

**A NEW VARIANT OF RUSANOV SCHEME: β -RUSANOV
FOR NUMERICAL RESOLUTION OF SHALLOW
WATER FLOWS**

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Abstract: In this article, we are interested in the numerical resolution of the shallow water equation. We present a new modified version of the Rusanov scheme, called β -Rusanov. A mathematical analysis of this new scheme is done. For the approximation of the source term we use the hydrostatic reconstruction method. Some, numerical tests are performed to show the efficiency of our new scheme and compare this one with the classical Rusanov scheme.

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1. Introduction

The Saint-Venant equations derive from the Navier-Stokes equations by performing an average operation along the vertical, the hydrostatic and the impermeability assumptions, [13]. In one-dimensional, the governing equations are given by:

$$\left\{ \begin{array}{l} \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0, \\ \frac{\partial hu}{\partial t} + \frac{\partial}{\partial x} \left(hu^2 + \frac{gh^2}{2} \right) = 0, \end{array} \right. \quad (1)$$

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where h is the water depth, u is the velocity in the x -direction, g is the gravity. The conservation form of this hyperbolic system (1) is:

$$\frac{\partial \mathbf{U}}{\partial t}(x, t) + \frac{\partial(\mathbf{f}(\mathbf{U}))}{\partial x}(x, t) = 0, \quad (2)$$

$$\text{with } \mathbf{U} = \begin{pmatrix} h \\ hu \end{pmatrix}, \mathbf{f}(\mathbf{U}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}.$$

These equations are very often used to describe many physical phenomena. For example, like the runoff [5], the transport of pollutants in the river [3, 6]. These equations are based on certain physical laws, namely the mass conservation and the momentum conservation [23]. The difficulties linked to this hyperbolic and conservative system is the obtain of the analytical solution. In general, it is not possible to derive exact solutions to this system, where from hence the need to proceed by numerical methods to approximate the solution. The most commonly used method to approximate these Saint-Venant system is the finite volume method [17] which consists in discretization of the domain of the flow into a multitude of control volumes, then integrated the nonlinear systems of partial differential equations (PDEs) on each cell. The advantage of the finite volume method lies in the fact that it ensures the continuity of the mass, an important property to be observed for all calculations of fluid flows. The essential point of this numerical method being the calculation of interfacial flux. Several numerical scheme have been developed to be able to calculate the flux. As an example, A. Harten, P.D. Lax, B. Van Leer introduced the HLL solver scheme in [12], E.F. Toro, M. Spruce and W. Spreace have presented the improved version HLLC of the Harten-Lax-van Leer Riemann solver by incorporating the contact surface into the wave pattern, [24], Roe scheme based on approximate Riemann solver [21]. In 1999, T. Buffard, T. Gallouët and J.-M. Hérard have developed an extension of the VFRoe scheme where the hyperbolic system is expressed in non conservative variables in the linearization part of the scheme (in short VFRoe-ncv Scheme), [4]. Similarly, in 1998, J.-M. Ghidaglia presented a reformulation of VFFC (Volume Finis à Flux Caractéristique) scheme - which is another approximate Riemann solver-, for solving nonlinear systems of conservation laws [9], Godunov scheme has been developped by Godunov [11], Lax-Friedrich fluxes [16], based on the average flux at the bondary, but very diffusive, Rusanov flux introduced by Rusanov [22]. The latter is a modified version of the Lax-Friedrich scheme, very robust and widely used in the numerical simulation of monophasic as well as two-phase flow problems. It is also too diffusive and imprecise, which makes it an expensive scheme in terms of computation time. We present in this work, a

new modification on the Rusanov scheme. The objective is to reduce the time computation and also make it more precise. This modified form of Rusanov flux depends on the absolute value of the maximum eigenvalue of the hyperbolic system in the definition of numerical flux. In the construction of this new scheme reform, our main idea is inspired by the works as [21, 20, 19, 15, 18] and we named it β -Rusanov, with β a parameter such as $\beta \in]-1; 1[$ along a straight path connecting two cells.

The paper is organised as follows. In Section 2 we make a brief presentation of the finite volume methods where the discrete flux is obtained by the β -Rusanov flux and Rusanov flux non-modified. We also present the theoretical study of β -Rusanov flux where we show the properties of conservativity, of consistency, Total Variation Diminishing (in short TVD), local maximum principle, L^∞ -stability of scheme, convergence and monotony properties. The topography term is processed by the hydrostatic reconstruction method combined with the well-balanced scheme. This scheme preserves the positivity of the height. Finally, in Section 3 some of the numerical simulation of the flooding phenomenon on a domain whose topography is not completely flat and comparison of numerical results obtained by the β -Rusanov flux and Rusanov classical.

2. Numerical Resolution

2.1. Finite Volume Method

The numerical discretization of the equation (2) is done by using the finite volume method [17], so let us consider a finite volume mesh: with mean a union of disjoint cells $C_i =]x_{i-\frac{1}{2}}; x_{i+\frac{1}{2}}[$ of size $|C_i| = \Delta x_i$, centered at $x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$, $i = 1, \dots, N$. Let us denote Δt^n the time step and U_i^n the approximation of the cell average of the exact solution $u(t, x)$ at time $t^n = n\Delta t^n$.

The finite volume method consists to integrate the system on $[t^n, t^{n+1}] \times C_i$, which gives:

$$\mathbf{U}_i^{n+1} - \mathbf{U}_i^n + \frac{\Delta t^n}{\Delta x_i} (\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n) = 0, \quad (3)$$

$$\begin{aligned} \mathcal{F}(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n) &= \mathbf{F}_{i+1/2}^n \simeq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{U}(t, x_{i+1/2})) dt, \\ \mathcal{F}(\mathbf{U}_{i-1}^n, \mathbf{U}_i^n) &= \mathbf{F}_{i-1/2}^n \simeq \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{U}(t, x_{i-1/2})) dt, \end{aligned}$$

$\mathbf{F}_{i+1/2}^n$ and $\mathbf{F}_{i-1/2}^n$ are numerical fluxes at the cell interfaces.

We recall some definitions and propositions that can be found in [10].

Definition 1. A numerical scheme is said to be write in conservative form, if there exists $k \in \mathbb{Z}$ and a continuous function $\mathcal{G} : \mathbb{R}^{2k} \rightarrow \mathbb{R}$ such that: $\forall i \in \mathbb{Z}$ and $n \geq 0$

$$u_i^{n+1} = u_i^n - h \left(\mathcal{G}(u_{i-k+1}^n, \dots, u_{i+k}^n) - \mathcal{G}(u_{i-k}^n, \dots, u_{i+k-1}^n) \right), \quad (4)$$

where $u_i^n = u(x_i, t^n)$, $\mathcal{G}(u_{i-k+1}^n, \dots, u_{i+k}^n)$ is the numerical flux.

Definition 2. The numerical scheme is consistent in the sense of Finite Volumes, with the equation (2), if the numerical flux verifies the following condition: $\mathcal{F}(\mathbf{U}, \mathbf{U}) = \mathbf{f}(\mathbf{U})$.

Definition 3. A numerical scheme is said to be continuously Lipschitzian flux, if there exists a constant $K > 0$ such that: $|\mathcal{F}(u_{i-k+1}^n, u_{i-k+2}^n, \dots, u_{i+k}^n) - \mathbf{f}(u)| \leq K \max_{-k+1 \leq j \leq k} |u_{i+j} - u|$, where

Definition 4. A scheme is monotonic if the operator $\mathcal{H}$ is an increasing function of each of its arguments.

Definition 5. A scheme can be put in incremental form if there exist coefficients A, B such that setting

$$A_{i+\frac{1}{2}}^n = A_{i+\frac{1}{2}}(u_{i-k+1}^n, \dots, u_{i+k}^n) \quad \text{and} \quad B_{i-\frac{1}{2}}^n = B_{i-\frac{1}{2}}(u_{i-k}^n, \dots, u_{i+k-1}^n),$$

$$\text{and, we have: } u_i^{n+1} = u_i^n + A_{i+\frac{1}{2}}^n (u_{i+1}^n - u_i^n) - B_{i-\frac{1}{2}}^n (u_i^n - u_{i-1}^n). \quad (5)$$

Proposition 6 ([10]). A numerical scheme, written in the form (5) is Total Variation Diminishing under the sufficient condition:

$A_{i+\frac{1}{2}}^n \geq 0$, $B_{i+\frac{1}{2}}^n \geq 0$, and $A_{i+\frac{1}{2}}^n + B_{i+\frac{1}{2}}^n \leq 1$, $\forall (i, n) \in \mathbb{Z} \times \mathbb{N}$. Moreover, it satisfies the local maximum principle, and is therefore L^∞ -stable under the sufficient condition:

$$A_{i+\frac{1}{2}}^n \geq 0, \quad B_{i-\frac{1}{2}}^n \geq 0, \quad \text{and} \quad A_{i+\frac{1}{2}}^n + B_{i-\frac{1}{2}}^n \leq 1, \quad \forall (i, n) \in \mathbb{Z} \times \mathbb{N}.$$

2.2. Flux approximation

Several methods exist in the literature to approximate the flux, but each would not lack drawbacks. For example, the Godunov scheme [11] satisfies the entropy conditions and preserves the positivity of the variables but requires solving the Riemann problem on each interface at each time step, which makes it very expensive. Van Leer's FVS scheme [1] makes it possible to calculate strong shocks or strong expansions without oscillation. This robust scheme is too diffusive for the integration of the Navier-Stokes equations. Roe's scheme [21] is robust and stable, resolves shocks but requires entropy correction. As we said in the introduction, our objective is to make an improvement on the Rusanov scheme which will allow us to gain time and precision. We reconstruct the variables assuming that there is some level of regularity between neighboring cells and such that there is no interaction between waves emanating from one interface with waves from adjacent interfaces. The maximum wave velocity is calculated as in the case of the classical Rusanov scheme, ie by calculating the maximum modular of the eigenvalue of the local Jacobian matrix. Before presenting our new scheme, called β -Rusanov, we recall the classical Rusanov flux then we make a link between the two schemes.

The classical Rusanov flux [22], it's a generalization of the Lax-Friedrich flux [8]. Let us note $\mathbf{U}_L$ and $\mathbf{U}_R$ the data in the two neighboring cells, then the Rusanov flux is gives by:

$$\mathcal{F}(\mathbf{U}_L, \mathbf{U}_R) = \frac{1}{2}(\mathbf{f}(\mathbf{U}_L) + \mathbf{f}(\mathbf{U}_R)) - \frac{1}{2}\kappa(\mathbf{U}_R - \mathbf{U}_L); \quad (6)$$

where κ is the Rusanov velocity given by: $\kappa = \max_{j \in \{1;2\}} (|\lambda_j(\mathbf{U}_L)|, |\lambda_j(\mathbf{U}_R)|)$ and

λ_j are eigenvalues of the matrix: $\frac{\partial \mathbf{f}}{\partial \mathbf{U}} = \begin{pmatrix} 0 & 1 \\ gh - u^2 & 2u \end{pmatrix}$,

one obtains: $\lambda_1(\mathbf{U}) = u - \sqrt{gh}$ and $\lambda_2(\mathbf{U}) = u + \sqrt{gh}$.

Let us take a real β such as $\beta \in]-1; 1[$. Thus, the new reconstructed data are obtained by:

$$U_R^* = U_R - \frac{\beta}{2}(U_R - U_L) \quad \text{and} \quad U_L^* = U_L + \frac{\beta}{2}(U_R - U_L).$$

The flux $\mathcal{F}_\beta(\mathbf{U}_L^*, \mathbf{U}_R^*)$ at the interfaces between cells is determined always by the formula (6) with

$$\mathbf{U}_R^* = \begin{pmatrix} h_R^* \\ u_R^* \end{pmatrix} \quad \text{and} \quad \mathbf{U}_L^* = \begin{pmatrix} h_L^* \\ h_L^* u_L^* \end{pmatrix},$$

$$\mathbf{f}(\mathbf{U}_R^*) = \begin{pmatrix} h_R^* u_R^{*2} \\ h_R^* u_R^{*2} + \frac{1}{2} g h_R^{*2} \end{pmatrix}, \quad \mathbf{f}(\mathbf{U}_L^*) = \begin{pmatrix} h_L^* u_L^{*2} \\ h_L^* u_L^{*2} + \frac{1}{2} g h_L^{*2} \end{pmatrix}.$$

In all the following, we replace U^* by V . Let us recall the Rusanov scheme in its conservative form:

$u_i^{n+1} = u_i^n - \delta_i^n (\mathcal{F}(u_i^n, u_{i+1}^n) - \mathcal{F}(u_{i-1}^n, u_i^n))$, where $\delta_i^n = \frac{\Delta t^n}{\Delta x_i}$. The Rusanov numerical flux is given by:

$$\mathcal{F}(u_i^n, u_{i+1}^n) = \frac{1}{2}(\mathbf{f}(u_i^n) + \mathbf{f}(u_{i+1}^n)) - \frac{1}{2}\kappa_{i+\frac{1}{2}}^n(u_{i+1}^n - u_i^n), \quad (7)$$

where $\kappa_{i+\frac{1}{2}}^n = \max(|\mathbf{f}'(u_i^n)|, |\mathbf{f}'(u_{i+1}^n)|)$, represents the local Rusanov velocity.

One of the advantage of the Rusanov scheme its robustness, so we want to conserve it and in addition make this scheme more efficient and faster. Thus, our β -Rusanov scheme is defined as follows.

Definition 7. Let us take a real β such as $|\beta| < 1$, the β -Rusanov scheme is defined by:

$$v_i^{n+1} = v_i^n - \alpha^n (\mathcal{F}_\beta(v_i^n, v_{i+1}^n) - \mathcal{F}_\beta(v_{i-1}^n, v_i^n)), \quad (8)$$

where $\alpha^n = (1 - \beta)\delta_i^n$ and the numerical flux given by:

$$\mathcal{F}_\beta(v_i^n, v_{i+1}^n) = \frac{1}{2}(\mathbf{f}(v_i^n) + \mathbf{f}(v_{i+1}^n)) - \frac{1}{2}\kappa_{i+\frac{1}{2}}^n(v_{i+1}^n - v_i^n), \quad (9)$$

$$\text{with } \begin{cases} v_i^n = u_i^n + \frac{\beta}{2}(u_{i+1}^n - u_i^n), \\ v_{i+1}^n = u_{i+1}^n - \frac{\beta}{2}(u_{i+1}^n - u_i^n). \end{cases} \quad (10)$$

In the following we assume that:

Assumptions: $\begin{cases} (H1) & \text{The function } \mathbf{f} \text{ is of class } \mathcal{C}^1(\mathbb{R}^2); \\ (H2) & \text{The function } \mathbf{f} \text{ is strictly monotonic in } \mathbb{R}^2. \end{cases}$

Notation:

$$\begin{cases} \Delta\omega_{i+\frac{1}{2}}^n = \omega_{i+1}^n - \omega_i^n, \\ \lambda_{i+\frac{1}{2}}^n = \frac{\alpha^n}{2} \left(\kappa_{i+\frac{1}{2}}^n - \frac{\Delta\mathbf{f}_{i+\frac{1}{2}}^n}{\Delta v_{i+\frac{1}{2}}^n} \right) \text{ and } \gamma_{i-\frac{1}{2}}^n = \frac{\alpha^n}{2} \left(\frac{\Delta\mathbf{f}_{i-\frac{1}{2}}^n}{\Delta v_{i-\frac{1}{2}}^n} + \kappa_{i-\frac{1}{2}}^n \right). \end{cases} \quad (11)$$

Now we are enable to announce and prove the most qualitative properties of our scheme.

Proposition 8. *The β -Rusanov numerical flux is conservative.*

Proposition 9. *The β -Rusanov scheme is consistent with (2).*

Proof. $\mathcal{F}_\beta(v_i^n, v_i^n) = \frac{1}{2}(\mathbf{f}(v_i^n) + \mathbf{f}(v_i^n)) - \frac{1}{2}\kappa_{i+\frac{1}{2}}^n(v_i^n - v_i^n) = \mathbf{f}(v_i^n)$. $\square$

Lemma 10. *The β -Rusanov scheme admits an incremental form.*

Proof. Using (8) and (9), we have:

$$\begin{aligned}
 v_i^{n+1} &= v_i^n - \frac{\alpha^n}{2} \left(\left(\mathbf{f}(v_i^n) + \mathbf{f}(v_{i+1}^n) - \kappa_{i+\frac{1}{2}}^n(v_{i+1}^n - v_i^n) \right) \right. \\
 &\quad \left. - \left(\mathbf{f}(v_i^n) + \mathbf{f}(v_{i-1}^n) - \kappa_{i-\frac{1}{2}}^n(v_i^n - v_{i-1}^n) \right) \right), \\
 &= v_i^n - \frac{\alpha^n}{2} \left(\left(\mathbf{f}(v_{i+1}^n) - \mathbf{f}(v_i^n) - \kappa_{i+\frac{1}{2}}^n(v_{i+1}^n - v_i^n) \right) \right. \\
 &\quad \left. + \left(\mathbf{f}(v_i^n) - \mathbf{f}(v_{i-1}^n) + \kappa_{i-\frac{1}{2}}^n(v_i^n - v_{i-1}^n) \right) \right), \\
 &= v_i^n - \frac{\alpha^n}{2} \left(\left(\frac{\Delta \mathbf{f}_{i+\frac{1}{2}}^n}{\Delta v_{i+\frac{1}{2}}^n} - \kappa_{i+\frac{1}{2}}^n \right) \Delta v_{i+\frac{1}{2}}^n + \left(\frac{\Delta \mathbf{f}_{i-\frac{1}{2}}^n}{\Delta v_{i-\frac{1}{2}}^n} + \kappa_{i-\frac{1}{2}}^n \right) \Delta v_{i-\frac{1}{2}}^n \right), \\
 &= v_i^n + \frac{\alpha^n}{2} \left(\kappa_{i+\frac{1}{2}}^n - \frac{\Delta \mathbf{f}_{i+\frac{1}{2}}^n}{\Delta v_{i+\frac{1}{2}}^n} \right) \Delta v_{i+\frac{1}{2}}^n - \frac{\alpha^n}{2} \left(\frac{\Delta \mathbf{f}_{i-\frac{1}{2}}^n}{\Delta v_{i-\frac{1}{2}}^n} + \kappa_{i-\frac{1}{2}}^n \right) \Delta v_{i-\frac{1}{2}}^n.
 \end{aligned}$$

Using (11),

we have: $v_i^{n+1} = v_i^n + \lambda_{i+\frac{1}{2}}^n \Delta v_{i+\frac{1}{2}}^n - \gamma_{i-\frac{1}{2}}^n \Delta v_{i-\frac{1}{2}}^n$.

Thus, we can rewrite the previous form as follows:

$$v_i^{n+1} = \gamma_{i-\frac{1}{2}}^n v_{i-\frac{1}{2}}^n + (1 - \lambda_{i+\frac{1}{2}}^n - \gamma_{i-\frac{1}{2}}^n) v_i^n + \lambda_{i+\frac{1}{2}}^n v_{i+\frac{1}{2}}^n.$$

$\square$

Proposition 11. *Let us assume that: $\alpha^n \kappa_{i+\frac{1}{2}}^n \leq 1$ then the β -Rusanov scheme is TVD.*

Proof. According to (H2.), $\alpha^n > 0$ and the definition of $\kappa_{i+\frac{1}{2}}^n$, we have:
 $\lambda_{i+\frac{1}{2}}^n \geq 0$ and $\gamma_{i+\frac{1}{2}}^n \geq 0$.

In addition, $\gamma_{i+\frac{1}{2}}^n + \lambda_{i+\frac{1}{2}}^n = \frac{\alpha^n}{2} (2\kappa_{i+\frac{1}{2}}^n) = \alpha^n \kappa_{i+\frac{1}{2}}^n$,

where $\gamma_{i+\frac{1}{2}}^n + \lambda_{i+\frac{1}{2}}^n \leq 1$. $\square$

Proposition 12. *Under the following condition:*

$$\frac{\alpha^n}{2} \left(\kappa_{i+\frac{1}{2}}^n + \kappa_{i-\frac{1}{2}}^n \right) + \alpha^n M \leq 1,$$

where $M = \max |(\mathbf{f}'(\theta_{i+\frac{1}{2}}^n), \mathbf{f}'(\theta_{i-\frac{1}{2}}^n))|$ with $\theta_{i+\frac{1}{2}}^n \in [\min(v_i^n, v_{i+1}^n); \max(v_i^n, v_{i+1}^n)]$ and $\theta_{i-\frac{1}{2}}^n \in [\min(v_{i-1}^n, v_i^n); \max(v_{i-1}^n, v_i^n)]$, then the β -Rusanov scheme satisfies the local maximum principle and is the L^∞ -stable.

Proof. According to (H2.), $\alpha^n > 0$ and the definition of $\kappa_{i+\frac{1}{2}}^n$, we have:
 $\lambda_{i+\frac{1}{2}}^n \geq 0$ and $\gamma_{i-\frac{1}{2}}^n \geq 0$.

According to (H1.), $\mathbf{f}$ is continuously differentiable on $[\min(v_i^n, v_{i+1}^n); \max(v_i^n, v_{i+1}^n)]$ and $[\min(v_{i-1}^n, v_i^n); \max(v_{i-1}^n, v_i^n)]$. By the mean value theorem, we have the existence of $(\theta_{i+\frac{1}{2}}^n, \theta_{i-\frac{1}{2}}^n) \in [\min(v_i^n, v_{i+1}^n); \max(v_i^n, v_{i+1}^n)] \times [\min(v_{i-1}^n, v_i^n); \max(v_{i-1}^n, v_i^n)]$ such as

$$\mathbf{f}(v_{i+1}^n) - \mathbf{f}(v_i^n) = \mathbf{f}'(\theta_{i+\frac{1}{2}}^n)(v_{i+1}^n - v_i^n);$$

$$\mathbf{f}(v_i^n) - \mathbf{f}(v_{i-1}^n) = \mathbf{f}'(\theta_{i-\frac{1}{2}}^n)(v_i^n - v_{i-1}^n);$$

and $\mathbf{f}'$ is bounded.

$$\begin{aligned} \text{Therefore, } \lambda_{i+\frac{1}{2}}^n &= \frac{\alpha^n}{2} \left(\kappa_{i+\frac{1}{2}}^n - \mathbf{f}'(\theta_{i+\frac{1}{2}}^n) \right) \leq \frac{\alpha^n}{2} \left(\kappa_{i+\frac{1}{2}}^n + M \right); \\ \gamma_{i-\frac{1}{2}}^n &= \frac{\alpha^n}{2} \left(\kappa_{i-\frac{1}{2}}^n + \mathbf{f}'(\theta_{i-\frac{1}{2}}^n) \right) \leq \frac{\alpha^n}{2} \left(\kappa_{i-\frac{1}{2}}^n + M \right); \end{aligned}$$

from where $\lambda_{i+\frac{1}{2}}^n + \gamma_{i-\frac{1}{2}}^n \leq 1$.

Finally, the β -Rusanov scheme satisfies the local maximum principle and induce the L^∞ -stable. $\square$

Proposition 13. *The β -Rusanov numerical flux is Lipschitz continuous.*

Proof.

$$\begin{aligned} \left| \mathcal{F}_\beta(v_i^n, v_{i+1}^n) - \mathbf{f}(v) \right| &= \left| \frac{1}{2}(\mathbf{f}(v_i^n) + \mathbf{f}(v_{i+1}^n)) - \mathbf{f}(v) - \frac{1}{2}\kappa_{i+\frac{1}{2}}^n(v_{i+1}^n - v_i^n) \right|, \\ &= \frac{1}{2} \left| (\mathbf{f}(v_{i+1}^n) - \mathbf{f}(v)) + (\mathbf{f}(v_i^n) - \mathbf{f}(v)) \right. \\ &\quad \left. - \kappa_{i+\frac{1}{2}}^n((v_{i+1}^n - v) - (v_i^n - v)) \right|. \end{aligned}$$

According to (H1.), $\mathbf{f}$ is continuously differentiable, therefore $\mathbf{f}$ being a Lipschitzian function, which implies the existence of two constants

$$(k_1, k_2) \in [\min(v_{i+1}^n, v); \max(v_{i+1}^n, v)] \times [\min(v_i^n, v); \max(v_i^n, v)],$$

$$\begin{aligned} \text{such as, } \left| \mathcal{F}_\beta(v_i^n, v_{i+1}^n) - \mathbf{f}(v) \right| &\leq \frac{1}{2} \left[k_1 |v_{i+1}^n - v| + k_2 |v_i^n - v| + \right. \\ &\quad \left. \kappa_{i+\frac{1}{2}}^n (|v_{i+1}^n - v| + |v_i^n - v|) \right], \\ &\leq \frac{1}{2} (k_1 + \kappa_{i+\frac{1}{2}}^n) |v_{i+1}^n - v| + \frac{1}{2} (k_2 + \kappa_{i+\frac{1}{2}}^n) |v_i^n - v|, \\ &\leq \max \left(\frac{1}{2} (k_1 + \kappa_{i+\frac{1}{2}}^n); \frac{1}{2} (k_2 + \kappa_{i+\frac{1}{2}}^n) \right) \max_{j \in \{0;1\}} |v_{i+j}^n - v|, \\ &\leq k \max_{j \in \{0;1\}} |v_{i+j}^n - v|. \quad \square \end{aligned}$$

Theorem 14. *The β -Rusanov scheme is convergent.*

Proof. According to Lax's equivalence theorem [10], a numerical scheme under conservative form, with continuously Lipschitzian flux, TVD and L^∞ -stable, is convergent. $\square$

Proposition 15. *We assume that (H1.), (H2.) holds and in addition the assumptions that: $\alpha^n \kappa_{i \pm \frac{1}{2}}^n \leq 1$, then β -Rusanov scheme is monotonic.*

Proof. The β -Rusanov scheme can be write as follows:
 $v_i^{n+1} = \mathcal{H}(v_{i-1}^n, v_i^n, v_{i+1}^n)$, when $\mathcal{H}$ is given by:

$$\mathcal{H}(v_{i-1}^n, v_i^n, v_{i+1}^n) = v_i^n - \alpha^n (\mathcal{F}_\beta(v_i^n, v_{i+1}^n) - \mathcal{F}_\beta(v_{i-1}^n, v_i^n)),$$

$$\text{we have: } \frac{\partial \mathcal{H}}{\partial v_{i-1}^n} = \frac{\alpha^n}{2} (\kappa_{i-\frac{1}{2}}^n + M) \geq 0,$$

$$\frac{\partial \mathcal{H}}{\partial v_i^n} = 1 - \frac{\alpha^n}{2} (\kappa_{i+\frac{1}{2}}^n + M) - \frac{\alpha^n}{2} (\kappa_{i-\frac{1}{2}}^n + M) \geq 0, \text{ and}$$

$$\frac{\partial \mathcal{H}}{\partial v_{i+1}^n} = \frac{\alpha^n}{2} (\kappa_{i+\frac{1}{2}}^n + M) \geq 0. \quad \square$$

Theorem 16. *Under the conditions of the proposition 15, the numerical solution given by the β -Rusanov scheme converges to the unique entropic solution.*

Proof. The β -Rusanov scheme is a three-point scheme conservative, con-

sistent, L^∞ – stable and monotonic, which allow us to conclude. $\square$

2.3. Traitement of the bottom topography

As done in a previous work [14], we add the topography in the system (2). We use the well-balanced method that is based on the hydrostatic reconstruction method [7, 2], consisting to decenter the source term at the interfaces. These schemes have the advantage to preserve the equilibrium of fluid. Which mean that the height and the velocity being constant in time. Using this property we obtain the relation: $\frac{\partial}{\partial x} \left(\frac{gh^2}{2} \right) = -gh \frac{\partial z}{\partial x}$, called hydrostatic equilibrium.

Applying the differentiation rule, we show that $h + z$ is constant. The hydrostatic reconstruction method is based on the idea that near the equilibrium, the flows are almost hydrostatic.

The reconstructed water heights on either side of the interfaces adjust to satisfy equation:

$$\frac{\partial}{\partial x} \left(\frac{g\bar{h}^2}{2} \right) = -g\bar{h} \frac{\partial z}{\partial x}. \quad (12)$$

Integrating on a cell we obtain a following discretization of the source term:

$$- \int_{x_{i-1/2}}^{x_{i+1/2}} g\bar{h} \frac{\partial z}{\partial x} = \frac{g}{2} \bar{h}_{i+1/2,L}^2 - \frac{g}{2} \bar{h}_{i-1/2,R}^2,$$

where $\bar{h}_{i+1/2,L} = h_{i,r} + z_{i,r} - z_{i+1/2}$ and $\bar{h}_{i+1/2,R} = h_{i+1,l} + z_{i+1,l} - z_{i+1/2}$ with $z_{i+1/2}$ is the topography at the interface gives by: $z_{i+1/2} = \max(z_{i,r}; z_{i+1,l})$.

The discrete system (3) with the topography become:

$$\mathbf{U}_i^{n+1} - \mathbf{U}_i^n + \frac{\Delta t}{\Delta x} (\mathcal{F}_\beta(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n) - \mathcal{F}_\beta(\mathbf{U}_{i-1}^n, \mathbf{U}_i^n)) = \Delta t \mathbf{S}_i^n, \text{ where } \mathbf{S}_i^n \text{ is given}$$

$$\text{by: } \mathbf{S}_i^n = \begin{pmatrix} 0 \\ g \frac{h_{i+1/2G}^2}{2} - g \frac{h_{i-1/2D}^2}{2} \end{pmatrix}, \text{ with:}$$

$$h_{i+1/2G} = \max(0; \bar{h}_{i+1/2G,r}) \quad ; \quad h_{i+1/2D} = \max(0; \bar{h}_{i+1/2D,l}).$$

3. Numerical results

In this section, we present numerical results for several free surface flow problems with non-flat topography to check the accuracy and performance of the β –Rusanov scheme. We also compare our results with those obtained by the classical Rusanov. The first objective of this simulation is to verify the accuracy and performance of our β –Rusanov scheme. The second is to measure the risk of flooding in urban environment with non-flat topography with a high rainfall

without taking into account the rate of the infiltration. For that, we imposed a very important discharge of water to upstream and a closure on downstream.

3.1. Simulation test without obstacle

The bottom topography is given by:

$$z(x) = \begin{cases} 0.5, & \text{if } x \leq 0.25; \\ \frac{-10x}{7} + \frac{6}{7}, & \text{if } x \in [0, 25; 0, 6]; \\ 0, & \text{otherwise;} \end{cases}$$

with initial condition given by: $h + z = 0.55 \text{ m}$ and $q = 0.6 \text{ m}^3/\text{s}$. The numerical tests are performed on a domain $[0; 1]$, subdivided into 150 cells, and $\beta = 0.4$.

In Figures 1-4: the water gradually submerges the area but not completely and the height approximated by the β -Rusanov flux advances faster than that calculated with the classical Rusanov flux. We see the return of water because of the blockade downstream. Thus, the water has almost submerged the area and we observe a ridge on the surface due to the superposition of the progressive and regressive waves. In Figures 5-6: we can see that the water has completely flooded the zone after 0.055s with the β -Rusanov flux but the classical Rusanov has reached its state equilibrium after 0.059s.

3.2. Simulation test with obstacle

In this second test, we add an obstacle in the lower zone. The bottom topography is given by:

$$z(x) = \begin{cases} 0.5, & \text{if } x \leq 0.25; \\ \frac{-10x}{7} + \frac{6}{7}, & \text{if } x \in [0, 25; 0, 6] \\ 0, & \text{if } x \in [0, 6; 1, 2]; \\ \sqrt{0,01 - (x - 1,2)^2} & \text{if } x \in [1, 2; 1, 4]; \\ 0, & \text{otherwise;} \end{cases}$$

with initial condition given by: $h + z = 0.55 \text{ m}$ and $q = 0.6 \text{ m}^3/\text{s}$. The numerical tests are performed on a domain $[0; 2]$, subdivided into 150 cells, and $\beta = -0.6$.

The question here is: how long time the water will take to invade lower zone?

In the various figures, we note a progression of the height of water whose that calculated by the β -Rusanov advances faster than by the classical Ru-

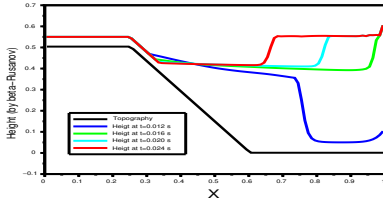


Figure 1: Variation of the water height with β -Rusanov.

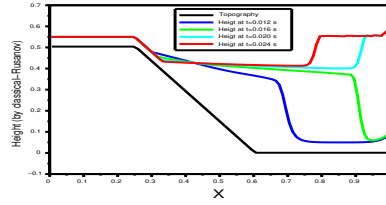


Figure 2: Variation of the water height with classical Rusanov.

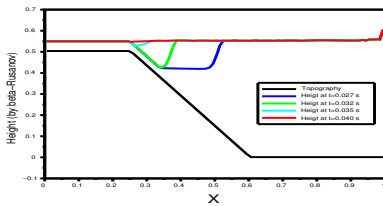


Figure 3: Variation of the water height with β -Rusanov.

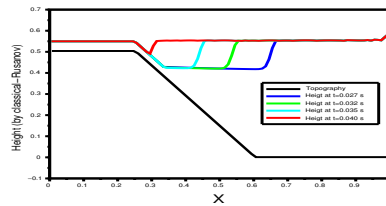


Figure 4: Variation of the water height with classical Rusanov.

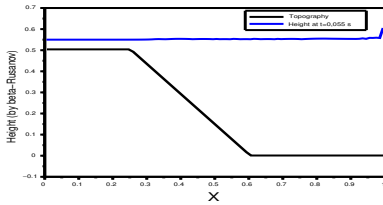


Figure 5: Variation of the water height with β -Rusanov.

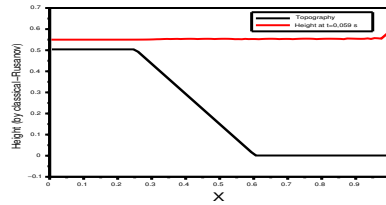


Figure 6: Variation of the water height with classical Rusanov.

sanov flux. Following the blockade placed downstream, at time $t = 0.13$ s, we notice a return of the water height approached with our scheme while the classical Rusanov took more than $t = 0.24$ s. Indeed, the ridge observed on some figures is linked to the superposition of the progressive and regressive waves and the morphology of the terrain at this level. Finally, in Figures 11-12, the β -Rusanov numerical flux (respectively the classical Rusanov) reaches its state of equilibrium after 4.17s (respectively $t = 5.83$ s) and also the lower zone is totally invaded by water.

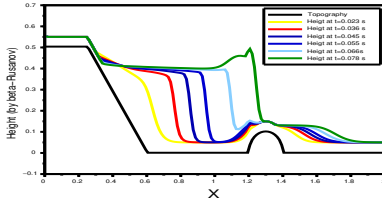


Figure 7: Variation of the water height with β -Rusanov.

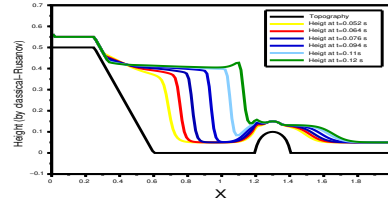


Figure 8: Variation of the water height with classical Rusanov.

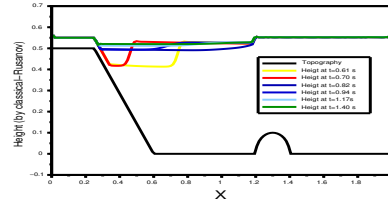
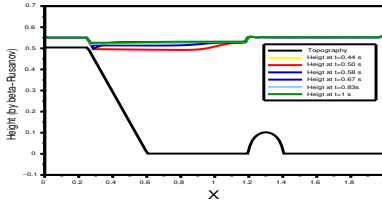
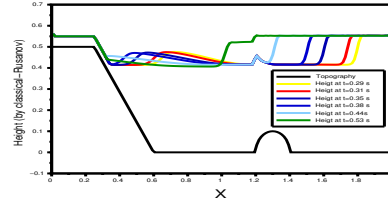
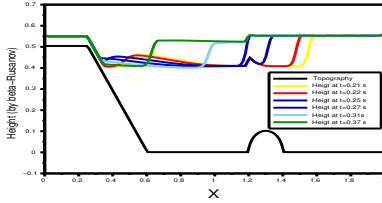
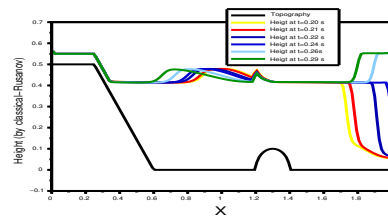
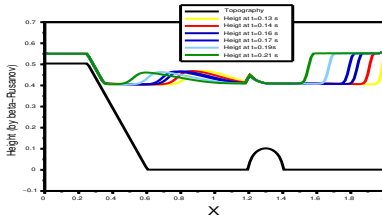


Figure 9: Variation of the water height with β -Rusanov.

Figure 10: Variation of the water height with classical Rusanov.

4. Conclusion

We have proposed and analysed a new variant of the Rusanov scheme called β -Rusanov. This scheme combine accuracy and performances. With our scheme we win in time compared within classical Rusanov. The numerical results performed, on various problems indicate that this new scheme is a ro-

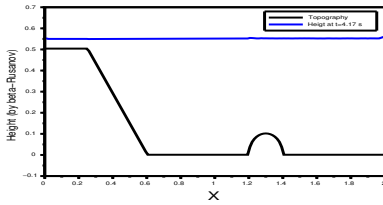


Figure 11: Variation of the water height with β -Rusanov.

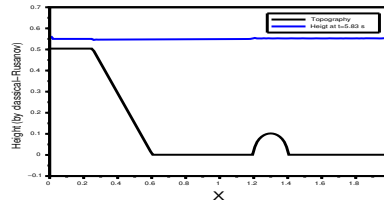


Figure 12: Variation of the water height with classical Rusanov.

bust way of treating various environmental problem such as runoff, erosion and so on.

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