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ON A STUDY OF THE HAMBURGER TRUNCATED MOMENT PROBLEM VIA JACOBI OPERATORS

Amar Rhazi[1 §], Abdelaziz El Boukili[1]
and Bouazza El Wahbi[1]

[1]Laboratory of Analysis, Geometry and Applications
(LAGA), Department of Mathematics
Faculty of Sciences, Ibn Tofail University
B.P. 133, Kenitra, MOROCCO

Abstract: In this paper, we deal with solving the Hamburger truncated moment problem without distinction between even and odd case. We establish an algorithm which allows to confirm if a finite sequence is a Hamburger moment sequence or not. This algorithm is generated by linking between the moment problem and Jacobi operators.

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1. Introduction

Hamburger proved in [1] that any infinite sequence of real numbers $s = (s_j)_j \geq 0$, can be represented as follows

$$s_j = \int_{\mathbb{R}} x^j d\mu(x), \quad j \geq 0, \tag{1}$$

with a positive measure μ on the real line, if and only if all Hankel matrices $H_n = (s_{i+j})_{0\leq i,j\leq n}$, $(n \geq 0)$, are positive semidefinite.

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§Correspondence author

The sequences defined in (1) are called Hamburger moment sequences.

If $(s_j)_{j\geq 0}$ is a Hamburger moment sequence, then Hankel determinants $D_n =\mid H_n \mid$ are nonnegative and only two possibilities can occur, either $D_n > 0$ for all $n \in \mathbb{N}$, and in this case any μ satisfying (1) has infinite support, or there exists r such that $D_n > 0$ for $n \leq r$ and $D_n = 0$ elsewhere.

In this latter case, μ is uniquely determined and it is a discrete measure concentrated in $r+1$ points on the real axis, [10].

It follows, from a general theorem about the leading principal minors of real symmetric matrices, that if $D_n > 0$ for $n \leq n_0$ then the Hankel matrix H_{n_0} is positive definite [3, p.70]. On the other hand, if $D_n \geq 0$ for $n \leq n_0$, we can not conclude that H_{n_0} is positive semidefinite. On this point, we recall the following interesting theorem due to C. Berg and R. Szwarc [4].

Theorem 1. *Let $s = (s_j)_{j\geq 0}$ be a real sequence. If the sequence of Hankel determinants satisfies $D_n > 0$ for $n \leq n_0$ and $D_n = 0$ for $n > n_0$, then $(s_j)_{j\geq 0}$ is a Hamburger moment sequence of a uniquely determinant measure μ concentrated in $n_0 + 1$ points.*

In the present work, we are interested in the Hamburger moment problem for finite sequences, that is the Hamburger truncated moment problem. We deal with a determinant characterization for the Hamburger truncated moment sequences.

Let $s = (s_j)_{0\leq j\leq m}$, $m \geq 0$ be a real finite sequence and consider the following Hamburger truncated moment problem

$$s_j = \int_{\mathbb{R}} x^j \, d\mu(x), \quad 0 \leq j \leq m. \tag{2}$$

Thereafter, we use the term even case if m is even, and odd case otherwise.

Note that the theory of complete moment problem does not provide a solution for the truncated one. In contrary, the solution of the truncated moment problem can be used to solve the complete one, [2, 7, 9].

If $H_{E(\frac{m}{2})}(s)$ is invertible, its positivity is necessary and sufficient to solve the problem (2). However, such condition is, in general, not sufficient if $H_{E(\frac{m}{2})}(s)$ is singular, and the determinant characterization of moment sequences with finely many mass-points given in [4] is no longer valid when dealing with finite sequences. In Example 9, we illustrate this fact.

The usual approach to solve the problem (2) consists in treating the even non-singular case and then reducing the even singular case to the non-singular one by the use of quasi-orthogonal polynomials and gaussian quadrature [6].

Another approach, revolves around the notion of recursivity [5] of Hankel matrices.

In this paper, we aim essentially to solve (2) without distinction between the even and the odd case. We establish some necessary and sufficient conditions based on the calculation of the leading principal minors of the matrix $H_k(s)$ where $k = E(\frac{m}{2})$.

If all leading principal minors are positive, then s is positive definite if $m = 2k$ or it can be extended to a positive definite sequence if $m = 2k + 1$. In both cases s is a Hamburger moment sequence. Otherwise, we focus on the first determinant which vanishes. So, let us assume that $D_0 > 0, D_1 > 0, \cdots, D_r > 0$ and $D_{r+1} = 0$ where $r < E(\frac{m}{2})$, then we construct a representative measure μ for the subsequence $(s_0, s_1, \cdots, s_{2r+1})$ and we show that s is a Hamburger moment sequence if and only if it can be represented by μ.

In practice, this approach has limits since the construction of the measure μ is based on the resolution of an equation of degree $r+1$ and the resolution of such equation is not always easy. This is why we have investigated a link between the Jacobi operators and the truncated moments sequences. This enable us to establish an algorithm allowing to conclude if the problem (2) is soluble or not by reducing the problem to a calculations of the matrix product of a row matrix of size r and a square matrix of size $r \times r$.

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries that will be used to establish our results. Section 3 is devoted to the statement of our findings.

2. Preliminaries

In this section, we present a resolution for the problem 2 when $n = 2r + 1$ and $(s_j)_{0 \leq j \leq 2r}$ is positive definite. Then, we provide a link between Jacobi operators and Hamburger moment problem.

2.1. Resolution of the problem (2) when $s = (s_0, s_1, \cdots, s_{2r+1})$ with $(s_0, s_1, \cdots, s_{2r})$ positive definite

Let $s = (s_j)_{0 \leq j \leq 2r+1}$ be a real finite sequence such that the sequence $(s_j)_{0 \leq j \leq 2r}$ is positive definite, s can be extended to a $2r + 2$ positive definite sequence s^* see [8, Lemma 9.1]. We can then define on $\mathbb{R}_{r+1}[x]$ an inner product by

$$< P, Q >= L_s^*(PQ),$$

where L_s^* is the Riesz functional associated to s^*, i.e $L_s^*(x^j) = s_j, 0 \leq j \leq 2r+2$.

Let $(P_0, P_1, \cdots, P_{r+1})$ be the family of unitary orthogonal polynomials associated to this inner product. It is defined by

$$P_n = \frac{1}{D_{n-1}} \begin{vmatrix} s_0 & \cdots & s_n \\ \cdots & \cdots & \cdots \\ s_{n-1} & \cdots & s_{2n-1} \\ 1 & \cdots & x^n \end{vmatrix},$$

for all $n \in \{0, \cdots, r+1\}$, with $D_{-1} = 1$, (note that this family does not depend on s_{2r+2}, unless we want to normalize it).

If $\lambda_1, \lambda_2, \cdots, \lambda_{r+1}$ denote the roots of P_{r+1} then using Lemma 9.4 and Lemma 9.6 [8], we obtain the following theorem.

Theorem 2. *For all $f \in \mathbb{R}_{2r+1}[x]$, we have*

$$L_s^*(f) = \int_{\mathbb{R}} f(x) d\mu_{P_{r+1}}(x),$$

where $\mu_{P_{r+1}} = \sum_{i=1}^{r+1} m_i \delta_{\lambda_i}$, $\quad m_i = \left(\sum_{k=0}^{r} (P_k(\lambda_i))^2 \right)^{-1}$ *and δ_{λ_i} denotes the Dirac measure at the point λ_i.*

In particular, $\forall j \in \{1, \cdots, 2r+1\}, s_j = \int_{\mathbb{R}} x^j d\mu_{P_{r+1}}(x)$.

The following example illustrates this theorem.

Example 3. Let s be the sequence $(1, 1, 2, 6, 24, 104)$. The subsequence $(1, 1, 2, 6, 24)$ is positive definite.

$$D_0 = 1, \quad D_1 = 1, \quad D_2 = 4.$$

The orthogonal polynomials are

$$P_0 = 1, \quad P_1 = x - 1, \quad P_2 = x^2 - 4x + 2 \quad \text{and} \quad P_3 = x^3 - 5x^2 + 2x + 2.$$

The roots of P_3 are

$$\lambda_1 = 1, \quad \lambda_2 = 2 - \sqrt{6} \quad \text{and} \quad \lambda_3 = 2 + \sqrt{6}.$$

The coefficients m_i are

$$m_1 = \frac{4}{5}, \quad m_2 = \frac{6 + \sqrt{6}}{60} \quad \text{and} \quad m_3 = \frac{6 - \sqrt{6}}{60}.$$

So, the measure μ_{P_3} is given by

$$\mu_{P_3} = \sum_{i=1}^{3} m_i \delta_{\lambda_i}.$$

One can easily check that $s_j = \int_{\mathbb{R}} x^j d\mu_{P_3}(x) \quad \text{for} \quad 0 \leq j \leq 5.$

In what follows, we will refer to the polynomial P_{r+1} and to the measure $\mu_{P_{r+1}}$ whenever we have a sequence $(s_0, s_1, \cdots, s_{2r+1})$ such that the sequence $(s_0, s_1, \cdots, s_{2r})$ is positive definite.

2.2. Moment sequences and Jacobi operator

Let $s = (s_j)_{j=0}^{\infty}$ be an infinite sequence of real numbers. Assume that s is positive definite.

The formula

$$< P, Q >_s = L_s(PQ), P, Q \in \mathbb{R}[x],$$

defines an inner product on the vector space $\mathbb{R}[x]$.

Applying the Gram-Schmidt procedure to the basis $\{1, x, x^2, \cdots\}$, we obtain an orthonormal basis $(p_n)_{n\in\mathbb{N}}$ given by

$$p_0 = \frac{1}{\sqrt{s_0}} \quad \text{and} \quad p_n = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & \cdots & s_n \\ \vdots & \vdots & \vdots \\ s_{n-1} & \cdots & s_{2n-1} \\ 1 & \cdots & x^n \end{vmatrix},$$

where $D_{-1} = 1$, and $D_n = |H_n(s)|, n \geq 0$.

The family $(p_n)_{n\in\mathbb{N}}$ is characterized by the following iterative relation

$$\begin{cases} p_{-1} = 0 \\ xp_n = a_n p_{n+1} + b_n p_n + a_{n-1} p_{n-1}, n \geq 0, \end{cases} \tag{3}$$

with $a_{-1} = 1, a_n = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}$, and $b_n = L_s(xp_n^2), n \geq 0$.

The relation (3) links the moment problem to Jacobi operators.

Let H_s denotethe Hilbert space completion of the unitary space $(\mathbb{R}[x]; <, >_s)$, and X the multiplication operator by the variable x with domain $\mathbb{R}[x]$ on H_s. Namely, $Xp(x) = xp(x)$, $p(x) \in \mathbb{R}[x]$. Then, X is a densely defined symmetric operator with domain $\mathbb{R}[x]$ on the Hilbert space H_s, since

$$< Xp, q >_s = L_s(xpq) = L_s(pxq) =< p, Xq >_s, \quad p, q \in \mathbb{R}[x].$$

Let $\{e_n, n \in \mathbb{N}\}$ be the standard orthonormal basis of the Hilbert space $l^2(\mathbb{N})$ given by $e_n = (\delta_{k,n})_{k\in\mathbb{N}}$.

Since, $\{p_n, n \in \mathbb{N}\}$ is an orthonormal basis of H_s, there is an unitary isomorphism U from H_s to $l^2(\mathbb{N})$ defined by $Up_n = e_n$.

Then, by (3) $T = UXU^{-1}$ is a symmetric operator on $l^2(\mathbb{N})$, which acts by

$$Te_n = a_n e_{n+1} + b_n e_n + a_{n-1} e_{n-1},$$

where $e_{-1} = 0$.

The domain $D(T) = U(\mathbb{R}[x])$ is the linear span of vector e_n, that is $D(T)$ is the vector space d of finite real sequences $(\beta_0, \cdots, \beta_n, 0, 0 \cdots)$. For any finite sequence $\beta = (\beta_n) \in d$, we obtain

$$T\left(\sum_{n\in\mathbb{N}} \beta_n e_n\right) = \sum_{n\in\mathbb{N}} (a_n\beta_{n+1} + \beta_n b_n + a_{n-1}\beta_{n-1})e_n.$$

Or equivalently

$$\begin{cases} (T\beta)_0 = a_0\beta_1 + b_0\beta_0 \\ (T\beta)_n = a_n\beta_{n+1} + b_n\beta_n + a_{n-1}\beta_{n-1}, \end{cases} \tag{4}$$

where we take $\beta_{-1} = 0$.

Relations (4) mean that the operator T acts on a sequence β by multiplication with the following infinite matrix

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ \cdots & \cdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

J is called Jacobi matrix, and the corresponding operator $T = T_J$ is called a Jacobi operator.

The terms s_n can also be computed from Jacobi operator T by

$$s_n(s_0)^{-1} = (s_0)^{-1} < x^n 1, 1 >_s = < X^n p_0, p_0 >_s = < T^n e_0, e_0 >_s .$$

So, if $s_0 = 1$ the term s_k is the entry in the left upper corner of the matrix J^k.

Remark 4. Assume that $s = (s_j)_{j\geq 0}$ is a definite positive sequence, to recover the terms of the sequence $(s_0, \cdots, s_{2n+1})$ it is sufficient to calculate J_n^k, $0 \leq k \leq 2n+1$ where

$$J_n = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & 0 & 0 & 0 \\ 0 & a_1 & b_2 & a_2 & 0 & 0 \\ \cdots & \cdots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & a_{n-1} \\ \cdots & \cdots & \cdots & \cdots & a_{n-1} & b_n \end{pmatrix}.$$

J_n will be called the Jacobi matrix associated to $s = (s_i)_{0\leq i\leq 2n+1}$. The entry in the left upper corner of J_n^k will be denoted $c_{n,k}$.

Note that if L_1^k denotes the first row of J_n then $L_1^k = L_1^{k-1} \times J_n$, $1 \leq k \leq 2n+1$, with L_1^0 the first row of I_n(the unit matrix) and we have

$$c_{n,k} = L_1^{k-1} \times J_n \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

In the remainder of the paper, we will refer to the matrix J_n whenever $s^* = (s_0, s_1, \cdots, s_{2n})$ is positive definite.

In the following example we illustrate this remark.

Example 5. Let $(s_n)_{n\in\mathbb{N}}$ be the infinite sequence defined by $s_n = n!$. To recover the terms of the sequence $(1, 1, 2, 6, 24, 120)$, it suffices to calculate the power of the matrix

$$J_2 = \begin{pmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & a_1 \\ 0 & a_1 & b_2 \end{pmatrix},$$

with $a_{-1} = 1, a_n = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}$, $0 \leq n \leq 1$, and $b_n = L_s(xp_n^2)$, $0 \leq n \leq 2$.

We have

$$D_{-1} = 1, \quad D_0 = 1, \quad D_1 = 1, \quad D_2 = 4 \quad \text{and} \quad D_3 = 144,$$

$$P_0 = 1, \quad P_1 = x - 1, \quad P_2 = \frac{1}{2}x^2 - 2x + 1 \quad \text{and} \quad P_3 = \frac{1}{6}x^3 - \frac{3}{2}x^2 + 3x - 1,$$

$$a_0 = 1, \quad a_1 = 2, \quad b_0 = 1, \quad b_1 = 3, \quad \text{and} \quad b_2 = 5,$$

and

$$J_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 5 \end{pmatrix}.$$

Using (5) and by simple calculation, we get

$$c_{2,1} = 1 = s_1, \quad c_{2,2} = 2 = s_2,$$

$$c_{2,3} = 6 = s_3, \quad c_{2,4} = 24 = s_4 \quad \text{and} \quad c_{2,5} = 120 = s_5.$$

3. Main results

Let $s = (s_j)_{j=0}^{m}$ be a real finite sequence, to solve the problem (2), we start by calculating the determinants of the leading principal minors of the Hankel matrix $H_{E(\frac{m}{2})}(s)$.

Let r be the smallest integer such that $D_k > 0$ for $k \leq r$ and $D_{r+1} = 0$, then we prove that s is a Hamburger moment sequence, if and only if s is a Hamburger moment sequence for the measure $\mu_{P_{r+1}}$ (see Theorem 2). To state this result, we need the following two lemmas

Lemma 6. *Let $(s_j)_{j=0}^{j=2r+1}$ be a $(2r+1)$-real sequence such that the sequence $(s_j)_{j=0}^{j=2r}$ is positive definite. The following two assertions are equivalent*:

(i) $P_{r+1} = x^{r+1} - a_0 - a_1 x - \cdots - a_r x^r$;

(ii) $v_{r+1} = \sum_{i=0}^{i=r} a_i v_i$;

where $v_0, v_1, \cdots,$ and v_r denote the columns of $H_r(s)$, and $v_{r+1} = (s_{r+1}, s_{r+2}, \cdots, s_{2r+1})^t$.

Proof. Let $j \in \{0, 1, \cdots, r\}$. By orthogonality, we have

$$L_s(P_{r+1} x^j) = 0.$$

Hence,

$$\forall j \in \{0, 1, \cdots, r\}, \quad s_{r+j+1} - a_0 s_j - a_1 s_{j+1} - \cdots - a_r s_{j+r} = 0.$$

Thus, $v_{r+1} = \sum\limits_{i=0}^{i=r} a_i v_i$.

Conversely, let λ be a root of the polynomial P_{r+1}, we have

$$P_{r+1}(\lambda) = \frac{1}{D_r} \begin{vmatrix} s_0 & \cdots & s_r & s_{r+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_r & \cdots & s_{2r} & s_{2r+1} \\ 1 & \cdots & \lambda^r & \lambda^{r+1} \end{vmatrix}.$$

Expanding according to the last row, we obtain

$$P_{r+1}(\lambda) = \frac{1}{D_r}\left[\lambda^{r+1} D_r + \sum_{k=0}^{r} (-1)^{k+1} \lambda^{r-k} D_{r,k}\right],$$

with

$$D_{r,k} = \begin{vmatrix} s_0 & \cdots & s_{r-k-1} & s_{r-k+1} & \cdots & s_{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_r & \cdots & s_{2r-k-1} & s_{2r-k+1} & \cdots & s_{2r+1} \end{vmatrix}, \quad 0 \leq k \leq r-1,$$

and

$$D_{r,r} = \begin{vmatrix} s_1 & \cdots & s_{r+1} \\ \vdots & \vdots & \vdots \\ s_{r+1} & \cdots & s_{2r+1} \end{vmatrix}.$$

On the other hand,

$$v_{r+1} = \sum_{i=0}^{i=r} a_i v_i \Rightarrow v_{r+1} - \sum_{i=0, i \neq j}^{i=r} a_i v_i = a_j v_j.$$

Thus, replacing the last column v_{r+1} by $v_{r+1} - \sum\limits_{i=0, i \neq r-k}^{i=r} a_i v_i$, we obtain for $0 \leq k \leq r-1$,

$$D_{r,k} = \begin{vmatrix} s_0 & \cdots & s_{r-k-1} & s_{r-k+1} & \cdots & a_{r-k} s_{r-k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_r & \cdots & s_{2r-k-1} & s_{2r-k+1} & \cdots & a_{r-k} s_{2r-k} \end{vmatrix} = (-1)^k D_r a_{r-k},$$

and

$$D_{r,r} = \begin{vmatrix} s_1 & \cdots & s_r & a_0 s_0 \\ \vdots & \vdots & \vdots & \vdots \\ s_{r+1} & \cdots & s_{2r} & a_0 s_r \end{vmatrix} = (-1)^r D_r a_0.$$

Hence,

$$\begin{aligned}P_{r+1}(\lambda)=0 &\Leftrightarrow \frac{1}{D_r}\left[\lambda^{r+1}D_r+\sum_{k=0}^{r}(-1)^{k+1}\lambda^{r-k}(-1)^k a_{r-k}D_r\right]=0\\ &\Leftrightarrow \lambda^{r+1}+\sum_{k=0}^{r}(-1)^{2k+1}\lambda^{r-k}a_{r-k}=0\\ &\Leftrightarrow \lambda^{r+1}=\sum_{k=0}^{r}\lambda^{r-k}a_{r-k}\\ &\Leftrightarrow \lambda^{r+1}=\sum_{i=0}^{r}\lambda^{i}a_{i}.\end{aligned}$$

We deduce that λ is a root of P_{r+1} if and only if λ is a root of the polynomial $Q=x^{r+1}-\sum_{i=0}^{i=r}a_ix^i$.

The polynomials Q and P_{r+1} have the same degree, the same roots and are both unitary. Therefore $P_{r+1}=Q$ and the proof is ended. □

Lemma 7. *Let $n\geq 1$ and $s=(s_j)_{j=0}^{2n}$ be a finite real sequence such that*

$$D_k>0 \text{ for all } k\in\{0,\cdots,n-1\} \text{ and } D_n=0.$$

Then, s is a Hamburger moment sequence.

Proof. Using Theorem 2 and since the sequence $(s_j)_0^{2n-2}$ is positive definite, then the sequence $(s_j)_0^{2n-1}$ is a Hamburger moment sequence for the measure μ_{p_n}, where

$$P_n=\frac{1}{D_{n-1}}\begin{vmatrix} s_0 & \cdots & s_n\\ \cdots & \cdots & \cdots\\ s_{n-1} & \cdots & s_{2n-1}\\ 1 & \cdots & x^n\end{vmatrix}.$$

It suffices therefore to show that, $s_{2n}=\int_{\mathbb{R}}x^{2n}d\mu_{P_n}(x)$.

Let us put $P_n=x^n-\sum_{k=0}^{n-1}a_kx^k$. By Lemma 7, we have $v_n=\sum_{k=0}^{n-1}a_kv_k$, where $v_0,v_1,\cdots,v_{n-1}$ denote the columns of $H_{n-1}(s)$ and $v_n=(s_n,s_{n+1},\cdots,s_{2n})^t$, i.e., $s_{n+j}=\sum_{k=0}^{n-1}a_ks_{j+k},\ j\in\{0,\cdots,n-1\}$.

We have

$$D_n = 0 \Leftrightarrow \begin{vmatrix} s_0 & \cdots & s_{n-1} & s_n \\ \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & \cdots & s_{2n-2} & s_{2n-1} \\ s_n & \cdots & s_{2n-1} & s_{2n} \end{vmatrix} = 0.$$

By replacing the last column v_n by $v_n - \sum_{i=0}^{n-1} a_i v_i$, we obtain

$$D_n = 0 \Leftrightarrow \begin{vmatrix} s_0 & \cdots & s_{n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & \cdots & s_{2n-2} & 0 \\ s_n & \cdots & s_{2n-1} & s_{2n} - \sum_{i=0}^{n-1} a_i s_{n+i} \end{vmatrix} = 0.$$

Then

$$\begin{aligned} D_n = 0 &\Leftrightarrow \left(s_{2n} - \sum_{i=0}^{n-1} a_i s_{n+i} \right) D_{n-1} = 0 \\ &\Leftrightarrow s_{2n} = \sum_{i=0}^{n-1} a_i s_{n+i}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{\mathbb{R}} x^{2n} d\mu_{P_n}(x) &= \sum_{i=1}^{n} m_i \lambda_i^{2n} \\ &= \sum_{i=1}^{n} m_i \lambda_i^{n} \sum_{j=0}^{n-1} a_j \lambda_i^{j} \\ &= \sum_{i=1}^{n} \sum_{j=0}^{n-1} a_j \lambda_i^{j+n} m_i \\ &= \sum_{j=0}^{n-1} a_j \sum_{i=1}^{n} m_i \lambda_i^{j+n} \\ &= \sum_{j=0}^{n-1} a_j \int_{\mathbb{R}} x^{n+j} d\mu_{P_n}(x) \\ &= \sum_{j=0}^{n-1} a_j s_{n+j} \end{aligned}$$

$$= s_{2n}.$$

With this, the proof is completed. □

Now, we are in a position to state our first main result.

Theorem 8. *Let $s = (s_j)_{j=0}^m, m \geq 1$, be a finite sequence such that*

$$\exists r \in \mathbb{N}, \quad \forall \alpha \leq r, \quad D_\alpha > 0 \quad \text{and} \quad D_{r+1} = 0.$$

Then, s is a Hamburger moment sequence if and only if the following formulas hold

$$s_j = \int_{\mathbb{R}} x^j d\mu_{P_{r+1}}(x), \quad 0 \leq j \leq m.$$

Proof. The condition is obviously sufficient. Let us show that it is necessary.

Assume that s is a Hamburger moment sequence for a measure $\tilde{\mu}$.

The sequence $(s_j)_{j=0}^{2r}$ is positive definite. So, the sequence $(s_j)_{j=0}^{2r+1}$ is a Hamburger moment sequence for the measure $\mu_{P_{r+1}}$, i.e $\forall j \in \{0, \cdots, 2r+1\}$,

$$s_j = \int_{\mathbb{R}} x^j d\mu_{P_{r+1}}(x).$$

It remains to show that

$$\forall k \in \{2r+2, \cdots, m\}; \quad s_k = \int_{\mathbb{R}} x^k d\mu_{P_{r+1}}(x).$$

For $k = 2r+2$, applying Lemma 7, we get the result.

For $k \in \{2r+2, \cdots, m\}$, let us assume that

$$\forall l \in \{0, \cdots, k\}; \quad s_l = \int_{\mathbb{R}} x^l d\mu_{P_{r+1}}(x).$$

We have,

$$\begin{aligned}
\int_{\mathbb{R}} x_{k+1} d\mu_{P_{r+1}}(x) &= \int_{\mathbb{R}} x^{r+1} x^{k-r} d\mu_{P_{r+1}}(x) \\
&= \int_{\mathbb{R}} (\sum_{i=0}^{r} a_i x^i) x^{k-r} d\mu_{P_{r+1}}(x) \\
&= \int_{\mathbb{R}} \sum_{i=0}^{r} a_i x^{i+k-r} d\mu_{P_{r+1}}(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{r} a_i \int_{\mathbb{R}} x^{i+k-r} d\mu_{P_{r+1}}(x) \\
&= \sum_{i=0}^{r} a_i s_{i+k-r} \\
&= \sum_{i=0}^{r} a_i \int_{\mathbb{R}} x^{i+k-r} d\tilde{\mu}(x) \\
&= \int_{\mathbb{R}} \sum_{i=0}^{r} a_i x^{i+k-r} d\tilde{\mu}(x) \\
&= \int_{\mathbb{R}} (\sum_{i=0}^{r} a_i x^i) x^{k-r} d\tilde{\mu}(x) \\
&= \int_{\mathbb{R}} x^{r+1} x^{k-r} d\tilde{\mu}(x) \\
&= \int_{\mathbb{R}} x^{k+1} d\tilde{\mu}(x)
\end{aligned}
$$

So, $\int_{\mathbb{R}} x_{k+1} d\mu_{P_{r+1}}(x) = s_{k+1}$, which concludes the proof. □

To make more understandable Theorem 8, we present the following example.

Example 9. Consider the sequence

$$s = (s_j)_{0 \leq j \leq 6} = (1, 2, 5, 12, 29, 70, 160).$$

We have

$$D_0 = 1 > 0, \quad D_1 = 1 > 0, \quad D_2 = 0.$$

So, $r = 1$.

The orthogonal polynomials are

$$P_0 = 1, \quad P_1 = x - 2 \quad \text{and} \quad P_2 = x^2 - 2x - 1.$$

The roots of P_2 are

$$\lambda_1 = 1 - \sqrt{2} \quad \text{and} \quad \lambda_2 = 1 + \sqrt{2}.$$

The measure is given bay

$$\mu_{P_2} = m_1 \delta_{\lambda_1} + m_2 \delta_{\lambda_2},$$

where
$$m_1 = \left(\sum_{k=0}^{2} [P_k(\lambda_1)]^2 \right)^{-1} = \frac{2-\sqrt{2}}{4}$$
and
$$m_2 = \left(\sum_{k=0}^{2} [P_k(\lambda_2)]^2 \right)^{-1} = \frac{2+\sqrt{2}}{4}.$$

A simple calculation shows that
$$s_j = \int_{\mathbb{R}} x^j d\mu_{P_2}(x) \quad \text{for all} \quad 0 \leq j \leq 5.$$
But $\int_{\mathbb{R}} x^6 d\mu_{P_2}(x) = 169 \neq s_6 = 160$.

So, s is not a Hamburger moment sequence.

Note that for the previous sequence, we have: $D_0 > 0, D_1 > 0$, and $D_2 = D_3 = 0$, but s is not a Hamburger moment sequence. So, the determinant characterization of moment sequence with finely many mass-point, given in [4], is no longer valid when dealing with truncated sequences.

Now, we focus on establishing an algorithm to conclude about the solvability of (2).

Let $s = (s_j)_{j=0}^m$ be a finite sequence, where $m \geq 3$. Without loss of generality, we take $s_0 = 1$.

Assume that
$$\exists r \in \mathbb{N}, \quad \forall l \leq r, \quad D_l > 0 \quad \text{and} \quad D_{r+1} = 0.$$

Then, $(s_0, \cdots, s_{2r})$ is positive definite.

Let J_r be the Jacobi matrix associated with $(s_0, \cdots, s_{2r+1})$, and $c_{r,j}$ the left upper corner of J_r^k

The definition of the algorithm is based on the following theorem.

Theorem 10. *The following statements are equivalent*:

(i) s *is a Hamburger moment sequence;*

(ii) $\forall j \in \{1, \cdots, m\}, s_j = c_{r,j}$.

Proof. $i) \Rightarrow ii)$ By Theorem 8, s is a Hamburger moment sequence for the atomic measure $\mu_{P_{r+1}}$, we then have
$$\forall j \in \{1, \cdots, r+1\}, s_j = c_{r,j}, \quad \text{(see Remark 4)}.$$

Assume that $P_{r+1} = x^{r+1} - a_0 - a_1 x - \cdots - a_r x^r$ so by Lemma 6, $(a_0, a_1, \cdots, a_r)$ is a solution of the system

$$\begin{cases} x_0 + x_1 s_1 + x_2 s_2 + \cdots + x_r s_r = s_{r+1} \\ x_0 s_1 + x_1 s_2 + x_2 s_3 + \cdots + x_r s_{r+1} = s_{r+2} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_0 s_r + x_1 s_{r+1} + x_2 s_{r+2} + \cdots + x_r s_{2r} = s_{2r+1} \end{cases},$$

which is a unique solution, because the determinant of this system is $D_r > 0$.

If we suppose that

$$J_r^{r+1} = b_0 I_r + b_1 J_r + \cdots + b_r J_r^r,$$

then $(b_0, b_1, \cdots, b_r)$, is a solution for the same system.

Thus, $b_k = a_k$ for all $k \in \{0, \cdots, r\}$.

Hence, $J_r^{r+1} = a_0 I_r + a_1 J_r + \cdots + a_r J_r^r$ and we e deduce that

$$J_r^{2r+2} = a_0 J_r^{r+1} + a_1 J_r^{r+2} + \cdots + a_r J_r^{2r+1}.$$

So,

$$\begin{aligned} c_{r,2r+2} &= a_0 c_{r,r+1} + a_1 c_{r,r+2} + \cdots + a_r c_{r,2r+1} \\ &= a_0 s_{r+1} + a_1 s_{r+2} + \cdots + a_r s_{2r+1} \\ &= s_{2r+2}, \end{aligned}$$

and by induction we conclude that $c_{r,k} = s_k$ for all $2r + 2 \leq k \leq m$.

$ii) \Rightarrow i)$ The sequence $(s_j)_0^{2r+1}$, is a Hamburger moment sequence for the measure $\mu_{P_{r+1}}$, and we have $J_r^{r+1} = a_0 I_r + a_1 J_r + \cdots + a_r J_r^r$.

So,

$$\begin{aligned} s_{2r+2} &= c_{r,2r+2} \\ &= a_0 c_{r,r+1} + a_1 c_{r,r+2} + \cdots + a_r c_{r,2r+1} \\ &= a_0 s_{r+1} + a_1 s_{r+2} + \cdots + a_r s_{2r+1} \\ &= a_0 \int_{\mathbb{R}} x^{r+1} d\mu_{P_{r+1}}(x) + \cdots + a_r \int_{\mathbb{R}} x^{2r+1} d\mu_{P_{r+1}}(x) \\ &= \int_{\mathbb{R}} x^{r+1} (a_0 + a_1 x + \cdots + a_r x^r) d\mu_{P_{r+1}}(x) \\ &= \int_{\mathbb{R}} x^{r+1} x^{r+1} d\mu_{P_{r+1}}(x) \end{aligned}$$

$$= \int_{\mathbb{R}} x^{2r+2} d\mu_{P_{r+1}}(x).$$

And by induction, we proof that for all $j \in \{2r+2, \cdots, m\}$, $s_j = \int_{\mathbb{R}} x^j d\mu_{P_{r+1}}(x)$. □

The previous theorem provides a simple and practical algorithm which allows to deduce whether a finite sequence is a Hamburger moment sequence or not.

Given a real sequence $s = (s_0, s_1, \cdots, s_m)$, we calculate successively the principal minors, $D_0, D_1, \cdots$. There are three possibilities:

1) All the determinants $D_0, \cdots, D_{E(\frac{m}{2})}$ are positive, in this case, s is a Hamburger moment sequence because it can be extended to an infinite positive definite sequence.

2) If one of the determinants is negative then s is not a Hamburger moment sequence.

3) $D_0 > 0, D_1 > 0, \cdots, D_r > 0, D_{r+1} = 0$ where $r \leq E(m/2) - 1$. In this case, we construct the Jacobi matrix

$$J_r = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & 0 & 0 & 0 \\ 0 & a_1 & b_2 & a_2 & 0 & 0 \\ \cdots & \cdots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \ddots & \ddots & a_{r-1} \\ \cdots & \cdots & \cdots & \cdots & a_{r-1} & b_r \end{pmatrix}.$$

Then, we compare $c_{r,k}$ and s_k where $c_{r,k}$ is the entry in the left upper corner of the matrix J_r^k. Let us note that there is no need to calculate J_r^k. In fact if L_1^k is the first row of J_r^k, then $L_1^k = L_1^{k-1} \times J_2$ and $c_{r,k} = L_1^{k-1} \times J_r \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

If $c_{r,k} = s_k$ for all $k \in \{1, \cdots, m\}$ then s is a Hamburger moment sequence, otherwise s is not a Hamburger moment sequence.

To end this paper, we propose an example to illustrate the efficiency of this algorithm.

Example 11. Consider the following finite sequence

$$s = \left(1, 1, 2, \frac{13}{4}, \frac{47}{8}, \frac{159}{16}, \frac{561}{32}, \frac{1927}{64}, \frac{6733}{128}, \frac{23271}{256}\right).$$

We have

$$D_0 = s_0 = 1, \quad D_1 = 1, \quad D_2 = 5/16, \quad \text{and} \quad D_3 = 0.$$

The orthogonal polynomials associated to the positive definite sequence $s^* = \left(1, 1, 2, \frac{13}{4}, \frac{47}{8}\right)$, are

$$P_0 = 1, \quad P_1 = x - 1, \quad \text{and} \quad P_2 = \frac{4\sqrt{5}}{5}(x^2 - \frac{5}{4}x - \frac{3}{4}),$$

and

$$a_0 = 1, \quad a_1 = \frac{\sqrt{5}}{4}, \quad b_0 = 1, \quad b_1 = \frac{1}{4}, \quad \text{and} \quad b_2 = \frac{-3}{4}.$$

The Jacobi matrix is

$$J_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1/4 & \sqrt{5}/4 \\ 0 & \sqrt{5}/4 & -3/4 \end{pmatrix}.$$

Now, we compare $c_{2,k}$ and s_k, for $1 \leq k \leq 9$.
$c_{2,1} = 1 = s_1$,

$l_1^1 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$, then $c_{2,2} = l_1^1 \times J_2 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2 = s_2$,

$l_1^2 = \begin{pmatrix} 2 & \frac{5}{4} & \frac{\sqrt{5}}{4} \end{pmatrix}$, then $c_{2,3} = l_1^2 \times J_2 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{13}{4} = s_3$,

$l_1^3 = \begin{pmatrix} \frac{13}{4} & \frac{21}{8} & \frac{\sqrt{5}}{8} \end{pmatrix}$, then $c_{2,4} = l_1^3 \times J_2 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{47}{8} = s_4$,

$l_1^4 = \begin{pmatrix} \frac{47}{8} & \frac{130}{32} & \frac{9\sqrt{5}}{16} \end{pmatrix}$, then $c_{2,5} = l_1^4 \times J_2 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{159}{16} = s_5$,

$l_1^5 = \begin{pmatrix} \frac{159}{16} & \frac{243}{32} & \frac{19\sqrt{5}}{32} \end{pmatrix}$, then $c_{2,6} = l_1^5 \times J_2 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{561}{32} = s_6$,

$$l_1^6 = \begin{pmatrix} \frac{561}{32} & \frac{805}{64} & \frac{93\sqrt{5}}{64} \end{pmatrix}, \text{ then } c_{2,7} = l_1^6 \times J_2 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1927}{64} = s_7,$$

$$l_1^7 = \begin{pmatrix} \frac{1927}{64} & \frac{2879}{128} & \frac{263\sqrt{5}}{128} \end{pmatrix}, \text{ then } c_{2,8} = l_1^7 \times J_2 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{6733}{128} = s_8,$$

$$l_1^8 = \begin{pmatrix} \frac{6733}{128} & \frac{9805}{256} & \frac{1045\sqrt{5}}{256} \end{pmatrix}, \text{ then } c_{2,9} = l_1^8 \times J_2 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{23271}{256} = s_9.$$

Thus, s is a Hamburger moment sequence.

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