

## ONE CLASS OF GENERALIZED CONVEX FUNCTIONS IN THE SENSE OF BECKENBACH

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**Abstract:** The present study is mainly concerned with one class of generalized convex functions in the sense of Beckenbach. The existence of the support curves is presented for this class, which leads to its generalized convexity. In addition, an extremum property of these functions is given. Furthermore, Hadamard's inequality for this class is obtained.

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### 1. Introduction

Convex functions have played an essential part in the development of several fields of pure and applied sciences. Convexity theory shows a wide spectrum of very interesting developments involving a link among different fields of mathematics, physics, engineering sciences and economics. For example, in 2019, Malyuta and Açikmeşe [14] presented a novel convex optimization-based system

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for finding the globally optimal solutions of a class of mixed-integer non-convex optimal control problems which was used in rockets and other applications. Additionally, they considered problems with non-convex constraints that restrict the input norms to be either zero or lower- and upper-bounded. The non-convex problem was relaxed to a convex one whose optimal solution was proved to be optimal almost anywhere for the main problem, a procedure known as lossless convexification.

In this paper, we suppose that  $I$  is a nonempty, connected, and bounded subset of  $\mathbf{R}$ . A real valued function  $f(x)$  of a single real variable  $x$  defined on  $I$  is called convex if, for every  $x_1, x_2 \in I$  and  $\gamma \in [0, 1]$ , we have the next inequality:

$$f(\gamma x_1 + (1 - \gamma)x_2) \leq \gamma f(x_1) + (1 - \gamma)f(x_2). \quad (1)$$

Since the beginning of the 20th century, many generalizations of convexity have been extensively presented and discussed in numerous ways by many authors in the past and present. One way to generalize the notion of the convex function is to relax the convexity condition (1) (see [13]).

As known, the definition of the classical convexity can be explained in terms of linear functions. A significant direction for generalization of the ordinary convexity replaced linear functions by another family of functions. For instance, Beckenbach and Bing [6] generalized this status by substituting the linear functions by a family of continuous functions in such a way that for each pair of points  $p_1(x_1, f(x_1))$  and  $p_2(x_2, f(x_2))$  of the plane, there exists exactly one member of the family with a graph joining these points (see also [5]).

To be more exact, let  $\{F(x)\}$  be a family of continuous functions and  $F(x)$  be defined on a real interval  $I$ . A function  $f : I \rightarrow \mathbf{R}$  is called a sub  $F$ -function if, for all  $x_1, x_2 \in I$  with  $x_1 < x_2$ , there is a unique member of  $\{F(x)\}$  satisfying the following conditions:

1.  $F(x_1) = f(x_1)$  and  $F(x_2) = f(x_2)$ ,
2.  $f(x) \leq F(x)$  for all  $x \in [x_1, x_2]$ .

The sub  $F$ -functions possess numerous characteristics analogous to those of ordinary convex functions (see [1], [5], [6], [8], [10], [11], [21]). For example, let  $f : I \rightarrow \mathbf{R}$  be a sub  $F$ -function, then for every  $x_1, x_2 \in I$ , the inequality

$$f(x) \geq F(x) \quad (2)$$

holds outside the interval  $(x_1, x_2)$ .

**Theorem 1.** [21] *A sub  $F$ -function  $f : I \rightarrow \mathbf{R}$  has finite left and right derivatives  $f'_-(x), f'_+(x)$  at any point  $x \in I$ .*

**Property 2.** [21] *Under the assumptions of Theorem 1, the function  $f$  is continuously differentiable on  $I$  with the exception of an at-most countable set.*

The current work is only concerned with generalized convexity in the sense of Beckenbach. For particular choices of the two-parameters family  $\{F(x)\}$ , we consider one class of generalized convex functions as follows

$$F(x) = L(x) := \ln(A + Bx).$$

The following double inequality

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(x)dx \leq \frac{f(u) + f(v)}{2} \quad (3)$$

is well-known in research references as Hadamard's inequality or, as it is quoted for historical reasons [16], the Hermite-Hadamard inequality, where  $f : I \rightarrow \mathbf{R}$  is a convex function and  $u, v \in I$  with  $u < v$ . This inequality attracted the interest of numerous mathematicians; for new generalization, extensions, and many applications, see [9], [12].

## 2. Definitions and preliminary results

From these investigations, let us now show the main definitions and results for the provided class of generalized convex functions in the sense of Beckenbach as they are used later in this article.

Now, we introduce a definition of the sub  $L$ -convex function and some of its properties.

**Definition 3.** The function  $f : I \rightarrow \mathbf{R}$  is said to be sub  $L$ -convex function on  $I$  if, for every  $x_1, x_2 \in I$  with  $x_1 < x_2$ , the graph of  $f(x)$  for  $x_1 \leq x \leq x_2$  lies on or under the function

$$L(x) = \ln(A + Bx), \quad (4)$$

where  $A$  and  $B$  are taken in such a way that  $L(x_1) = f(x_1)$ , and  $L(x_2) = f(x_2)$ .

Equivalently, for all  $x \in [x_1, x_2]$

$$\begin{aligned} f(x) &\leq L(x) \\ &= \ln \left[ \frac{(x_2 - x)e^{f(x_1)} + (x - x_1)e^{f(x_2)}}{x_2 - x_1} \right]. \end{aligned} \quad (5)$$

It is should be noted that:

(1) There is another formula for the function  $L(x)$  rather than that stated in (5), for example,

$$L(x) = \ln(e^{f(x_1)} + (x - x_1)B); \quad B = \frac{e^{f(x_2)} - e^{f(x_1)}}{x_2 - x_1}. \quad (6)$$

(2) If  $f : I \rightarrow \mathbf{R}$  is a two-time continuously differentiable function. Then,  $f$  is a sub  $L$ -convex function on  $I$  if and only if

$$f''(x) + f'^2(x) \geq 0 \quad \forall x \in I. \quad (7)$$

**Definition 4.** A function  $f : I \rightarrow \mathbf{R}$  is said to be sub  $L$ -concave function on  $I$  if, for every  $x_1, x_2 \in I$  with  $x_1 < x_2$

$$f(x) \geq \ln \left[ \frac{(x_2 - x)e^{f(x_1)} + (x - x_1)e^{f(x_2)}}{x_2 - x_1} \right], \quad x \in [x_1, x_2]. \quad (8)$$

**Remark 5.** A function  $f : I \rightarrow \mathbf{R}$  is said to be exponentially convex function (see [17, 18, 19]), if

$$e^{f(\gamma x_1 + (1-\gamma)x_2)} \leq \gamma e^{f(x_1)} + (1 - \gamma)e^{f(x_2)}, \quad (9)$$

for all  $x_1, x_2 \in I$  and  $\gamma \in [0, 1]$ .

We use the substitution:

$$\gamma = \frac{x_2 - x}{x_2 - x_1}. \quad (10)$$

Hence, from (5), the function  $L(x)$  has the following form

$$L(x) = L(\gamma x_1 + (1 - \gamma)x_2) = \ln[\gamma e^{f(x_1)} + (1 - \gamma)e^{f(x_2)}]. \quad (11)$$

Thus, from Definition 3, it is revealed that a function  $f : I \rightarrow \mathbf{R}$  is a sub  $L$ -convex function on  $I$  if, for all  $x_1, x_2 \in I$  and  $\gamma \in [0, 1]$ , we have the next inequality:

$$f(\gamma x_1 + (1 - \gamma)x_2) \leq L(x)$$

$$= \ln[\gamma e^{f(x_1)} + (1 - \gamma)e^{f(x_2)}].$$

Then,

$$e^{f(\gamma x_1 + (1-\gamma)x_2)} \leq \gamma e^{f(x_1)} + (1 - \gamma)e^{f(x_2)}. \quad (12)$$

Therefore, from (9) and (12), we deduce that the sub  $L$ -convex functions are precisely the standard exponentially convex functions.

In fact, the topic of exponentially convex functions really originated in the paper of Bernstein [7]. In 1929, Bernstein introduced a function  $f(x)(u < x < v)$  belonging to the class  $\varphi_{u,v}$  if it is continuous and

$$\sum_{i,j=1}^n f\left(\frac{x_i + x_j}{2}\right) \alpha_i \alpha_j \geq 0 \quad (13)$$

for all  $n \in \mathbf{N}$  and all choices  $\alpha_i \in \mathbf{R}$ ,  $x_i$  in  $(u, v)$ ,  $i = 1, \dots, n$ . Additionally, he called these functions  $f(x) \in \varphi_{u,v}$  exponentially convex functions. This concept was studied by some researchers (see [20] and the references therein) after Benstein [7]. It should be noted that this concept is different from Definition 3. In 1972, Avriel [4] introduced and studied the definition of  $r$ -convex functions. Let  $r$  be a real number. A real function  $f$  defined on a convex set  $C \subset \mathbf{R}^n$  is said to be  $r$ -convex function if for any  $x_1, x_2 \in C$ ,  $\gamma \geq 0$ ,  $\delta \geq 0$ , such that  $\gamma + \delta = 1$  we have

$$f(\gamma x_1 + \delta x_2) \leq \begin{cases} \log\{\gamma e^{rf(x_1)} + \delta e^{rf(x_2)}\}^{1/r}, & \text{if } r \neq 0 \\ \gamma f(x_1) + \delta f(x_2), & \text{if } r = 0. \end{cases} \quad (14)$$

A function  $f$  is  $r$ -convex ( $r$ -concave) if  $r > 0$  ( $r < 0$ ). The subject of  $r$ -convex attracted the interest of some researchers such as Antczak [3] and Zhao et al. [23]. They studied some properties of  $r$ -convex that have important applications in mathematical programming and optimization. This definition is the same as Definition 3. In 2018, Alirezai and Mathar [2] showed the concept of exponentially concave functions in  $\mathbf{R}^n$  and presented some of their properties and their impact on information theory. This notion agrees with Definition 4 in  $\mathbf{R}$ . In 2019, Noor and Noor [17], [18], [19] proposed the concept of exponentially convex functions as there is in the Remark 5. They showed many of the properties of exponentially convex functions and discussed their relations with convex functions.

Based on continued research, this worksheet just deals with generalized convexity in the sense of Beckenbach. By choosing the family  $\{F(x)\}$  equal

$\{L(x) = \ln[A + Bx]\}$  as shown in Definition 3. So again, the concept of sub  $L$ -convex functions is considered. Some properties of sub  $L$ -convex functions are proposed. These properties are different from those on the class of exponentially convex functions. The existence of support curves is presented, which implies their generalized convexity. In addition, an extremum property of these functions is obtained, and the relation between these functions and increasing functions is yielded. Furthermore, Hadamard's inequality is established for sub  $L$ -convex functions.

Now, we show the form of supporting functions in the class of sub  $L$ -convex functions.

**Definition 6.** Let  $f : I \rightarrow \mathbf{R}$  be a sub  $L$ -convex function. Then, a function

$$\psi_{x_1}(x) = \ln(A + Bx) \quad (15)$$

is called a supporting function for  $f(x)$  at the point  $x_1 \in I$ , if

$$\psi_{x_1}(x_1) = f(x_1), \quad (16)$$

and

$$\psi_{x_1}(x) \leq f(x), \quad \forall x \in I. \quad (17)$$

This is to say that if  $f(x)$  and  $\psi_{x_1}(x)$  agree at  $x = x_1$ , then the graph of  $f(x)$  lies on or above the support curve.

**Proposition 7.** Let  $f : I \rightarrow \mathbf{R}$  be a differentiable sub  $L$ -convex function, then the supporting function for  $f(x)$  at the point  $x_1 \in I$  has the form

$$\psi_{x_1}(x) = f(x_1) + \ln[1 + (x - x_1)f'(x_1)]. \quad (18)$$

*Proof.* The supporting function  $\psi_{x_1}(x)$  for  $f(x)$  at the point  $x_1 \in I$  can be described as follows:

$$\psi_{x_1}(x) = \lim_{x_2 \rightarrow x_1} L(x), \quad (19)$$

where  $x_2 \in I$  and

$$f(x) \geq L(x), \quad \forall x \in I \setminus (x_1, x_2). \quad (20)$$

Then, taking the limit of both sides as  $x_2 \rightarrow x_1$  and from (6), one obtains

$$\begin{aligned} f(x) &\geq \psi_{x_1}(x) \\ &= \lim_{x_2 \rightarrow x_1} L(x) \end{aligned}$$

$$\begin{aligned}
&= \lim_{x_2 \rightarrow x_1} [\ln(e^{f(x_1)} + (x - x_1)B)] \\
&= f(x_1) + \ln[1 + (x - x_1)f'(x_1)].
\end{aligned} \tag{21}$$

Thus, the claim follows.  $\square$

### 3. Results

In [22], a basic theorem in the theory of convex functions refers to a necessary and sufficient condition in order that the function  $f : I \rightarrow \mathbf{R}$  be convex, that is there must be at least one line of support for  $f$  at each point  $x$  in  $I$ . In the following theorem, we prove analogs of this result for sub  $L$ -functions.

**Theorem 8.** *A function  $f : I \rightarrow \mathbf{R}$  is a sub  $L$ -convex function on  $I$  if and only if there exist a supporting functions for  $f(x)$  at each point  $x$  in  $I$ .*

*Proof.* The necessity is an instant result of Bonsall [8].

To show the sufficiency, assume that  $x$  is an arbitrary point in  $I$  and  $f$  has a supporting function at this point. For suitability, we write the supporting function in the following form:

$$\psi_x(y) = f(x) + \ln \left[ 1 + (y - x)S_{x,f} \right], \tag{22}$$

where  $S_{x,f}$  is a fixed real number depending on  $x$  and  $f$ .

From Definition 6, we have

$$\psi_x(x) = f(x),$$

and

$$\psi_x(y) \leq f(y) \quad \forall y \in I. \tag{23}$$

It follows that

$$f(x) + \ln \left[ 1 + (y - x)S_{x,f} \right] \leq f(y) \quad \forall y \in I. \tag{24}$$

Consequently,

$$1 + (y - x)S_{x,f} \leq e^{f(y) - f(x)} \quad \forall y \in I. \tag{25}$$

For every  $x_1, x_2 \in I$  with  $x_1 < x_2$  and  $\gamma, \delta \geq 0$  with  $\gamma + \delta = 1$ , let

$$x = \gamma x_1 + \delta x_2. \quad (26)$$

we apply (25) twice at  $y = x_1$  and at  $y = x_2$  and this yields

$$1 + (x_1 - x)S_{x,f} \leq e^{f(x_1) - f(x)}, \quad (27)$$

and

$$1 + (x_2 - x)S_{x,f} \leq e^{f(x_2) - f(x)}. \quad (28)$$

Multiplying the first inequality by  $\gamma(x_2 - x_1)$  and the second inequality by  $\delta(x_2 - x_1)$  and then adding them, we get

$$\begin{aligned} & (\gamma + \delta)(x_2 - x_1) + [\gamma(x_2 - x_1)(x_1 - x) + \delta(x_2 - x_1)(x_2 - x)]S_{x,f} \\ & \leq \frac{\gamma(x_2 - x_1)e^{f(x_1)} + \delta(x_2 - x_1)e^{f(x_2)}}{e^{f(x)}}. \end{aligned}$$

Consequently,

$$f(x) \leq \ln \left[ \frac{(x_2 - x)e^{f(x_1)} + (x - x_1)e^{f(x_2)}}{x_2 - x_1} \right] \quad \forall x \in [x_1, x_2],$$

which is to prove that the function  $f(x)$  is a sub  $L$ -convex function on  $I$ . Then, the theorem is true.  $\square$

**Remark 9.** For a sub  $L$ -convex function  $f : I \rightarrow \mathbf{R}$ , the constant  $S_{x,f}$  in the foregoing theorem is equal to  $f'(x)$  if  $f$  is differentiable at the point  $x \in I$ . Otherwise,  $f'_-(x) \leq S_{x,f} \leq f'_+(x)$ .

Mils [15] presented an extremum property of convex functions which gave the minimum integral to subtract the convex function from its support. In this paper, we get an extremum property to the sub  $L$ -convex functions which are introduced in the following theorem.

**Theorem 10.** If  $f : I \rightarrow \mathbf{R}$  is a sub  $L$ -convex function, with  $u, v \in I$  and if  $\psi_{x_1}(x)$  is a supporting function for  $f(x)$  at the point  $x_1 \in [u, v]$ . Then, the function

$$K(x_1) = \int_u^v [f(x) - \psi_{x_1}(x)]dx \quad (29)$$

has a minimum value at  $x_1 = (u + v)/2$ .



*Proof.* Using Definition 6, we get

$$\psi_{x_1}(x_1) = f(x_1), \quad (30)$$

$$\psi_{x_1}(x) \leq f(x) \quad \forall x \in [u, v], \quad (31)$$

and  $\psi_{x_1}(x)$  can be written in the following formula:

$$\psi_{x_1}(x) = f(x_1) + \ln[1 + (x - x_1)S_{x_1,f}]; \quad \forall x \in [u, v]. \quad (32)$$

From (32), we get

$$\begin{aligned} \int_u^v \psi_{x_1}(x) dx &= f(x_1)(v - u) + v \ln[1 + (v - x_1)S_{x_1,f}] \\ &\quad - u \ln[1 + (u - x_1)S_{x_1,f}] \\ &\quad - \int_u^v \frac{xS_{x_1,f}}{1 + (x - x_1)S_{x_1,f}} dx, \\ &= f(x_1)(v - u) + v \ln[1 + (\frac{u+v}{2} - x_1)S_{x_1,f} + (\frac{v-u}{2})S_{x_1,f}] \\ &\quad - u \ln[1 + (\frac{u+v}{2} - x_1)S_{x_1,f} - (\frac{v-u}{2})S_{x_1,f}] - (v - u) \\ &\quad + \frac{1 - x_1S_{x_1,f}}{S_{x_1,f}} \ln \left[ \frac{1 + (v - x_1)S_{x_1,f}}{1 + (u - x_1)S_{x_1,f}} \right], \end{aligned}$$

consequently,

$$\begin{aligned} \int_u^v \psi_{x_1}(x) dx &= \\ &= f(x_1)(v - u) + (v - u) \ln[1 + (\frac{u+v}{2} - x_1)S_{x_1,f}] \\ &\quad + v \ln \left[ 1 + \frac{(\frac{v-u}{2})S_{x_1,f}}{1 + (\frac{u+v}{2} - x_1)S_{x_1,f}} \right] \\ &\quad - u \ln \left[ 1 - \frac{(\frac{v-u}{2})S_{x_1,f}}{1 + (\frac{u+v}{2} - x_1)S_{x_1,f}} \right] - (v - u) \\ &\quad + \frac{1 - x_1S_{x_1,f}}{S_{x_1,f}} \ln \left[ \frac{1 + (\frac{u+v}{2} - x_1)S_{x_1,f} + (\frac{v-u}{2})S_{x_1,f}}{1 + (\frac{u+v}{2} - x_1)S_{x_1,f} - (\frac{v-u}{2})S_{x_1,f}} \right], \\ &= f(x_1)(v - u) + (v - u) \ln[1 + (\frac{u+v}{2} - x_1)S_{x_1,f}] \\ &\quad + (\frac{u+v}{2} + \frac{v-u}{2}) \ln \left[ 1 + \frac{(\frac{v-u}{2})S_{x_1,f}}{1 + (\frac{u+v}{2} - x_1)S_{x_1,f}} \right] \end{aligned}$$

$$\begin{aligned}
& -\left(\frac{u+v}{2} - \frac{v-u}{2}\right) \ln \left[ 1 - \frac{\left(\frac{v-u}{2}\right) S_{x_1, f}}{1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}} \right] \\
& -(v-u) + 2 \frac{1 - x_1 S_{x_1, f}}{S_{x_1, f}} \tanh^{-1} \left[ \frac{\left(\frac{v-u}{2}\right) S_{x_1, f}}{1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}} \right], \\
= & f(x_1)(v-u) + (v-u) \ln \left[ 1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f} \right] \\
& + 2 \frac{u+v}{2} \tanh^{-1} \left[ \frac{\left(\frac{v-u}{2}\right) S_{x_1, f}}{1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}} \right] \\
& + \frac{v-u}{2} \ln \left[ 1 - \frac{\left[\left(\frac{v-u}{2}\right) S_{x_1, f}\right]^2}{\left[1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}\right]^2} \right] \\
& -(v-u) + 2 \frac{1 - x_1 S_{x_1, f}}{S_{x_1, f}} \tanh^{-1} \left[ \frac{\left(\frac{v-u}{2}\right) S_{x_1, f}}{1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}} \right].
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& \int_u^v \psi_{x_1}(x) dx = \\
= & f(x_1)(v-u) + (v-u) \ln \left[ 1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f} \right] - (v-u) \\
& + \frac{v-u}{2} \ln \left[ 1 - \frac{\left[\left(\frac{v-u}{2}\right) S_{x_1, f}\right]^2}{\left[1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}\right]^2} \right] \\
& + 2 \frac{1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}}{S_{x_1, f}} \tanh^{-1} \left[ \frac{\left(\frac{v-u}{2}\right) S_{x_1, f}}{1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}} \right]. \quad (33)
\end{aligned}$$

We apply (32) at  $x = (u+v)/2$ , to obtain

$$\psi_{x_1}\left(\frac{u+v}{2}\right) - f(x_1) = \ln\left(1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}\right) \quad \forall x_1 \in [u, v]. \quad (34)$$

By substituting (34) in (33), we observe that

$$\begin{aligned}
& \int_u^v \psi_{x_1}(x) dx = (v-u) \psi_{x_1}\left(\frac{u+v}{2}\right) - (v-u) \\
& + \frac{v-u}{2} \ln \left[ 1 - \frac{\left[\left(\frac{v-u}{2}\right) S_{x_1, f}\right]^2}{\left[1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}\right]^2} \right] \\
& + 2 \frac{1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}}{S_{x_1, f}} \tanh^{-1} \left[ \frac{\left(\frac{v-u}{2}\right) S_{x_1, f}}{1 + \left(\frac{u+v}{2} - x_1\right) S_{x_1, f}} \right] \\
= & (v-u) \left[ \psi_{x_1}\left(\frac{u+v}{2}\right) + H \right], \quad (35)
\end{aligned}$$

where

$$H = \frac{1}{2} \ln \left[ 1 - \frac{[(\frac{v-u}{2})S_{x_1,f}]^2}{[1 + (\frac{u+v}{2} - x_1)S_{x_1,f}]^2} \right] + 2 \frac{1 + (\frac{u+v}{2} - x_1)S_{x_1,f}}{(v-u)S_{x_1,f}} \tanh^{-1} \left[ \frac{(\frac{v-u}{2})S_{x_1,f}}{1 + (\frac{u+v}{2} - x_1)S_{x_1,f}} \right] - 1. \quad (36)$$

Consequently,

$$K(x_1) = \int_u^v f(x)dx - (v-u) \left[ \psi_{x_1}(\frac{u+v}{2}) + H \right]. \quad (37)$$

One might directly notice that  $H$  does not depend on  $x_1$ , that is  $H$  depends only on  $u$  and  $v$ . Using (30) at  $x_1 = \frac{u+v}{2}$ , the function  $K(x_1)$  becomes

$$K(\frac{u+v}{2}) = \int_u^v f(x)dx - (v-u) \left[ f(\frac{u+v}{2}) + H \right]. \quad (38)$$

But from (31), we note that

$$\psi_{x_1}(\frac{u+v}{2}) \leq f(\frac{u+v}{2}) \quad \forall x_1 \in (u, v). \quad (39)$$

Then, from (39), (37) and (38), we obtain

$$K(x_1) \geq K(\frac{u+v}{2}) \quad \forall x_1 \in (u, v). \quad (40)$$

Thus, the minimum value at function  $K(x_1)$  occurs at  $x_1 = (u+v)/2$ .  $\square$

In the next theorem, we establish Hadamard's inequality for sub  $L$ -convex functions.

**Theorem 11.** Let  $f : I \rightarrow \mathbf{R}$  be a sub  $L$ -convex function, with  $u, v \in I$  with  $u < v$ .

Then, one obtains the following inequality

$$f(\frac{u+v}{2}) \leq \frac{1}{v-u} \int_u^v f(x)dx \leq \frac{[f(v) - 1]e^{f(v)} - [f(u) - 1]e^{f(u)}}{e^{f(v)} - e^{f(u)}}. \quad (41)$$

*Proof.* Let  $x_1$  be an arbitrary point in  $(u, v)$ . As  $f(x)$  is a sub  $L$ -convex function, then from Definitions 3 and 6 we note that the graph of  $f(x)$  lies nowhere above the function

$$L(x) = \ln[e^{f(u)} + (x - u)B], \quad \text{where } B = \frac{e^{f(v)} - e^{f(u)}}{v - u}, \quad (42)$$

and nowhere below any supporting function

$$\psi_{x_1}(x) = f(x_1) + \ln[1 + (x - x_1)S_{x_1, f}], \quad (43)$$

at the point  $x_1 \in [u, v]$ .

Thus,

$$\psi_{x_1}(x) \leq f(x) \leq L(x) \quad x \in [u, v],$$

and

$$\frac{1}{v - u} \int_u^v \psi_{x_1}(x) dx \leq \frac{1}{v - u} \int_u^v f(x) dx \leq \frac{1}{v - u} \int_u^v L(x) dx. \quad (44)$$

From (42), we get

$$\begin{aligned} \frac{1}{v - u} \int_u^v L(x) dx &= \frac{1}{v - u} \int_u^v \ln[e^{f(u)} + (x - u)B] dx, \\ &= \frac{1}{v - u} \left[ vf(v) - uf(u) \right. \\ &\quad \left. - \int_u^v \frac{x B}{1 + (x - u)B} dx \right], \\ &= \frac{1}{v - u} \left[ vf(v) - uf(u) - (v - u) \right. \\ &\quad \left. + \frac{e^{f(u)} - uB}{B} [f(v) - f(u)] \right] \\ &= \frac{f(v)e^{f(v)} - f(u)e^{f(u)}}{e^{f(v)} - e^{f(u)}} - 1, \\ &= \frac{[f(v) - 1]e^{f(v)} - [f(u) - 1]e^{f(u)}}{e^{f(v)} - e^{f(u)}}. \end{aligned} \quad (45)$$

By using (43), (35) and (36), it follows that

$$\begin{aligned} \frac{1}{v - u} \int_u^v \psi_{x_1}(x) dx &= \psi_{x_1}\left(\frac{u + v}{2}\right) \\ &+ 2 \frac{1 + \left(\frac{u+v}{2} - x_1\right)S_{x_1, f}}{(v - u)S_{x_1, f}} \tanh^{-1} \left[ \frac{\left(\frac{v-u}{2} S_{x_1, f}\right)}{1 + \left(\frac{u+v}{2} - x_1\right)S_{x_1, f}} \right] - 1 \end{aligned} \quad (46)$$

$$+\frac{1}{2}\ln\left[1-\frac{((\frac{v-u}{2})S_{x_1,f})^2}{(1+(\frac{u+v}{2}-x_1)S_{x_1,f})^2}\right]. \quad (47)$$

Since,  $\tanh^{-1}y > y$  for  $0 < y < 1$ , we have

$$\begin{aligned} & \frac{1}{v-u} \int_u^v \psi_{x_1}(x) dx > \psi_{x_1}\left(\frac{u+v}{2}\right) \\ & + \frac{1+(\frac{u+v}{2}-x_1)S_{x_1,f}}{(\frac{v-u}{2})S_{x_1,f}} \times \left[ \frac{(\frac{v-u}{2})S_{x_1,f}}{1+(\frac{u+v}{2}-x_1)S_{x_1,f}} \right] - 1 \\ & + \frac{1}{2} \ln \left[ 1 - \frac{((\frac{v-u}{2})S_{x_1,f})^2}{(1+(\frac{u+v}{2}-x_1)S_{x_1,f})^2} \right] \\ & > \psi_{x_1}\left(\frac{u+v}{2}\right) + \frac{1}{2} \ln \left[ 1 - \frac{((\frac{v-u}{2})S_{x_1,f})^2}{(1+(\frac{u+v}{2}-x_1)S_{x_1,f})^2} \right]. \end{aligned}$$

We take the maximum of the term

$$(1/(v-u)) \int_u^v \psi_{x_1}(x) dx \leq (1/(v-u)) \int_u^v f(x) dx$$

for  $x_1 \in (u, v)$  and from (48), it follows that

$$\begin{aligned} & \frac{1}{v-u} \int_u^v f(x) dx \geq \max_{u < x_1 < v} \left\{ \frac{1}{v-u} \int_u^v \psi_{x_1}(x) dx \right\}, \\ & > \max_{u < x_1 < v} \left\{ \psi_{x_1}\left(\frac{u+v}{2}\right) + \frac{1}{2} \ln \left[ 1 - \frac{((\frac{v-u}{2})S_{x_1,f})^2}{(1+(\frac{u+v}{2}-x_1)S_{x_1,f})^2} \right] \right\}, \\ & = \max_{u < x_1 < v} \left\{ \psi_{x_1}\left(\frac{u+v}{2}\right) \right\} \\ & + \max_{u < x_1 < v} \left\{ \frac{1}{2} \ln \left[ 1 - \frac{((\frac{v-u}{2})S_{x_1,f})^2}{(1+(\frac{u+v}{2}-x_1)S_{x_1,f})^2} \right] \right\}. \end{aligned}$$

Actually, we observe

$$\frac{1}{2} \ln \left[ 1 - \frac{((\frac{v-u}{2})S_{x_1,f})^2}{(1+(\frac{u+v}{2}-x_1)S_{x_1,f})^2} \right] \leq 0,$$

then

$$\max_{u < x_1 < v} \left\{ \frac{1}{2} \ln \left[ 1 - \frac{((\frac{v-u}{2})S_{x_1,f})^2}{(1+(\frac{u+v}{2}-x_1)S_{x_1,f})^2} \right] \right\} = 0.$$

Consequently,

$$\frac{1}{v-u} \int_u^v f(x) dx \geq \max_{u < x_1 < v} \left\{ \psi_{x_1}\left(\frac{u+v}{2}\right) \right\}. \quad (48)$$

From (31) at  $x_1 = (u + v)/2$ , we have

$$\psi_{x_1}\left(\frac{u+v}{2}\right) \leq f\left(\frac{u+v}{2}\right) \quad \forall x_1 = (u, v). \quad (49)$$

Using (48) and (49), one obtains

$$\begin{aligned} \frac{1}{v-u} \int_u^v f(x) dx &\geq \max_{u < x_1 < v} \left\{ \psi_{x_1}\left(\frac{u+v}{2}\right) \right\} \\ &= f\left(\frac{u+v}{2}\right). \end{aligned} \quad (50)$$

Hence, from (44), (45) and (50), we have the required inequality (41).  $\square$

**Theorem 12.** Let  $f : (u, v) \rightarrow \mathbf{R}$  be a twice differentiable on  $(u, v)$ . Then  $f$  is a sub  $L$ -convex function if and only if there is an increasing function  $w : (u, v) \rightarrow \mathbf{R}$  and a point  $c \in (u, v)$  such that for all  $x \in (u, v)$ ,

$$e^{f(x)} - e^{f(c)} = \int_c^x w(t) dt. \quad (51)$$

*Proof.* Let  $f$  be a sub  $L$ -convex function. Since  $f$  is a differentiable function on  $I$ , then we can choose

$$w(x) = f'(x)e^{f(x)} \quad \forall x \in (u, v). \quad (52)$$

By differentiating the above equation with respect to  $x$ , we get

$$\begin{aligned} w'(x) &= f''(x)e^{f(x)} + f'^2(x)e^{f(x)}, \\ &= [f''(x) + f'^2(x)]e^{f(x)}. \end{aligned}$$

Since  $f$  is a sub  $L$ -convex function and from (7), we have

$$w'(x) = [f''(x) + f'^2(x)]e^{f(x)} \geq 0 \quad \forall x \in (u, v). \quad (53)$$

Then  $w$  is an increasing function. For any  $c \in (u, v)$ , and by integrating (52) from  $c$  to  $x$ , we get

$$\int_c^x w(t) dt = e^{f(x)} - e^{f(c)}.$$

Conversely, let  $w$  be an increasing function and the equation (51) be realized, we observe that

$$f(x) = \ln \left[ e^{f(c)} + \int_c^x w(t) dt \right]. \quad (54)$$

Then, for  $x_1 < x_2$  in  $(u, v)$ , we have

$$\begin{aligned} f(x) &= \ln \left[ e^{f(c)} + \frac{(x_2 - x) + (x - x_1)}{x_2 - x_1} \int_c^x w(t) dt \right], \\ &= \ln \left[ e^{f(c)} + \frac{x_2 - x}{x_2 - x_1} \int_c^{x_1} w(t) dt + \frac{x_2 - x}{x_2 - x_1} \int_{x_1}^x w(t) dt \right. \\ &\quad \left. + \frac{x - x_1}{x_2 - x_1} \int_c^{x_2} w(t) dt - \frac{x - x_1}{x_2 - x_1} \int_x^{x_2} w(t) dt \right]. \end{aligned}$$

Since  $w$  is an increasing, then  $w(t) \leq w(x)$  in  $\int_{x_1}^x w(t) dt$  and  $-w(t) \leq -w(x)$  in  $-\int_x^{x_2} w(t) dt$ , and therefore, we obtain

$$\begin{aligned} f(x) &\leq \ln \left[ e^{f(c)} + \frac{(x_2 - x) + (x - x_1)}{x_2 - x_1} \int_c^x w(t) dt \right] \\ &= \ln \left[ e^{f(c)} + \frac{x_2 - x}{x_2 - x_1} \int_c^{x_1} w(t) dt + \frac{x_2 - x}{x_2 - x_1} w(x) \int_{x_1}^x dt \right. \\ &\quad \left. + \frac{x - x_1}{x_2 - x_1} \int_c^{x_2} w(t) dt - \frac{x - x_1}{x_2 - x_1} w(x) \int_x^{x_2} dt \right] \\ &= \ln \left[ \frac{(x_2 - x) + (x - x_1)}{x_2 - x_1} e^{f(c)} + \frac{x_2 - x}{x_2 - x_1} \int_c^{x_1} w(t) dt \right. \\ &\quad \left. + \frac{x_2 - x}{x_2 - x_1} w(x)(x - x_1) + \frac{x - x_1}{x_2 - x_1} \int_c^{x_2} w(t) dt \right. \\ &\quad \left. - \frac{x - x_1}{x_2 - x_1} w(x)(x_2 - x) \right] \\ &= \ln \left[ \frac{(x_2 - x)(e^{f(c)} + \int_c^{x_1} w(t) dt) + (x - x_1)(e^{f(c)} + \int_c^{x_2} w(t) dt)}{x_2 - x_1} \right]. \end{aligned} \tag{55}$$

Applying (54) at  $x_1$  and  $x_2$ , we get

$$f(x_1) = \ln \left[ e^{f(c)} + \int_c^{x_1} w(t) dt \right],$$

and

$$f(x_2) = \ln \left[ e^{f(c)} + \int_c^{x_2} w(t) dt \right].$$

Consequently,

$$e^{f(x_1)} = e^{f(c)} + \int_c^{x_1} w(t) dt, \tag{56}$$

and

$$e^{f(x_2)} = e^{f(c)} + \int_c^{x_2} w(t) dt. \quad (57)$$

Substituting (56) and (57) in (55), we have

$$f(x) \leq \ln \left[ \frac{(x_2 - x)e^{f(x_1)} + (x - x_1)e^{f(x_2)}}{x_2 - x_1} \right] \quad \forall x \in [x_1, x_2].$$

Which means that the function  $f(x)$  is a sub  $L$ -convex function on  $(u, v)$ .

Hence, the theorem follows.  $\square$

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