s13660-015-0565-z.

[14] T. Śalát, On statistical convergent sequences of real numbers, *Math. Slovaca* **30** (1980), 139-150.

[15] B. C. Tripathy, Matrix transformation between some classes of sequences, *J. Math. Anal. Appl.*, **206** (1997), 448-450.

[16] B. C. Tripathy, On statistical convergence, *Proc. Estonian Acad. Sci. Phy. Math. Analysis*, **47** (1998), 299-303.

[17] B. C. Tripathy, A. Esi, B. Tripathy, On a new type of generalized difference Cesàro sequence spaces, *Soochow J. Math.*, **31**, No 3 (2005), 333-340.

[18] B. C. Tripathy, B. Hazarika, Paranorm $\mathcal{I}$-convergent sequence spaces, *Math. Slovaca*, **59** (2009), 485-494.

[19] B. C. Tripathy, B. Hazarika, Some $\mathcal{I}$-convergent sequence spaces defined by Orlicz functions, *Acta Math. Appl. Sin. Engl. Ser.*, **27**, No 1 (2011), 149-154.