

**A MODIFIED ACCELERATED OVERRELAXATION (MAOR)
SCHEME FOR SOLVING THE SHIFTED GRÜN WALD
ESTIMATION FRACTIONAL POISSON EQUATION**

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Abstract: This research studies the Modified Accelerated Overrelaxation (MAOR) scheme on fractional Poisson equation. The equation is discretized using the fractional finite difference method with the shifted Grünwald estimate. The superiority of this scheme is shown through comparison with previous well known relaxation schemes. This research also presents the convergence analysis for this scheme. Then the numerical result is used to compare and discuss all the schemes.

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1. Introduction

Fractional calculus is very important in mathematical modeling of various phenomena in quantum mechanics, engineering, hydrology, viscoelasticity, control systems, bioscience, and other sciences [2, 6, 7, 11, 18, 21, 22, 23, 24, 32, 42]. The fractional derivative possesses memory and nonlocal properties, simultaneously, which assists in accurately and efficiently describing different nonlinear phenomena in comparison to the derivative of integer-order [19]. This makes the fractional derivative a powerful tool when modeling complex dynamical systems.

In the past few years, there have been various approximations or numerical techniques developed to solve fractional equations. Research in [10] utilized a group iterative method to solve the two-dimensional time fractional advection-diffusion equation. Furthermore, some time-fractional diffusion equations [4, 9, 43] and space-fractional diffusion equations [33, 34, 35] have sometimes been solved using various sweeping iterative methods. Another equation includes the two-dimensional time-fractional telegraph equation which has been solved on the rotated iterative method [5] and by utilizing the explicit group iterative method [3].

These equations are discretized using suitable finite difference approximations that generate a system of linear equations that is large and sparse in the form of $Au = b$, where there is a sparse matrix A with known non-singular values, a column vector b that is constant, and a column vector u with the unknown values. When solving a system of linear equations, it is more suitable to apply iterative methods due to the sparsity of the matrix A .

Additionally, there also have been many relaxation schemes developed which increase the convergence rate of the iterative methods. These begin with the Gauss Seidel (GS) benchmark scheme. The Successive Overrelaxation (SOR) scheme was then developed [15, 40], where a weighted parameter ω is added to speed up the convergence rate [26, 37]. Moreover, researchers in [20] invented the Modified SOR (MSOR) scheme, which combines the concept of red-black ordering with the SOR scheme, hence the weighted parameters ω_r and ω_b are added. The MSOR scheme has a faster convergence rate compared to the SOR scheme, as shown in [13]. In addition, Hadjidimos [16] developed an

Accelerated Overrelaxation (AOR) scheme that generalizes the SOR scheme using two weighted parameters, which are θ and ω . Some research has validated the superiority of the AOR scheme in terms of convergence rate when compared to the SOR scheme [8, 36, 38]. Thus, this research aims to experiment on a relaxation scheme that has even more weighted parameters and validate its superiority when compared to the MSOR and AOR schemes.

2. Derivation of fractional Poisson equation

In this research, the finite difference methods together with various relaxation schemes are examined to solve the following fractional Poisson equation:

$$\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial y^\alpha} = f(x, y). \quad (1)$$

The parameter α is the fractional order of the spatial derivative, where it is considered to be $1 \leq \alpha \leq 2$, and the $f(x, y)$ function in Equation (1) is of real values such that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Riemann-Liouville fractional derivative

The Riemann-Liouville fractional derivative of order α is defined as

$$D_x^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_0^x (x - t)^{n - \alpha - 1} f(t) dt \quad (2)$$

such that the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \rightarrow f(x)$ has continuous derivatives of the integer-order n where $n - 1 \leq \alpha \leq n$ [1, 27, 30] and the order is $\alpha \geq 0$.

Shifted Grünwald estimate

The shifted Grünwald formula for the case of $1 \leq \alpha \leq 2$ is defined as

$$\frac{d^\alpha f}{dx^\alpha} = \lim_{M \rightarrow \infty} \frac{1}{h^\alpha} \sum_{k=0}^M g_k \cdot f(x - (k - 1)h)$$

which further defines the shifted Grünwald estimate to the fractional derivative as follows:

$$\frac{d^\alpha f}{dx^\alpha} = \frac{1}{h^\alpha} \sum_{k=0}^M g_k \cdot f(x - (k - 1)h) + O(h^\alpha) \quad (3)$$

given that $h = \frac{1}{M}$, M is a positive integer and g_k is the normalized Grünwald weight defined as

$$g_k = (-1)^k \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - k + 1)\Gamma(k + 1)} \quad \text{for } k = 0, 1, 2, \dots,$$

where Γ is the Gamma function. The normalized weights are dependent on the index k and the order α [25, 30, 31].

Consider the fractional Poisson equation (1) on the unit square $\Omega = \{(x, y) | (x, y) \in [0, 1] \times [0, 1]\}$ with Dirichlet boundary conditions. The uniform Cartesian grid consisting of grid points (x_i, y_j) where $x_i = ih$, $y_j = jh$ for $i, j = 1, 2, \dots, (n - 1)$ and denote $u_{i,j}$ as the approximation to $u(x_i, y_j)$. In order to discretize Equation (1), both the x -derivatives and y -derivatives are replaced with shifted Grünwald finite differences, which gives

$$\frac{1}{h^\alpha} \sum_{s=0}^{i+1} g_s \cdot u_{i-s+1,j} + \frac{1}{h^\alpha} \sum_{z=0}^{j+1} g_z \cdot u_{i,j-z+1} = f_{i,j} \quad (4)$$

and can be rearranged to

$$u_{i,j} = -\frac{1}{2g_1} \left(g_0 u_{i+1,j} + g_0 u_{i,j+1} + \sum_{s=2}^{i+1} g_s \cdot u_{i-s+1,j} + \sum_{z=2}^{j+1} g_z \cdot u_{i,j-z+1} - h^\alpha f_{i,j} \right). \quad (5)$$

The approximation of Equation (5) will form the following linear system of equations

$$Au = b. \quad (6)$$

Matrix A is then decomposed into

$$A = D - U - L, \quad (7)$$

where there are the diagonal matrix D , the upper triangular matrix U and the lower triangular matrix L . By substituting Equation (7) into Equation (6) and with some manipulation, the general formulae for some well-known relaxation schemes are developed.

3. Fractional finite difference method with relaxation schemes

As mentioned in Introduction, various relaxation schemes have been developed throughout the years that increase the convergence rate of iterative methods. Figures 1 and 2 will assist in explaining these schemes.

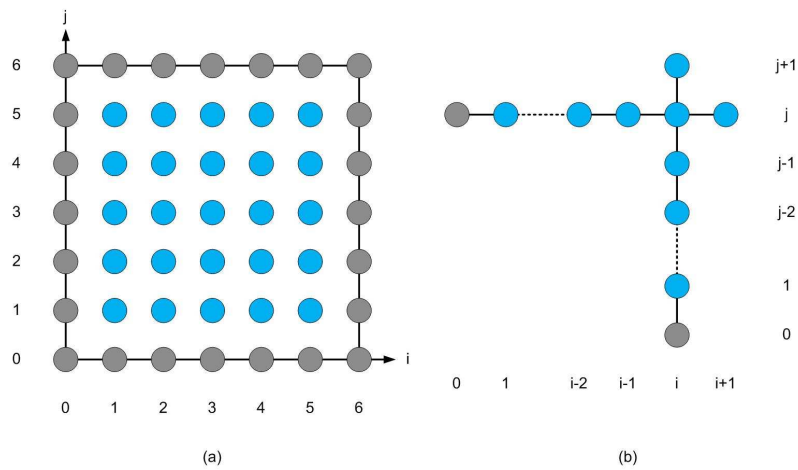


Figure 1: (a) The natural ordering system of the solution domain for $n = 6$; (b) The fractional stencil for every $\bullet$ point in the solution domain.

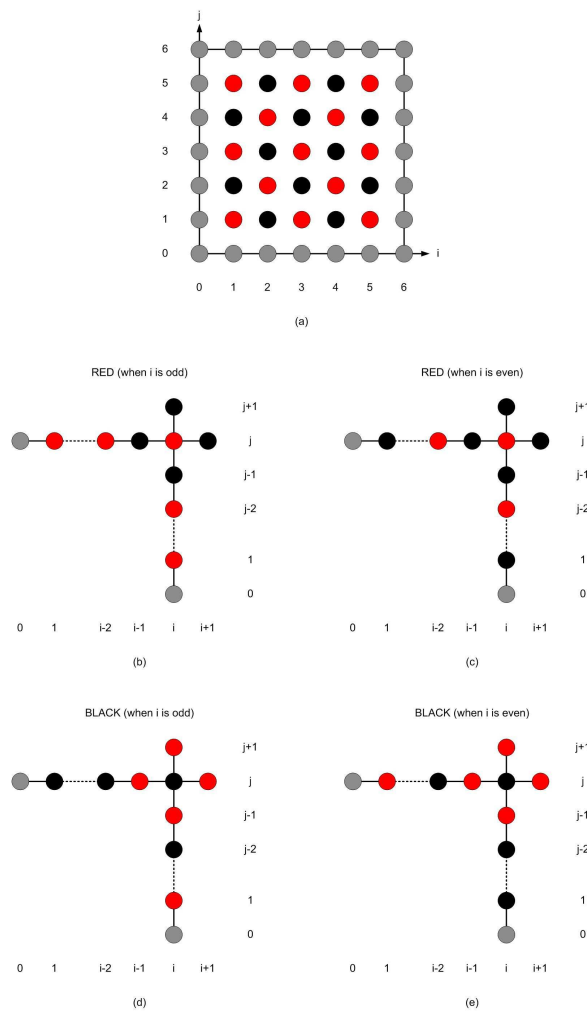


Figure 2: (a) The red-black ordering system of the solution domain for $n = 6$; The fractional stencil for every $\bullet$ point of the solution domain (b) when i is odd and (c) when i is even; The fractional stencil for every $\bullet$ point of the solution domain (d) when i is odd and (e) when i is even.

3.1. Gauss-Seidel (GS)

The general formula of the GS scheme is defined as

$$u^{k+1} = (D - L)^{-1} [Uu^k + b] \quad (8)$$

and by referencing the fractional stencil in Figure 1 (b), will yield

$$u_{i,j}^{k+1} = -\frac{1}{2g_1} \left(g_0 u_{i+1,j}^k + g_0 u_{i,j+1}^k + \sum_{s=2}^{i+1} g_s \cdot u_{i-s+1,j}^{k+1} + \sum_{z=2}^{j+1} g_z \cdot u_{i,j-z+1}^{k+1} - h^\alpha f_{i,j} \right). \quad (9)$$

3.2. Successive Overrelaxation (SOR)

The SOR scheme is an extension of the GS scheme, where it includes ω as a weighted parameter, as shown below:

$$u^{k+1} = (D - \omega L)^{-1} [((1 - \omega)D + \omega U) u^k + \omega b] \quad (10)$$

and by also referencing the fractional stencil in Figure 1 (b), will yield

$$u_{i,j}^{k+1} = -\frac{\omega}{2g_1} \left(g_0 u_{i+1,j}^k + g_0 u_{i,j+1}^k + \sum_{s=2}^{i+1} g_s \cdot u_{i-s+1,j}^{k+1} + \sum_{z=2}^{j+1} g_z \cdot u_{i,j-z+1}^{k+1} - h^\alpha f_{i,j} \right) + (1 - \omega) u_{i,j}^k. \quad (11)$$

3.3. Modified SOR (MSOR)

The MSOR scheme combines the concept of red-black ordering into the SOR scheme, which adds the weighted parameters ω_r and ω_b . Hence, the general formula for the MSOR scheme is as follows:

$$u^{k+1} = (D - \omega_{r,b} L)^{-1} [((1 - \omega_{r,b})D + \omega_{r,b} U) u^k + \omega_{r,b} b] \quad (12)$$

and by referencing the fractional stencils in Figures 2 (b) and (c) for every ● point and Figures 2 (d) and (e) for every ● point, will yield

$$\bullet \quad u_{i,j}^{k+1} = -\frac{\omega_r}{2g_1} \left(g_0 u_{i+1,j}^k + g_0 u_{i,j+1}^k + \sum_{s=2}^{i+1} g_s \cdot u_{i-s+1,j}^{k+1} + \sum_{z=2}^{j+1} g_z \cdot u_{i,j-z+1}^{k+1} - h^\alpha f_{i,j} \right) + (1 - \omega_r) u_{i,j}^k, \quad (13)$$

$$\bullet \quad u_{i,j}^{k+1} = -\frac{\omega_b}{2g_1} \left(g_0 u_{i+1,j}^{k+1} + g_0 u_{i,j+1}^{k+1} + \sum_{s=2}^{i+1} g_s \cdot u_{i-s+1,j}^{k+1} + \sum_{z=2}^{j+1} g_z \cdot u_{i,j-z+1}^{k+1} - h^\alpha f_{i,j} \right) + (1 - \omega_b) u_{i,j}^k. \quad (14)$$

3.4. Accelerated Overrelaxation (AOR)

This scheme was developed as a generalization of the SOR scheme using two weighted parameters, which are θ and ω . Hence, the general formula for the AOR scheme is as follows:

$$u^{k+1} = (D - \theta L)^{-1} \left[[(1 - \omega) D + (\omega - \theta) L + \omega U] u^k + \omega b \right] \quad (15)$$

and by referencing the fractional stencil in Figure 1 (b), will yield

$$\begin{aligned} u_{i,j}^{k+1} = & -\frac{\theta}{2g_1} \left(\sum_{s=2}^{i+1} g_s \cdot (u_{i-s+1,j}^{k+1} - u_{i-s+1,j}^k) \right. \\ & \left. + \sum_{z=2}^{j+1} g_z \cdot (u_{i,j-z+1}^{k+1} - u_{i,j-z+1}^k) \right) \\ & - \frac{\omega}{2g_1} \left(g_0 u_{i+1,j}^k + g_0 u_{i,j+1}^k + \sum_{s=2}^{i+1} g_s \cdot u_{i-s+1,j}^k \right. \\ & \left. + \sum_{z=2}^{j+1} g_z \cdot u_{i,j-z+1}^k - h^\alpha f_{i,j} \right) + (1 - \omega) u_{i,j}^k. \end{aligned} \quad (16)$$

The previous researches mentioned in the introduction found that the SOR scheme converges faster than the GS scheme. Correspondingly, both the MSOR

and AOR schemes converge faster than the SOR scheme. However, no comparison has been made between both schemes. In the following section, we will discuss a relaxation scheme that converges even faster than the previously mentioned schemes.

4. Fractional finite difference method with the MAOR scheme

Hadjidimos et al. [17] introduced a theory of the MAOR scheme which is a generalization of the AOR scheme. Their research stated that the MAOR scheme minimizes the extrapolation of previous schemes with distinctive parameters that correspond to the row blocks of matrices for determined choice of the acceleration and relaxation matrices. Additionally, the convergence analysis for two-cyclic matrices was also provided. The concept of two-cyclic matrices is similar to the idea of the red-black ordering system, as shown in Figure 2 (a), where both θ and ω are defined as

$$\theta = \begin{bmatrix} \theta_r I_r & 0 \\ 0 & \theta_b I_b \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_r I_r & 0 \\ 0 & \omega_b I_b \end{bmatrix}$$

with the weighted parameters on the ● points denoted as θ_r and ω_r , the weighted parameters on the ● points denoted as θ_b and ω_b , and the identity matrices I_r and I_b . Hence, the general formula for the MAOR scheme is defined as

$$u^{k+1} = (D - \theta_{r,b} L)^{-1} [(1 - \omega_{r,b}) D + (\omega_{r,b} - \theta_{r,b}) L + \omega_{r,b} U] u^k + \omega_{r,b} b \quad (17)$$

Research in [14, 39] utilized the MAOR scheme on their respective problems, and both concluded that this scheme produces the least number of iterations. Thus, this research proposes an MAOR scheme based on the fractional finite difference iterative method for solving the fractional Poisson equation (1). Since this scheme applies the red-black ordering system, the updated grid point on iteration $k + 1$ for the two stencils can be shown as in Figure 3.

By applying Equation (17) on Equation (6) with the stencils in Figure 3,

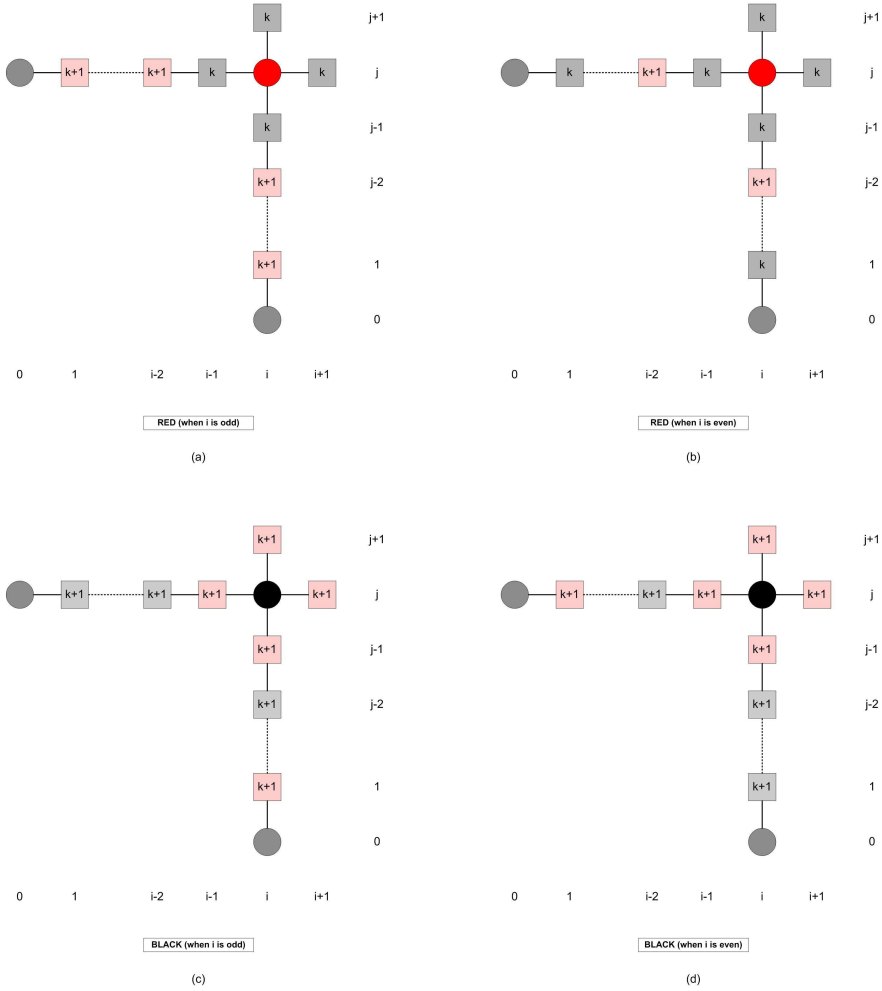


Figure 3: The stencils in variable k for (a) ● points when i is odd; (b) ● points when i is even; (c) ● points when i is odd and (d) ● points when i is even.

the following equations are developed:

$$\begin{aligned}
 \bullet \quad u_{i,j}^{k+1} = & -\frac{\theta_r}{2g_1} \left(\sum_{s=2}^{i+1} g_s \cdot (u_{i-s+1,j}^{k+1} - u_{i-s+1,j}^k) \right. \\
 & \left. + \sum_{z=2}^{j+1} g_z \cdot (u_{i,j-z+1}^{k+1} - u_{i,j-z+1}^k) \right) \\
 & - \frac{\omega_r}{2g_1} \left(g_0 u_{i+1,j}^k + g_0 u_{i,j+1}^k + \sum_{s=2}^{i+1} g_s \cdot u_{i-s+1,j}^k \right. \\
 & \left. + \sum_{z=2}^{j+1} g_z \cdot u_{i,j-z+1}^k - h^\alpha f_{i,j} \right) + (1 - \omega_r) u_{i,j}^k,
 \end{aligned} \tag{18}$$

$$\begin{aligned}
\bullet \quad u_{i,j}^{k+1} = & -\frac{\theta_b}{2g_1} \left(g_0(u_{i+1,j}^{k+1} - u_{i+1,j}^k) + g_0(u_{i,j+1}^{k+1} - u_{i,j+1}^k) \right. \\
& + \sum_{s=2}^{i+1} g_s \cdot (u_{i-s+1,j}^{k+1} - u_{i-s+1,j}^k) \\
& \left. + \sum_{z=2}^{j+1} g_z \cdot (u_{i,j-z+1}^{k+1} - u_{i,j-z+1}^k) \right) \\
& - \frac{\omega_b}{2g_1} \left(g_0 u_{i+1,j}^k + g_0 u_{i,j+1}^k + \sum_{s=2}^{i+1} g_s \cdot u_{i-s+1,j}^k \right. \\
& \left. + \sum_{z=2}^{j+1} g_z \cdot u_{i,j-z+1}^k - h^\alpha f_{i,j} \right) + (1 - \omega_b) u_{i,j}^k.
\end{aligned} \tag{19}$$

Every $\bullet$ and $\bullet$ point in the solution domain is arranged as displayed in Figure 2 (a). Using either Equations (18) or (19) will continuously generate the iteration on each point until it successfully achieves the convergence criteria. Algorithm 1 summarizes the iterative process for the MAOR scheme.

Algorithm 1: The iterative process for the MAOR scheme.

All points in the domain $\mathbb{R}^2$ are defined into $\bullet$ and $\bullet$ as shown in Figure 2 (a);
The initial values for all matrices and relaxation parameters $\omega_r, \omega_b, \theta_r$ and θ_b are set;
Set error tolerance ε ;
while $|u^{k+1} - u^k| > \varepsilon$ **do**
 | Compute all $\bullet$ points using Equation (18);
 | Compute all $\bullet$ points using Equation (19);
end

5. Convergence analysis

Based on the analysis done by Hadjidimos et al. [17], which was on second order finite difference, the functional relationship that connects the sets of eigenvalues for the fractional finite difference becomes

$$(\lambda + \omega_r - 1)(\lambda + \omega_b - 1) = (\omega_r - \theta_r + \theta_r \lambda)(\omega_b - \theta_b + \theta_b \lambda) \mu^2, \tag{20}$$

where $\lambda \in \sigma(\mathcal{L}_{\theta,\omega})$ with $\mathcal{L}_{\theta,\omega} := (I - \theta L)^{-1} [1 - \omega_r + (\omega_r - \theta_r)L + \omega_r U]$ and $\mu \in \sigma(A)$. Dropping the indices and simplifying the notation in Equation (20) yields

$$\lambda^2 - b\lambda + c = 0, \quad (21)$$

where

$$\begin{aligned} b &:= \frac{2 - \omega_1 - \omega_2 + (\omega_1 r_2 + \omega_2 r_1 - 2r_1 r_2) \mu^2}{1 - r_1 r_2 \mu^2}, \\ c &:= \frac{(\omega_1 - 1)(\omega_2 - 1) + (\omega_1 - r_1)(\omega_2 - r_2) \mu^2}{1 - r_1 r_2 \mu^2} \end{aligned} \quad (22)$$

and $\omega_1 := \omega_r$, $\omega_2 := \omega_b$, $r_1 := \theta_r$, $r_2 := \theta_b$. It is assumed that $\omega_1, \omega_2, r_1, r_2 \in \mathbb{R}$, $\omega_1 \omega_2 \neq 0$, $r_1 r_2 \neq 0$, and $\sigma(A) \in \mathbb{R}$. Let $\sigma(A) \subset [-\bar{\mu}, \bar{\mu}]$, $\bar{\mu} = \rho(A)$, and $\sigma(A^2) \subset [\underline{\mu}^2, \bar{\mu}^2] =: M$ where $0 \leq \underline{\mu}^2 \leq \bar{\mu}^2$. Hence, we will now prove Theorem 4 of [17].

Proof. The MAOR scheme converges if and only if for all $\mu^2 \in M$ the roots of Equation (21) are less than modulus one. With Lemma 2.1 of [41], this holds if and only if

$$|c| < 1, \quad |b| < 1 + c, \quad \text{for all } \mu^2 \in M. \quad (23)$$

In view of Equation (22), Equation (23) is equivalent to

$$\begin{aligned} \omega_1 \omega_2 (1 - \mu^2) &> 0, \\ -2 &< \omega_1 \omega_2 (1 - \mu^2) - \omega_1 - \omega_2 + (\omega_1 r_2 + \omega_2 r_1 - 3r_1 r_2) \mu^2 < 0, \\ 4 + \omega_1 \omega_2 (1 - \mu^2) - 2\omega_1 - 2\omega_2 + 2(\omega_1 r_2 + \omega_2 r_1 - 3r_1 r_2) \mu^2 &> 0, \\ \text{for all } \mu^2 \in M, \end{aligned}$$

or to

$$\begin{aligned} 0 &< \omega_1 \omega_2 (1 - \mu^2) < 4, \\ \omega_1 + \omega_2 - 2 - \frac{1}{2} \omega_1 \omega_2 (1 - \mu^2) &< (\omega_1 r_2 + \omega_2 r_1 - r_1 r_2) \mu^2 \\ &< \omega_1 + \omega_2 - \omega_1 \omega_2 (1 - \mu^2) \end{aligned} \quad (24)$$

for all $\mu^2 \in M$.

Thus, for convergence to exist, it must either be i) $\bar{\mu} < 1$ or ii) $\underline{\mu} > 1$.

Case i: Let $0 \leq \underline{\mu} \leq \bar{\mu} < 1$. By assuming that $\underline{\mu} > 0$, then Equation (24) is equivalent to either

$$\omega_2 > 0, \quad 0 < \omega_1 < \frac{4}{\omega_2(1 - \underline{\mu}^2)}, \quad (25)$$

$$\max_{\mu^2 \in M} B(\mu^2) < r_1, r_2 < \min_{\mu^2 \in M} C(\mu^2),$$

or

$$\begin{aligned} \omega_2 < 0, \quad \frac{4}{\omega_2(1 - \underline{\mu}^2)} < \omega_1 < 0, \\ \max_{\mu^2 \in M} C(\mu^2) < r_1, r_2 < \min_{\mu^2 \in M} B(\mu^2), \end{aligned} \quad (26)$$

where

$$\begin{aligned} B(\mu^2) &:= \frac{1}{(\omega_1 + \omega_2)\mu^2} \left(\omega_1 + \omega_2 - 2 - \frac{1}{2}\omega_1\omega_2(1 - \mu^2) \right) \\ C(\mu^2) &:= \frac{1}{(\omega_1 + \omega_2)\mu^2} (\omega_1 + \omega_2 - \omega_1\omega_2(1 - \mu^2)). \end{aligned} \quad (27)$$

It can be found that

$$\frac{\partial B(\mu^2)}{\partial \mu^2} = \frac{(2 - \omega_1)(2 - \omega_2)}{2(\omega_1 + \omega_2)\mu^4}, \quad \frac{\partial C(\mu^2)}{\partial \mu^2} = \frac{\omega_1\omega_2 - \omega_1 - \omega_2}{(\omega_1 + \omega_2)\mu^4}. \quad (28)$$

The extreme values for B and C in Equation (25) and Equation (26) are determined by the signs of the expressions in Equation (28). For these signs, not only the intervals for ω_1 and ω_2 defined in Equation (25) and Equation (26) must be taken into account, but also the relative positions of ω_2 with respect to (wrt) 2, that of ω_1 wrt 2 and $\frac{\omega_2}{\omega_2 - 1}$ and therefore the relative position of ω_2 wrt 1 and that of $\frac{\omega_2}{\omega_2 - 1}$ wrt 2. By considering all possible subdomains in the (ω_1, ω_2) -plane, the behavior of $B(\mu^2)$ and $C(\mu^2)$ can be studied and consequently define the ranges for r_1 and r_2 . The result of this analysis can be referred to in Table 1. From there, the domain of convergence can be defined by

$$0 < \omega_1 < 2, \quad 0 < \omega_2 < 2, \quad B(\mu^2) < r_1, r_2 < C(\mu^2). \quad (29)$$

Table 1: $0 \leq \underline{\mu} \leq \bar{\mu} < 1$. (Increasing (I), Decreasing (D)).

Case	Range of ω_2	Sub-case	Range of ω_1	Behavior of $B(\mu^2)$	Behavior of $C(\mu^2)$	Range of r_1	Range of r_2
1	$0 < \omega_2 \leq \frac{2}{1-\underline{\mu}}$	(i)	$0 < \omega_1 \leq \frac{2}{4}$	I	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(ii)	$2 \leq \omega_1 < \frac{\omega_2(1-\underline{\mu}^2)}{\omega_2(1-\underline{\mu}^2)}$	D	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
2	$\frac{2}{1-\underline{\mu}} \leq \omega_2 \leq 2$	(i)	$0 < \omega_1 \leq \frac{2}{4}$	I	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(ii)	$2 \leq \omega_1 \leq \frac{\omega_2}{\omega_2-1}$	D	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(iii)	$\frac{\omega_2}{\omega_2-1} \leq \omega_1 < \frac{\omega_2}{\omega_2(1-\underline{\mu}^2)}$	D	I	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
3	$2 \leq \omega_2 \leq \frac{2}{1-\underline{\mu}^2}$	(i)	$0 < \omega_1 \leq \frac{\omega_2}{\omega_2-1}$	D	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(ii)	$\frac{\omega_2}{\omega_2-1} \leq \omega_1 \leq 2$	D	I	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(iii)	$2 \leq \omega_1 < \frac{4}{\omega_2(1-\underline{\mu}^2)}$	I	I	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
4	$\frac{2}{1-\underline{\mu}^2} \leq \omega_2 \leq \frac{2}{1-\underline{\mu}}$	(i)	$0 < \omega_1 \leq \frac{\omega_2}{\omega_2-1}$	D	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(ii)	$\frac{\omega_2}{\omega_2-1} \leq \omega_1 < \frac{4}{\omega_2(1-\underline{\mu}^2)}$	D	I	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
5	$\frac{2}{1-\underline{\mu}} \leq \omega_2 < \infty$		$0 < \omega_1 < \frac{4}{\omega_2(1-\underline{\mu}^2)}$	D	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
6	$-\infty < \omega_2 < 0$		$\frac{4}{\omega_2(1-\underline{\mu}^2)} < \omega_1 < 0$	D	D	$C(\mu^2) < r_1 < B(\mu^2)$	$C(\mu^2) < r_2 < B(\mu^2)$

Table 2: $1 \leq \underline{\mu} \leq \bar{\mu}$. (Increasing (I), Decreasing (D)).

Case	Range of ω_2	Sub-case	Range of ω_1	Behavior of $B(\mu^2)$	Behavior of $C(\mu^2)$	Range of r_1	Range of r_2
1	$0 < \omega_2 \leq \frac{2}{1+\bar{\mu}}$	(i)	$\frac{\omega_2}{4} \leq \omega_1 < 0$	D	I	$C(\mu^2) < r_1 < B(\mu^2)$	$C(\mu^2) < r_2 < B(\mu^2)$
		(ii)	$\frac{\omega_2}{\omega_2(1-\bar{\mu}^2)} < \omega_1 \leq \frac{\omega_2}{\omega_2-1}$	D	D	$C(\mu^2) < r_1 < B(\mu^2)$	$C(\mu^2) < r_2 < B(\mu^2)$
2	$\frac{2}{1+\bar{\mu}} \leq \omega_2 \leq 2$		$\frac{\omega_2(1-\bar{\mu}^2)}{4} < \omega_1 < 0$	D	I	$C(\mu^2) < r_1 < B(\mu^2)$	$C(\mu^2) < r_2 < B(\mu^2)$
3	$2 \leq \omega_2 < \infty$		$\frac{\omega_2(1-\bar{\mu}^2)}{\omega_2-1} < \omega_1 < 0$	I	I	$C(\mu^2) < r_1 < B(\mu^2)$	$C(\mu^2) < r_2 < B(\mu^2)$
4	$\frac{2}{1-\bar{\mu}^2} \leq \omega_2 < 0$	(i)	$0 < \omega_1 \leq \frac{\omega_2}{\omega_2-1}$	I	I	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(ii)	$\frac{\omega_2}{\omega_2-1} \leq \omega_1 < 2$	I	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(iii)	$2 \leq \omega_1 < \frac{\omega_2}{\omega_2(1-\bar{\mu}^2)}$	D	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
5	$\frac{2}{1-\bar{\mu}} \leq \omega_2 < \frac{2}{1-\bar{\mu}^2}$	(i)	$0 < \omega_1 \leq \frac{\omega_2}{\omega_2-1}$	I	I	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
		(ii)	$\frac{\omega_2}{\omega_2-1} \leq \omega_1 < \frac{\omega_2(1-\bar{\mu}^2)}{4}$	I	D	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$
6	$-\infty < \omega_2 \leq \frac{2}{1-\bar{\mu}}$		$0 < \omega_1 < \frac{\omega_2(1-\bar{\mu}^2)}{\omega_2-1}$	I	I	$B(\mu^2) < r_1 < C(\mu^2)$	$B(\mu^2) < r_2 < C(\mu^2)$

Case ii: Let $1 \leq \underline{\mu} \leq \overline{\mu}$. Then Equation (24) is equivalent to either

$$\omega_2 > 0, \quad \frac{4}{\omega_2(1 - \underline{\mu}^2)} < \omega_1 < 0, \quad (30)$$

$$\max_{\mu^2 \in M} B(\mu^2) < r_1, r_2 < \min_{\mu^2 \in M} C(\mu^2),$$

or

$$\omega_2 < 0, \quad 0 < \omega_1 < \frac{4}{\omega_2(1 - \underline{\mu}^2)}, \quad (31)$$

$$\max_{\mu^2 \in M} C(\mu^2) < r_1, r_2 < \min_{\mu^2 \in M} B(\mu^2),$$

where $B(\mu^2)$ and $C(\mu^2)$ are given in Equation (27). Following an analysis similar to the one in **Case i**, its result can be referred to in Table 2. The derivation of both tables and the result of Equation (29) complete the proof. $\square$

6. Numerical results and discussions

A numerical test was executed to investigate the performance of the MAOR scheme for $\alpha = 1.25, 1.50, 1.75$. The tests were carried out using Code::Blocks for the C++ programming language on a laptop with an Intel(R) Core(TM) i5-7200U COU @ 2.50GHz and 4.00GB RAM running Windows 10. The proposed scheme was compared with the GS, SOR, MSOR and AOR schemes on fractional finite difference iterative methods. Throughout the computational experiments, a tolerance of $\varepsilon = 10^{-10}$ was utilized for the convergence criteria.

The computational cost is an important factor in developing fast iterative numerical schemes as it estimates the total number of arithmetic operations to be implemented per iteration. A higher computational cost would produce a higher computational time, which indicates slowness in the convergence. Here, the computational cost for each scheme is measured by computing the total of arithmetic operations involved, as illustrated in Table 3. From the table, the term *Ite* corresponds to the number of iterations for their respective α , and when $\alpha = 2.00$, Equation (1) will revert back to a second order Poisson equation [28, 29], which is solved using the second order finite difference method. Hence the difference in the total number of operations is compared to when $\alpha = 1.25, 1.50, 1.75$.

Table 3: The computational complexities between relaxation schemes ($\sigma = M - 1$).

Scheme	Total operations per iteration	
	$\alpha = 1.25, 1.50, 1.75$	$\alpha = 2.00$
GS	$\sigma^2(2\sigma + 7) * Ite$	$6\sigma^2 * Ite$
SOR	$\sigma^2(2\sigma + 11) * Ite$	$10\sigma^2 * Ite$
MSOR	$\sigma^2(2\sigma + 11) * Ite$	$10\sigma^2 * Ite$
AOR	$4\sigma^2(\sigma + 5) * Ite$	$16\sigma^2 * Ite$
MAOR	$2(\sigma^2(2\sigma + 11) - 1) * Ite$	$5(3\sigma^2 - 1) * Ite$

In order to validate the superiority of the proposed scheme, all five schemes were executed to solve the following example problem [12]

$$\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial y^\alpha} = \Gamma(\alpha + 1)(x^\alpha + y^\alpha)$$

on the solution domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and boundary conditions

$$\begin{aligned} u(x, 0) &= 0 & u(0, y) &= 0 \\ u(x, 1) &= x^\alpha & u(0, 1) &= y^\alpha \end{aligned}$$

with an exact solution $u(x, y) = (xy)^\alpha$. The error is identified as

$$Error = \frac{1}{(M-1)^2} \sqrt{\sum_{i,j=1}^{M-1} (U_{i,j} - u_{i,j})^2} \quad (32)$$

in which $U_{i,j}$ represents the exact solution and $u_{i,j}$ represents the numerical solution. The results are shown in Tables 4 and 5 for the above example problem with several mesh sizes of 10, 20, 40 and 80 and different values of α .

From Tables 4 and 5, the criteria examined are the number of iterations (Ite), the error which was calculated using Equation (32), the execution time (in seconds), and the total number of arithmetic operations (Total Ops). These criteria were used to compare the relaxation schemes executed on several mesh sizes and different values of α . The results for $\alpha = 2.00$ are considered benchmark results when compared with the other values of α . Through observing the results, it can be seen that as the mesh size M increases, the proposed MAOR scheme for fractional Poisson equation produces the least number of iterations. However, there was a 17 – 65% increase in execution time when the MAOR scheme was compared to the MSOR scheme, which was supported

by a higher number of complexities for the MAOR scheme that utilized four weighted parameters as compared to the MSOR scheme that utilized only two weighted parameters. Here, there can be an open problem to reduce the number of weighted parameters and still produce a high convergence rate.

7. Conclusion

This research has successfully developed a fractional finite difference method with a shifted Grünwald estimate with the MAOR scheme and has also successfully obtained the approximate solution for fractional Poisson equation. The results in Tables 4 and 5 show that in comparison with previous relaxation schemes, the MAOR scheme has the least number of iterations; however, there is a slight increase in execution time due to the higher number of computational complexities. For future work, this research will be implemented on different group iterative methods for solving fractional equations, which will be reported separately.

Table 4: Performance comparison between relaxation schemes at $\alpha = 1.25$ and $\alpha = 1.50$.

M	Scheme	$\alpha = 1.25$					$\alpha = 1.50$				
		Ite	Parameters	Error	Time	Total Ops	Ite	Parameters	Error	Time	Total Ops
10	GS	81	-	1.16E-05	0.016	164,025	116	-	2.46E-06	0.031	234,900
	SOR	56	$\omega = 1.23$	1.16E-05	0.015	131,544	56	$\omega = 1.41$	2.46E-06	0.016	131,544
	MSOR	28	$\omega_r = 1.15, \omega_b = 1.26$	1.16E-05	0.000	65,772	27	$\omega_r = 1.27, \omega_b = 1.42$	2.46E-06	0.000	63,423
	AOR	43	$\theta = 1.59, \omega = 1.12$	1.16E-05	0.016	195,048	44	$\theta = 1.58, \omega = 1.31$	2.46E-06	0.016	199,584
	MAOR	23	$\theta_r = 1.15, \theta_b = 1.31,$	1.16E-05	0.015	108,008	25	$\theta_r = 1.78, \theta_b = 1.42,$	2.46E-06	0.015	117,400
			$\omega_r = 1.05, \omega_b = 1.18$					$\omega_r = 1.19, \omega_b = 1.38$			
20	GS	198	-	2.00E-06	0.500	3,216,510	331	-	4.27E-07	0.765	5,377,095
	SOR	140	$\omega = 1.27$	2.00E-06	0.234	2,476,460	154	$\omega = 1.51$	4.27E-07	0.266	2,724,106
	MSOR	63	$\omega_r = 1.21, \omega_b = 1.35$	2.00E-06	0.109	1,114,407	56	$\omega_r = 1.47, \omega_b = 1.51$	4.27E-07	0.094	990,584
	AOR	99	$\theta = 1.98, \omega = 1.23$	2.00E-06	0.343	3,430,944	102	$\theta = 1.93, \omega = 1.34$	4.27E-07	0.375	3,534,912
	MAOR	47	$\theta_r = 1.11, \theta_b = 1.30,$	2.00E-06	0.171	1,662,672	48	$\theta_r = 1.20, \theta_b = 1.45,$	4.27E-07	0.172	1,698,048
			$\omega_r = 1.20, \omega_b = 1.04$					$\omega_r = 1.45, \omega_b = 1.30$			
40	GS	472	-	3.36E-07	10.844	61,022,520	915	-	6.98E-08	20.250	118,295,775
	SOR	347	$\omega = 1.26$	3.36E-07	6.109	46,973,043	454	$\omega = 1.51$	6.97E-08	7.985	61,457,526
	MSOR	156	$\omega_r = 1.25, \omega_b = 1.31$	3.36E-07	2.750	21,117,564	151	$\omega_r = 1.55, \omega_b = 1.57$	6.97E-08	2.656	20,440,719
	AOR	257	$\theta = 1.98, \omega = 1.29$	3.36E-07	9.078	68,797,872	281	$\theta = 1.99, \omega = 1.53$	6.97E-08	9.891	75,222,576
	MAOR	102	$\theta_r = 1.10, \theta_b = 1.42,$	3.36E-07	3.594	27,615,072	92	$\theta_r = 1.23, \theta_b = 1.47,$	6.97E-08	3.375	24,907,712
			$\omega_r = 1.16, \omega_b = 1.01$					$\omega_r = 1.52, \omega_b = 1.15$			
80	GS	1100	-	5.51E-08	273.937	1,132,741,500	2486	-	1.09E-08	614.984	2,559,995,790
	SOR	845	$\omega = 1.23$	5.51E-08	167.140	891,246,005	1350	$\omega = 1.46$	1.09E-08	266.922	1,423,884,150
	MSOR	388	$\omega_r = 1.25, \omega_b = 1.27$	5.51E-08	76.657	409,234,852	488	$\omega_r = 1.54, \omega_b = 1.53$	1.09E-08	95.828	514,707,752
	AOR	628	$\theta = 1.99, \omega = 1.29$	5.51E-08	248.078	1,316,900,928	817	$\theta = 1.99, \omega = 1.54$	1.09E-08	321.891	1,713,229,392
	MAOR	231	$\theta_r = 1.01, \theta_b = 1.59,$	5.51E-08	91.344	487,284,336	271	$\theta_r = 1.10, \theta_b = 1.46,$	1.09E-08	141.937	571,662,576
			$\omega_r = 1.02, \omega_b = 1.02$					$\omega_r = 1.56, \omega_b = 1.04$			

Table 5: Performance comparison between relaxation schemes at $\alpha = 1.75$ and $\alpha = 2.00$.

M	Scheme	$\alpha = 1.75$					$\alpha = 2.00$				
		Ite	Parameters	Error	Time	Total Ops	Ite	Parameters	Error	Time	Total Ops
10	GS	153	-	1.82E-06	0.047	309,825	192	-	5.71E-11	0.000	93,312
	SOR	50	$\omega = 1.51$	1.82E-06	0.016	117,450	42	$\omega = 1.53$	2.56E-12	0.000	34,020
	MSOR	30	$\omega_r = 1.41, \omega_b = 1.47$	1.82E-06	0.016	70,470	37	$\omega_r = 1.46, \omega_b = 1.61$	1.23E-12	0.000	29,970
	AOR	44	$\theta = 1.55, \omega = 1.47$	1.82E-06	0.016	199,584	42	$\theta = 1.53, \omega = 1.49$	4.84E-12	0.000	54,432
	MAOR	29	$\theta_r = 1.44, \theta_b = 1.56,$ $\omega_r = 1.26, \omega_b = 1.47$	1.82E-06	0.016	136,184	35	$\theta_b = 1.57,$ $\omega_r = 1.43, \omega_b = 1.50$	1.58E-11	0.000	42,350
20	GS	500	-	2.33E-07	1.234	8,122,500	712	-	1.08E-10	0.015	1,542,192
	SOR	132	$\omega = 1.69$	2.33E-07	0.234	2,334,948	84	$\omega = 1.73$	4.37E-12	0.000	303,240
	MSOR	59	$\omega_r = 1.59, \omega_b = 1.65$	2.33E-07	0.109	1,043,651	74	$\omega_r = 1.69, \omega_b = 1.78$	2.53E-12	0.000	267,140
	AOR	94	$\theta = 1.83, \omega = 1.52$	2.33E-07	0.344	3,257,664	83	$\theta = 1.73, \omega = 1.72$	6.54E-12	0.015	479,408
	MAOR	56	$\theta_r = 1.37, \theta_b = 1.67,$ $\omega_r = 1.52, \omega_b = 1.60$	2.33E-07	0.188	1,981,056	69	$\theta_b = 1.77,$ $\omega_r = 1.61, \omega_b = 1.68$	9.04E-12	0.000	373,290
40	GS	1596	-	2.95E-08	35.547	206,338,860	2610	-	2.12E-10	0.094	23,818,860
	SOR	376	$\omega = 1.78$	2.95E-08	6.562	50,898,744	169	$\omega = 1.86$	1.18E-12	0.016	2,570,490
	MSOR	121	$\omega_r = 1.73, \omega_b = 1.75$	2.95E-08	2.110	16,379,649	144	$\omega_r = 1.84, \omega_b = 1.87$	3.90E-12	0.015	2,190,240
	AOR	226	$\theta = 1.99, \omega = 1.62$	2.95E-08	7.875	60,499,296	165	$\theta = 1.86, \omega = 1.84$	1.65E-12	0.016	4,015,440
	MAOR	114	$\theta_r = 1.45, \theta_b = 1.47,$ $\omega_r = 1.74, \omega_b = 1.15$	2.95E-08	4.110	30,863,904	136	$\theta_b = 1.81,$ $\omega_r = 1.81, \omega_b = 1.72$	5.35E-12	0.015	3,102,160
80	GS	5030	-	3.60E-09	1247.360	5,179,717,950	9511	-	4.15E-10	0.906	356,148,906
	SOR	1333	$\omega = 1.75$	3.67E-09	264.562	1,405,953,757	335	$\omega = 1.93$	1.49E-12	0.047	20,907,350
	MSOR	280	$\omega_r = 1.79, \omega_b = 1.82$	3.69E-09	55.578	295,324,120	281	$\omega_r = 1.92, \omega_b = 1.93$	9.23E-13	0.031	17,537,210
	AOR	753	$\theta = 1.99, \omega = 1.80$	3.68E-09	299.469	1,579,022,928	325	$\theta = 1.93, \omega = 1.92$	2.65E-12	0.047	32,453,200
	MAOR	209	$\theta_r = 1.01, \theta_b = 1.64,$ $\omega_r = 1.79, \omega_b = 1.37$	3.69E-09	83.093	440,876,304	270	$\theta_b = 1.89,$ $\omega_r = 1.91, \omega_b = 1.84$	2.51E-12	0.031	25,274,700

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