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# ON THE GENERALIZED KOBER TYPE FRACTIONAL $q$-INTEGRAL OPERATOR INVOLVING A BASIC ANALOGUE OF $H$-FUNCTION

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**Abstract:** In this paper, the generalized fractional $q$-integral operator of the Kober type is applied to the basic analogue of the $H$-function. Results involving the basic hypergeometric functions $J_\nu(x;q)$, $Y_\nu(x;q)$, $K_\nu(x;q)$, $H_\nu(x;q)$, ${}_{r+1}\phi_r(\cdot)$, Mac-Robert's $E$-function and several elementary $q$-functions, have been deduced as particular cases of the main result.

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**Key Words:** $q$-fractional calculus, applications, fractional $q$-integral operator, basic analogue of the Fox $H-$function, elementary and special $q$-functions

## 1. Introduction

Many researchers have considered and developed operators in $q$-calculus, which are established as basic analogues of the well-known fractional calculus operators and have applied them to different $q$-functions to generate tables of integrals. The Riemann-Liouville fractional $q$-integral operator was applied in

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2004 by Yadav and Purohit [12] to certain q-hypergeometric functions and later in 2005 by Kalla et al. [8] to the basic analogue of the Fox $H$-function, deriving results that include elementary and special $q$-functions, as particular cases of the main result. Kober's fractional $q$-integral operator was used by Saxena et al. In 2005 [11] to establish an extension for the basic analog of Fox's $H$-function and derive various special cases, and in 2006 by Yadav and Purohit [13] to apply it to certain basic hypergeometric functions. In 2008 Delgado and Galué [3] defined the $L$ operator and applied it to the basic analogue of the Fox $H$-function to establish a table of integrals. In 2009 Galué [4] defines the generalized Erdélyi-Kober fractional $q$-integral operator and applies it to the basic analogue of the Fox $H$-function; from its result derive a table of integrals for various $q$-functions. In 2010 Yadav et al. [14] considered the L operator and applied it to the basic analogue of the multiple hypergeometric function; this result was used in 2012 by Galué [5] to obtain the $q$-fractional integral operator L of the generalized basic hypergeometric function ${}_r\phi_s(\cdot)$. From the main result, he derived several interesting special cases, including q-special functions.

### 1.1. Basic analogue of the Fox $H-$function

$$H_{A,B}^{m,n}\left[x;q\,\middle|\,\begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right]$$
$$=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\alpha_j s}\right)\pi x^s ds}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-\alpha_j s}\right)G\left(q^{1-s}\right)\sin\left(\pi s\right)}, \tag{1}$$

where $0\leq m\leq B, 0\leq n\leq A$; $\alpha_i\,(i=1,...,A)$ and $\beta_j\,(j=1,...,B)$ are all positive integers. The contour $C$ is a line parallel to $Re\,(ws)=0$, with indentations, if necessary, in such a manner that all the poles of $G\left(q^{b_j-\beta_j s}\right)$, $j=1,...,m$, are to its right, and those of $G\left(q^{1-a_j+\alpha_j s}\right)$, $j=1,...,n$, are to the left of $C$. The basic integral converges if $Re\left[s\log\left(x\right)-\log\sin\left(\pi s\right)\right]<0$ for large values of $|s|$ on the contour $C$, that is if $\left|\arg\left(x\right)-w_2w_1^{-1}\log|x|\right|<\pi$, where $0<|q|<1$, $\log q=-w=-\left(w_1+iw_2\right)$, $w$, $w_1$, $w_2$ are definite quantities, $w_1$ and $w_2$ being real and

$$G\left(q^a\right)=\frac{1}{\left\{\prod_{n=0}^{\infty}\left(1-q^{a+n}\right)\right\}}=\frac{1}{\left(q^a;q\right)_\infty}. \tag{2}$$

For $\alpha_i=\beta_j=1$ for all $i$ and $j$ in (1), it reduces to the basic analogue of

Meijer's $G$-function [9]:

$$H_{A,B}^{m,n}\left[x;q\left|\begin{array}{c}(a,1)\\(b,1)\end{array}\right.\right]=G_{A,B}^{m,n}\left[x;q\left|\begin{array}{c}a_1,...,a_A\\b_1,...,b_B\end{array}\right.\right]$$
$$=\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-s}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+s}\right)\pi x^s ds}{\prod_{j=m+1}^{B}G\left(q^{1-b_j+s}\right)\prod_{j=n+1}^{A}G\left(q^{a_j-s}\right)G\left(q^{1-s}\right)\sin\left(\pi s\right)},\qquad(3)$$

where $0\le m\le B,\ 0\le n\le A$ and $Re\left(s\log\left(x\right)-\log\sin\left(\pi s\right)\right)<0$, for large values of $|s|$ on the contour $C$.

Saxena et al. [11], expressed the following elementary $q$-functions in terms of the basic analogue of Meijer's $G$-function:

$$e_q\left(-x\right)=G\left(q\right)G_{0,2}^{1,0}\left[x\left(1-q\right);q\left|\begin{array}{c}-\\0,1\end{array}\right.\right];\qquad(4)$$

$$\sin_q\left(x\right)=\sqrt{\pi}\left(1-q\right)^{-1/2}\left[G\left(q\right)\right]^2G_{0,3}^{1,0}\left[\frac{x^2\left(1-q\right)^2}{4};q\left|\begin{array}{c}-\\\frac{1}{2},0,1\end{array}\right.\right];\qquad(5)$$

$$\cos_q\left(x\right)=\sqrt{\pi}\left(1-q\right)^{-1/2}\left[G\left(q\right)\right]^2G_{0,3}^{1,0}\left[\frac{x^2\left(1-q\right)^2}{4};q\left|\begin{array}{c}-\\0,\frac{1}{2},1\end{array}\right.\right];\qquad(6)$$

$$\sin h_q\left(x\right)=\frac{\sqrt{\pi}}{i}\left(1-q\right)^{-1/2}\left[G\left(q\right)\right]^2G_{0,3}^{1,0}\left[-\frac{x^2\left(1-q\right)^2}{4};q\left|\begin{array}{c}-\\\frac{1}{2},0,1\end{array}\right.\right];\qquad(7)$$

$$\cosh_q\left(x\right)=\sqrt{\pi}\left(1-q\right)^{-1/2}\left[G\left(q\right)\right]^2G_{0,3}^{1,0}\left[-\frac{x^2\left(1-q\right)^2}{4};q\left|\begin{array}{c}-\\0,\frac{1}{2},1\end{array}\right.\right].\qquad(8)$$

Saxena et al. [11] established the basic analog of the Mac-Robert's $E$-function due to Agarwal [1] in terms of the basic analogue of Meijer's $G$-function, by the relationship

$$E_q[B;b_j:A;a_j:x]=G_{A,B}^{B,0}\left[x;q\left|\begin{array}{c}a_1,...,a_A\\b_1,...,b_B\end{array}\right.\right],\qquad(9)$$

$Re\left(s\log\left(x\right)-\log\sin\left(\pi x\right)\right)<0$, for large values of $|s|$ on the contour $C$.

Saxena and Kumar [10] introduced the basic analogues of $J_\nu\left(x\right)$, $Y_\nu\left(x\right)$, $K_\nu\left(x\right)$, $H_\nu\left(x\right)$ in terms of the basic analogue of Meijer's $G$-function as follows:

$$J_\nu\left(x;q\right)=\left[G\left(q\right)\right]^2G_{0,3}^{1,0}\left[\frac{x^2\left(1-q\right)^2}{4};q\left|\begin{array}{c}-\\\frac{\nu}{2},\frac{-\nu}{2},1\end{array}\right.\right],\qquad(10)$$

where $J_\nu(x;q)$ denotes the $q$-analogue of Bessel function of first kind $J_\nu(x)$;

$$Y_\nu(x;q) = [G(q)]^2 G^{2,0}_{1,4}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \frac{-\nu-1}{2} \\ \frac{\nu}{2}, \frac{-\nu}{2}, \frac{-\nu-1}{2}, 1 \end{matrix}\right], \tag{11}$$

where $Y_\nu(x;q)$ denotes the $q$-analogue of the Bessel function $Y_\nu(x)$;

$$K_\nu(x;q) = (1-q)\, G^{2,0}_{0,3}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} - \\ \frac{\nu}{2}, \frac{-\nu}{2}, 1 \end{matrix}\right], \tag{12}$$

where $K_\nu(x;q)$ denotes the basic analogue of the Bessel function of the third kind $K_\nu(x)$;

$$H_\nu(x;q) = \left(\frac{1-q}{2}\right)^{1-\alpha} G^{3,1}_{1,4}\left[\frac{x^2(1-q)^2}{4};q \,\middle|\, \begin{matrix} \frac{1+\alpha}{2} \\ \frac{\nu}{2}, \frac{-\nu}{2}, \frac{1+\alpha}{2}, 1 \end{matrix}\right], \tag{13}$$

where $H_\nu(x;q)$ is the basic analogue of Struve's function $H_\nu(x)$.

On the other hand, the basic analog of the generalized hypergeometric function ${}_{r+1}\phi_r(\cdot)$, is expressed in terms of the basic analogue of $H$-function as follows ([6]):

$$\begin{aligned} {}_{r+1}\phi_r\left[\begin{matrix} q^{a_1}, \ldots, q^{a_{r+1}} \\ q^{b_1}, \ldots, q^{b_r} \end{matrix};q,x\right] &= \frac{\prod_{j=1}^{r} G\left(q^{b_j}\right) G(q)}{\prod_{j=1}^{r+1} G\left(q^{a_j}\right)} \\ &\times H^{1,r+1}_{r+1,r+2}\left[-\frac{x}{q};q \,\middle|\, \begin{matrix} (1-a_j,1) \\ (0,1),(1-b_j,1),(1,1) \end{matrix}\right], \end{aligned} \tag{14}$$

$$|x|<1, \ \left|\arg\left(-\frac{x}{q}\right)\right| < \pi,$$

by (3), (14) reduces to

$$\begin{aligned} {}_{r+1}\phi_r\left[\begin{matrix} q^{a_1}, \ldots, q^{a_{r+1}} \\ q^{b_1}, \ldots, q^{b_r} \end{matrix};q,x\right] &= \frac{\prod_{j=1}^{r} G\left(q^{b_j}\right) G(q)}{\prod_{j=1}^{r+1} G\left(q^{a_j}\right)} \\ &\times G^{1,r+1}_{r+1,r+2}\left[-\frac{x}{q};q \,\middle|\, \begin{matrix} 1-a_j \\ 0, 1-b_j, 1 \end{matrix}\right], \end{aligned} \tag{15}$$

$$|x|<1, \ \left|\arg\left(-\frac{x}{q}\right)\right| < \pi.$$

### 1.2. The operator $I_q^n(\cdot)$

In 2020 Castillo and Galué [2] presented the generalized fractional $q$-integral operator of the Kober type that contains the basic analogue of the Fox-Wright hypergeometric function in the following form:

$$I_q^n\left[\begin{array}{l}\rho,\ (M,1),\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]f(x)=\frac{x^{-\rho-1}}{\Gamma_q(M+1)}$$
$$\times\int_0^x t^{\rho}\ {}_r\psi_s^*\left[\begin{array}{l}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|\,q,q^n\frac{t}{x}\right]f(t)\,d_qt, \tag{16}$$

where

$$A_i,B_j\in\mathbb{R}^+,Re(\alpha_i)>0,\ Re(\beta_j)>0;i=1,...,r-1,j=1,...,s,$$

$$\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0,M\in\mathbb{N}_0,n\in\mathbb{N},\rho\in\mathbb{C},\ 0<q<1,\ \left|\frac{t}{x}\right|<1.$$

They also presented the series representation for (16) in the following form:

$$I_q^n\left[\begin{array}{l}\rho,\ (M,1),\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]f(x)=\frac{(1-q)}{\Gamma_q(M+1)}\sum_{k=0}^{\infty}q^{(\rho+1)k}$$
$$\times\ {}_r\psi_s^*\left[\begin{array}{l}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|\,q,q^{n+k}\right]f\left(xq^k\right), \tag{17}$$

$$A_i,B_j\in\mathbb{R}^+,Re(\alpha_i)>0,\ Re(\beta_j)>0;i=1,...,r-1,j=1,...,s,$$

$$M\in\mathbb{N}_0,n\in\mathbb{N},\rho\in\mathbb{C},\ 0<q<1,\ \sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0,$$

where the kernel ${}_r\psi_s^*\left[\begin{array}{l}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|\,q,z\right]$ was established in the following way

$${}_r\psi_s^*\left[\begin{array}{l}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\ (\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|\,q,z\right]$$
$$=\sum_{k=0}^{M}\frac{\left(q^{-M};q\right)_k}{(q;q)_k}\frac{\prod_{i=1}^{r-1}\left(q^{\alpha_i};q\right)_{A_ik}}{\prod_{j=1}^{s}\left(q^{\beta_j};q\right)_{B_jk}}\left[(1-q)^k\,q^{\frac{k(k-1)}{2}}\right]^{\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i}z^k, \tag{18}$$

$$A_i,B_j\in\mathbb{R}^+,M\in\mathbb{N}_0,Re(\alpha_i)>0,\ Re(\beta_j)>0;i=1,...,r-1,$$

$$j=1,...,s,\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0, 0<q<1,\ |z|<1.$$

## 2. Applications

This section envisages the applications of the generalized fractional $q$-integral operator of the Kober type that contains the basic analogue of the Fox-Wright hypergeometric function to the basic analogue of the $H$-function.

We operate the operator (17) to the function $x^{\mu}H_{p,v}^{m,n}(\cdot)$ using the definition (1) with argument as $x^{\lambda}$, $\lambda\in\mathbb{R}^{+}$. The result will be useful to evaluate the operator $I_q^n(\cdot)$ of the $q$-elementary and $q$-special functions of (4)$-$(13), (15) when $\mu=0$.

We consider

$$I=I_q^l\left[\begin{array}{l}\rho,(M,1),(\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]\left(x^{\mu}H_{p,v}^{m,n}\left[x^{\lambda};q\left|\begin{array}{c}(a,\delta)\\(b,\sigma)\end{array}\right.\right]\right),\ \mu\in\mathbb{R}.$$

According to (1) and (17) and changing $s$ for $S$, we have

$$\begin{aligned}I=&\frac{(1-q)}{\Gamma_q(M+1)}\frac{1}{2\pi i}\\&\times\sum_{k=0}^{\infty}q^{(\rho+1)k}\ {}_r\psi_s^{*}\left[\begin{array}{l}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\(\beta_1,B_1),...,(\beta_s,B_s)\end{array}\middle|q,q^{l+k}\right]\\&\times\left(xq^k\right)^{\mu}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\sigma_jS}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\delta_jS}\right)\pi\left(xq^k\right)^{\lambda S}dS}{\prod_{j=m+1}^{v}G\left(q^{1-b_j+\sigma_jS}\right)\prod_{j=n+1}^{p}G\left(q^{a_j-\delta_jS}\right)G\left(q^{1-S}\right)\sin(\pi S)},\end{aligned}$$

where $A_i,\ B_j\in\mathbb{R}^{+}$, $Re(\alpha_i)>0$, $Re(\beta_j)>0$; $i=1,...,r-1$, $j=1,...,s$, $M\in\mathbb{N}_0$, $l\in\mathbb{N}$, $\rho\in\mathbb{C}$, $\lambda\in\mathbb{R}^{+}$, $\mu\in\mathbb{R}$, $0<q<1$, $\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0$, $0\leq m\leq v$, $0\leq n\leq p$; $\delta_i$ $(i=1,...,p)$ and $\sigma_j$ $(j=1,...,v)$ are all positive integers, $Re\left[S\log(x)-\log\sin(\pi S)\right]<0$ for large values of $|S|$ on the contour $C$.

According to (18),

$$\begin{aligned}I=&\frac{(1-q)}{\Gamma_q(M+1)}\frac{1}{2\pi i}\sum_{k=0}^{\infty}q^{(\rho+1)k}\sum_{w=0}^{M}\frac{\left(q^{-M};q\right)_w\prod_{i=1}^{r-1}\left(q^{\alpha_i};q\right)_{A_iw}}{(q;q)_w\prod_{j=1}^{s}\left(q^{\beta_j};q\right)_{B_jw}}q^{(l+k)w}\\&\times\left[(1-q)^w q^{w(w-1)/2}\right]^{\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i}\left(xq^k\right)^{\mu}\end{aligned}$$

$$\times \int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-\sigma_j S}\right) \prod_{j=1}^{n} G\left(q^{1-a_j+\delta_j S}\right) \pi \left(xq^k\right)^{\lambda S} dS}{\prod_{j=m+1}^{v} G\left(q^{1-b_j+\sigma_j S}\right) \prod_{j=n+1}^{p} G\left(q^{a_j-\delta_j S}\right) G\left(q^{1-S}\right) \sin\left(\pi S\right)}.$$

Reversing the order of summation and integration is justified by the absolute convergence,

$$I = \frac{(1-q)\,x^{\mu}}{\Gamma_q(M+1)} \sum_{w=0}^{M} \frac{\left(q^{-M};q\right)_w \prod_{i=1}^{r-1} \left(q^{\alpha_i};q\right)_{A_i w}}{(q;q)_w \prod_{j=1}^{s} \left(q^{\beta_j};q\right)_{B_j w}}$$
$$\times q^{lw} \left[(1-q)^w q^{w(w-1)/2}\right]^{\sum_{j=1}^{s} B_j - \sum_{i=1}^{r-1} A_i}$$

$$\times \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-\sigma_j S}\right) \prod_{j=1}^{n} G\left(q^{1-a_j+\delta_j S}\right) \pi x^{\lambda S}}{\prod_{j=m+1}^{v} G\left(q^{1-b_j+\sigma_j S}\right) \prod_{j=n+1}^{p} G\left(q^{a_j-\delta_j S}\right) G\left(q^{1-S}\right) \sin\left(\pi S\right)}$$
$$\times \sum_{k=0}^{\infty} q^{(\rho+\mu+w+\lambda S+1)k} dS.$$

According to the $q$-binomial expansion, established in [7] in the following form:

$${}_1\phi_0 \left[ \begin{array}{c} \alpha \\ - \end{array} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(q;q)_n} x^n = \frac{(\alpha x;q)_\infty}{(x;q)_\infty}, \ |x| < 1, \ |q| < 1,$$

we obtain

$$I = \frac{(1-q)\,x^{\mu}}{\Gamma_q(M+1)} \sum_{w=0}^{M} \frac{\left(q^{-M};q\right)_w \prod_{i=1}^{r-1} \left(q^{\alpha_i};q\right)_{A_i w}}{(q;q)_w \prod_{j=1}^{s} \left(q^{\beta_j};q\right)_{B_j w}}$$
$$\times q^{lw} \left[(1-q)^w q^{w(w-1)/2}\right]^{\sum_{j=1}^{s} B_j - \sum_{i=1}^{r-1} A_i}$$
$$\times \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-\sigma_j S}\right) \prod_{j=1}^{n} G\left(q^{1-a_j+\delta_j S}\right) \pi x^{\lambda S}}{\prod_{j=m+1}^{v} G\left(q^{1-b_j+\sigma_j S}\right) \prod_{j=n+1}^{p} G\left(q^{a_j-\delta_j S}\right) G\left(q^{1-S}\right) \sin\left(\pi S\right)}$$
$$\times \frac{\left(q^{\rho+\mu+w+\lambda S+2};q\right)_\infty}{\left(q^{\rho+\mu+w+\lambda S+1};q\right)_\infty} dS.$$

Using (2), we have

$$I = \frac{(1-q)\,x^{\mu}}{\Gamma_q(M+1)} \sum_{w=0}^{M} \frac{\left(q^{-M};q\right)_w \prod_{i=1}^{r-1} \left(q^{\alpha_i};q\right)_{A_i w}}{(q;q)_w \prod_{j=1}^{s} \left(q^{\beta_j};q\right)_{B_j w}}$$

$$\times q^{lw}\left[(1-q)^{w}q^{w(w-1)/2}\right]^{\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i}$$
$$\times\frac{1}{2\pi i}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\sigma_jS}\right)\prod_{j=1}^{n}G\left(q^{1-a_j+\delta_jS}\right)}{\prod_{j=m+1}^{v}G\left(q^{1-b_j+\sigma_jS}\right)G\left(q^{\rho+\mu+w+\lambda S+2}\right)\prod_{j=n+1}^{p}G\left(q^{a_j-\delta_jS}\right)}$$
$$\times\frac{G\left(q^{\rho+\mu+w+\lambda S+1}\right)\pi x^{\lambda S}}{G\left(q^{1-S}\right)\sin(\pi S)}\,dS.$$

Considering

$$\begin{aligned}b_{w,q} &= \frac{(1-q)}{\Gamma_q(M+1)}\frac{\left(q^{-M};q\right)_w\prod_{i=1}^{r-1}\left(q^{\alpha_i};q\right)_{A_iw}}{(q;q)_w\prod_{j=1}^{s}\left(q^{\beta_j};q\right)_{B_jw}}\\ &\quad\times q^{lw}\left[(1-q)^{w}q^{w(w-1)/2}\right]^{\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i},\end{aligned}$$

and interpreting using (1), we get

$$\begin{aligned}&I_q^l\left[\begin{array}{l}\rho,(M,1),\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]\left(x^{\mu}H_{p,v}^{m,n}\left[x^{\lambda};q\left|\begin{array}{c}(a,\delta)\\(b,\sigma)\end{array}\right.\right]\right)\\ &=x^{\mu}\sum_{w=0}^{M}b_{w,q}\,H_{p+1,v+1}^{m,n+1}\left[x^{\lambda};q\left|\begin{array}{l}(-(\rho+\mu+w),\lambda),(a,\delta)\\(b,\sigma),(-(\rho+\mu+w+1),\lambda)\end{array}\right.\right],\end{aligned}\tag{19}$$

where $A_i,\ B_j\in\mathbb{R}^+,\ Re(\alpha_i)>0,\ Re(\beta_j)>0;\ i=1,...,r-1,\ j=1,...,s,$ $M\in\mathbb{N}_0,\ l\in\mathbb{N},\ \rho\in\mathbb{C},\ \lambda\in\mathbb{R}^+,\ \mu\in\mathbb{R},\ 0<q<1,\ \sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0,$ $0\leq m\leq v,\ 0\leq n\leq p;\ \delta_i\ (i=1,...,p)$ and $\sigma_j\ (j=1,...,v)$ are all positives integer, $Re[S\log(x)-\log\sin(\pi S)]<0$ for large values of $|S|$ on the contour $C$.

From (3) and (19) we obtain $I_q^n(\cdot)$ applied to the basic analog of the Meijer $G$-function, as follows:

$$\begin{aligned}&I_q^l\left[\begin{array}{l}\rho,(M,1),\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]\left(x^{\mu}G_{p,v}^{m,n}\left[x^{\lambda};q\left|\begin{array}{c}a_1,...,a_p\\b_1,...,b_v\end{array}\right.\right]\right)\\ &=x^{\mu}\sum_{w=0}^{M}b_{w,q}\,H_{p+1,v+1}^{m,n+1}\left[x^{\lambda};q\left|\begin{array}{l}(-(\rho+\mu+w),\lambda),(a,1)\\(b,1),(-(\rho+\mu+w+1),\lambda)\end{array}\right.\right],\end{aligned}\tag{20}$$

where $A_i,\ B_j\in\mathbb{R}^+,\ Re(\alpha_i)>0,\ \ Re(\beta_j)>0;\ i=1,...,r-1,\ j=1,...,s,$ $M\in\mathbb{N}_0,\ l\in\mathbb{N},\ \rho\in\mathbb{C},\ \lambda\in\mathbb{R}^+,\ \mu\in\mathbb{R},\ 0<q<1,\ \sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0,$ $0\leq m\leq v,\ 0\leq n\leq p,\ Re[S\log(x)-\log\sin(\pi S)]<0$ for large values of $|S|$

on the contour $C$ and

$$b_{w,q} = \frac{(1-q)}{\Gamma_q(M+1)} \frac{\left(q^{-M};q\right)_w \prod_{i=1}^{r-1}\left(q^{\alpha_i};q\right)_{A_i w}}{(q;q)_w \prod_{j=1}^{s}\left(q^{\beta_j};q\right)_{B_j w}} \times q^{lw}\left[(1-q)^w q^{w(w-1)/2}\right]^{\sum_{j=1}^{s} B_j - \sum_{i=1}^{r-1} A_i}.$$

### 2.1. Particular cases

Various elementary $q$-functions and the basic analogues of the special functions are expressed in terms of the basic analogue of $G$-function. The generalized fractional $q$-integral operator of Kober type, which contains the basic analogue of the Fox-Wright function applied to these functions, is obtained directly from (20), making appropriate changes in the values of the parameters.

In (20), making $m=1$, $n=0$, $p=0$, $v=2$, $b_1=0$, $b_2=1$, $\lambda=1$, $\mu=0$, we obtain the fractional q-integral operator $I_q^n(\cdot)$ of the $q$-exponential function $e_q(-x)$, given in (4) as follows:

$$\begin{aligned} &I_q^l\left[\begin{array}{l}\rho,(M,1),\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right] e_q(-x)\\ =&I_q^l\left[\begin{array}{l}\rho,(M,1),\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right] G(q)\, G_{0,2}^{1,0}\left[x(1-q);q\left|\begin{array}{c}-\\ 0,1\end{array}\right.\right]\\ =&G(q)\sum_{w=0}^{M} b_{w,q}\, G_{1,3}^{1,1}\left[x(1-q);q\left|\begin{array}{c}-\rho-w\\ 0,1,-\rho-w-1\end{array}\right.\right].\end{aligned}$$

The fractional $q$-integral operator $I_q^n(\cdot)$ of the elementary and especial $q$-functions given from (5)$-$(13), is shown in Table 1.

Considering the results (15) and (20) with $\mu=0$, we obtain

$$\begin{aligned} &I_q^l\left[\begin{array}{l}\rho,(M,1),\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]\left({}_{r+1}\phi_r\left[\begin{array}{l}q^{a_1},...,q^{a_{r+1}}\\ q^{b_1},...,q^{b_r}\end{array};q,x\right]\right)\\ =&\frac{\prod_{j=1}^{r} G\left(q^{b_j}\right)G(q)}{\prod_{j=1}^{r+1} G\left(q^{a_j}\right)}\\ &\times I_q^l\left[\begin{array}{l}\rho,(M,1),\ (\alpha_i,A_i)_{1,r-1}\\ (\beta_j,B_j)_{1,s}\end{array}\right]\left(G_{r+1,r+2}^{1,r+1}\left[-\frac{x}{q};q\left|\begin{array}{l}1-a_j\\ 0,1-b_j,1\end{array}\right.\right]\right)\\ =&\frac{\prod_{j=1}^{r} G\left(q^{b_j}\right)G(q)}{\prod_{j=1}^{r+1} G\left(q^{a_j}\right)}\sum_{w=0}^{M} b_{w,q}\end{aligned}$$

$$\times H^{1,r+2}_{r+2,r+3}\left[-\frac{x}{q};q\,\middle|\,\begin{array}{l}(-(\rho+w),1),(1-a_j,1)\\(0,1),(1,1),(-(\rho+w+1),1),(1-b_j,1)\end{array}\right],$$

$$|x|<1,\ \left|\arg\left(-\frac{x}{q}\right)\right|<\pi.$$

Using (3), we get

$$I^l_q\left[\begin{array}{l}\rho,(M,1),\ (\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{array}\right]\left({}_{r+1}\phi_r\left[\begin{array}{l}q^{a_1},...,q^{a_{r+1}}\\q^{b_1},...,q^{b_r}\end{array};q,x\right]\right)$$
$$=\frac{\prod_{j=1}^{r}G\left(q^{b_j}\right)G(q)}{\prod_{j=1}^{r+1}G\left(q^{a_j}\right)}\sum_{w=0}^{M}b_{w,q}$$
$$\times G^{1,r+2}_{r+2,r+3}\left[-\frac{x}{q};q\,\middle|\,\begin{array}{l}(-(\rho+w),1),(1-a_j,1)\\(0,1),(1,1),(-(\rho+w+1),1),(1-b_j,1)\end{array}\right],$$

where $A_i$, $B_j\in\mathbb{R}^+$, $Re(\alpha_i)>0$, $Re(\beta_j)>0$; $i=1,...,r-1$, $j=1,...,s$, $M\in\mathbb{N}_0$, $l\in\mathbb{N}$, $\rho\in\mathbb{C}$, $0<q<1$, $\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i\geq 0$, $|x|<1$, $\left|\arg\left(-\frac{x}{q}\right)\right|<\pi$; $b_{w,q}$ given above.

where

$$b_{w,q}=\frac{(1-q)}{\Gamma_q(M+1)}\frac{\left(q^{-M};q\right)_w\prod_{i=1}^{r-1}\left(q^{\alpha_i};q\right)_{A_iw}}{(q;q)_w\prod_{j=1}^{s}\left(q^{\beta_j};q\right)_{B_jw}}$$
$$\times\left[(1-q)^w q^{w(w-1)/2}\right]^{\sum_{j=1}^{s}B_j-\sum_{i=1}^{r-1}A_i}q^{lw}.$$

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Table 1: The fractional $q$-integral operator $I_q^n(\cdot)$ of some $q$-functions

| $f(x)$ | $I_q^n\left[\begin{matrix}\rho,(M,1),\ (\alpha_i,A_i)_{1,r-1}\\(\beta_j,B_j)_{1,s}\end{matrix}\right]f(x)=\frac{x^{-\rho-1}}{\Gamma_q(M+1)}\times$ $\int_0^x t^\rho\ {}_r\psi_s^*\left[\begin{matrix}(\alpha_1,A_1),...,(\alpha_{r-1},A_{r-1}),(-M,1)\\(\beta_1,B_1),...,(\beta_s,B_s)\end{matrix}\middle\vert q,q^n\frac{t}{x}\right]f(t)\,d_qt$ | No. |
|---|---|---|
| $e_q(-x)$ | $G(q)\sum_{w=0}^M b_{w,q}\ G_{1,3}^{1,1}\left[x(1-q);q\middle\vert\begin{matrix}-\rho-w\\0,1,-\rho-w-1\end{matrix}\right]$ | (21) |
| $\sin_q(x)$ | $\sqrt{\pi}(1-q)^{-1/2}[G(q)]^2\sum_{w=0}^M b_{w,q}\times$ $H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\rho-w,2)\\(\frac{1}{2},1),(0,1),(1,1),(-\rho-w-1,2)\end{matrix}\right]$ | (22) |
| $\cos_q(x)$ | $\sqrt{\pi}(1-q)^{-1/2}[G(q)]^2\sum_{w=0}^M b_{w,q}\times$ $H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\rho-w,2)\\(0,1),(\frac{1}{2},1),(1,1),(-\rho-w-1,2)\end{matrix}\right]$ | (23) |
| $\sin h_q(x)$ | $\frac{\sqrt{\pi}}{i}(1-q)^{-1/2}[G(q)]^2\sum_{w=0}^M b_{w,q}\times$ $H_{1,4}^{1,1}\left[-\frac{x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\rho-w,2)\\(\frac{1}{2},1),(0,1),(1,1),(-\rho-w-1,2)\end{matrix}\right]$ | (24) |
| $\cosh_q(x)$ | $\sqrt{\pi}(1-q)^{-1/2}[G(q)]^2\sum_{w=0}^M b_{w,q}\times$ $H_{1,4}^{1,1}\left[-\frac{x^2(1-q)^2}{4};q\middle\vert\begin{matrix}(-\rho-w,2)\\(0,1),(\frac{1}{2},1),(1,1),(-\rho-w-1,2)\end{matrix}\right]$ | (25) |

Table 2: Table 1, Continuation

| | | |
|---|---|---|
| $E_q\left[B;b_j:A;a_j:x\right]$ | $\sum_{w=0}^{M} b_{w,q}\, G_{A+1,B+1}^{B,1}\left[x;q \,\middle\vert\, \begin{matrix} -\rho-w, a_1,...,a_A \\ b_1,...,b_B,-\rho-w-1 \end{matrix}\right]$ | (26) |
| $J_\nu\left(x;q\right)$ | $\left[G\left(q\right)\right]^2 \sum_{w=0}^{M} b_{w,q}\; H_{1,4}^{1,1}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix} \left(-\rho-w,2\right) \\ \left(\frac{\nu}{2},1\right),\left(-\frac{\nu}{2},1\right),\left(1,1\right),\left(-\rho-w-1,2\right) \end{matrix}\right]$ | (27) |
| $Y_\nu\left(x;q\right)$ | $\left[G\left(q\right)\right]^2 \sum_{w=0}^{M} b_{w,q}\; H_{2,5}^{2,1}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix} \left(-\rho-w,2\right),\left(-\frac{\nu+1}{2},1\right) \\ \left(\frac{\nu}{2},1\right),\left(-\frac{\nu}{2},1\right),\left(-\frac{\nu+1}{2},1\right),\left(1,1\right),\left(-\rho-w-1,2\right) \end{matrix}\right]$ | (28) |
| $K_\nu\left(x;q\right)$ | $\left(1-q\right) \sum_{w=0}^{M} b_{w,q}\; H_{1,4}^{2,1}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix} \left(-\rho-w,2\right) \\ \left(\frac{\nu}{2},1\right),\left(-\frac{\nu}{2},1\right),\left(1,1\right),\left(-\rho-w-1,2\right) \end{matrix}\right]$ | (29) |
| $H_\nu\left(x;q\right)$ | $\left(\frac{1-q}{2}\right)^{1-\alpha} \sum_{w=0}^{M} b_{w,q}\; H_{2,5}^{3,2}\left[\frac{x^2(1-q)^2}{4};q \,\middle\vert\, \begin{matrix} \left(-\rho-w,2\right),\left(\frac{1+\alpha}{2},1\right) \\ \left(\frac{\nu}{2},1\right),\left(-\frac{\nu}{2},1\right),\left(\frac{1+\alpha}{2},1\right),\left(1,1\right),\left(-\rho-w-1,2\right) \end{matrix}\right]$ | (30) |
| ${}_{r+1}\phi_r\left[\begin{matrix} q^{a_1},...,q^{a_{r+1}} \\ q^{b_1},...,q^{b_r} \end{matrix};q,x\right]$ | $\frac{\prod_{j=1}^{r} G\left(q^{b_j}\right)G(q)}{\prod_{j=1}^{r+1} G\left(q^{a_j}\right)} \sum_{w=0}^{M} b_{w,q}\; \times$ $G_{r+2,r+3}^{1,r+2}\left[-\frac{x}{q};q \,\middle\vert\, \begin{matrix} -\left(\rho+w\right),1-a_j \\ 0,1-b_j,1,-\left(\rho+w+1\right) \end{matrix}\right]$ | (31) |

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