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A NOTE ON THE DECAY IN THE ENERGY SPACE OF THE RADIAL SOLUTION TO THE FOURTH-ORDER NLS ON THE WAVEGUIDES $\mathbb{R}^d \times \mathbb{T}$

Elena Nikolova[1], Mirko Tarulli[2 §], George Venkov[3]

[1]Department of Information Engineering, Computer
Science and Mathematics, University of L'Aquila
Via Vetoio, L'Aquila - 67100, ITALY

[2]Department of Mathematical Analysis and
Differential Equations, Technical University of Sofia
Kliment Ohridski Blvd. 8, Sofia - 1000, BULGARIA
& Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Block 8, Sofia - 1113, BULGARIA
& Department of Mathematics, University of Pisa
Largo Bruno Pontecorvo 5, Pisa - 56100, ITALY

[3]Department of Mathematical Analysis and
Differential Equations, Technical University of Sofia
Kliment Ohridski Blvd. 8, Sofia - 1000, BULGARIA

Abstract: We present, in any space dimension $d \geq 3$, the decay in the energy space for the defocusing biharmonic nonlinear Schrödinger equation (NLS4) with energy subcritical nonlinearity posed on the product manifold $\mathbb{R}^d \times \mathbb{T}$. We show also new nonlinear correlation inequalities improving the results appearing in [17].

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§Correspondence author

1. Introduction

Consider the Cauchy problem associated with the NLS4 with pure power energy subcritical nonlinearity, posed on the product space $\mathbb{R}^d \times \mathbb{T}$, with space-dimension $d \geq 3$ and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ being the one-dimensional standard torus endowed with the flat metric:

$$\begin{cases} i\partial_t u + \Delta_{x,z}^2 u + u|u|^\alpha = 0, & (t,x,z) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T}, \\ u(0,x,z) = u_0(x,z) \in H^2(\mathbb{R}^d \times \mathbb{T}), \end{cases} \tag{1}$$

where $\Delta_{x,z} = \Delta_{\mathbb{R}^d} + \Delta_{\mathbb{T}}$, and $u = u(t,x,z) : \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \to \mathbb{C}$. We will require the nonlinear parameter α to satisfy the following:

$$0 < \alpha < \alpha^*(d), \qquad \alpha^*(d) = \begin{cases} \frac{8}{d-3} & \text{if} \quad d \geq 4, \\ +\infty & \text{if} \quad d = 3. \end{cases} \tag{2}$$

This implies that $\alpha^*(d)$ corresponds to the H^1-critical nonlinearity in $\mathbb{R}^{d+1}$. We are in a position to state our main result, that is the decay of the solutions to (1). Namely,

Theorem 1. *Let $d \geq 3$ and assume (2) is satisfied. Let $u(t,x,z) \in \mathcal{C}(\mathbb{R}; H^2(\mathbb{R}^d \times \mathbb{T}))$ be a global solution to (1) with radial initial datum $u_0 \in H^2(\mathbb{R}^d \times \mathbb{T})$. Then,*

$$\lim_{t \to \pm\infty} \|u(t,x,z)\|_{L^r(\mathbb{R}^d \times \mathbb{T})} = 0, \qquad \text{with} \;\; 2 < r < 2 + \frac{8}{d-3}. \tag{3}$$

The NLS4 is important in several models of the mathematical physics. Introduced in [4] to describe small dispersion in the propagation of intense laser beams in a medium with Kerr nonlinearity, it was successively used in the context of the theory of motion of a vortex filament in an incompressible fluid in [7], [8] and [12]. As far as the NLS4 settled in $\mathbb{R}^d$ and the scattering in Sobolev spaces with various regularities are concerned, we mainly refer to the works [10], [11] and [9]. In the first, the author establishes global well-posedness and scattering in the whole energy space H^2 for the defocusing case when $d \geq 5$ and one has mass supercritical and energy subcritical nonlinearity. In the second, the cubic defocusing fourth-order NLS is studied and scattering in the energy space H^2 is achieved for $5 \leq d \leq 8$. In the last, the authors also take into account the cubic defocusing fourth-order NLS and scattering in H^s is obtained. In all these papers the main tool used is the interaction Morawetz estimates (see for example [3] and [5] for more details) and only the case $d \geq 5$ is discussed, because of the difficulties obtaining such inequalities in the low space dimensions. However, in [13] we presented a version of the interaction Morawetz estimates different than the results cited above. We treated in a unified and simple manner the fourth-order NLS in $\mathbb{R}^d$ without any distinction between the cases $d = 3, 4$ and $d \geq 5$. These estimates lead to the strong decay (3) which plays a key role in the study of the scattering properties of solutions to (1), as emphasized also in [2], [14], [15] and [16]. In this work, we present a generalization of the aforementioned [13] on the product manifold of type $\mathbb{R}^d \times \mathbb{T}$, not only simplifying the results displayed in [17] but shedding light on the unexplored scenario of low space dimensions $d = 3, 4$.

2. Preliminaries and well-posedness

We preface some useful notations. We indicate by $f \in L^r_{x,z}$, if

$$\|f\|^r_{L^r_{x,z}} = \int_{\mathbb{R}^d \times \mathbb{T}} |f(x,z)|^r \, dx \, dz < +\infty,$$

for $1 \leq r < \infty$ and obvious modification for $r = \infty$. We introduce also the Sobolev space $H^s_{x,z} = (1 - \Delta_{x,z})^{-\frac{1}{2}} L^2_{x,z}$. Moreover, from [17], one has the following proposition.

Proposition 2. *Assume $d \geq 1$ and (2) is satisfied. Then, for any radial initial datum $u_0 \in H^2_{x,z}$, the problem (1) has a unique global solution $u(t,x,z) \in \mathcal{C}(\mathbb{R}; H^2_{x,z})$. Furthermore, one has*

$$M(u(t)) = \|u_0\|^2_{L^2_{x,z}}, \quad E(u(t)) = E(u_0) \tag{4}$$

with

$$M(u(t)) = \int_{\mathbb{R}^d \times \mathbb{T}} |u(t,x,z)|^2 dx dz,$$

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{T}} |\Delta_{x,z} u(t,x,z)|^2 + \frac{1}{\alpha + 2} |u(t,x,z)|^{\alpha+2} \, dx \, dz.$$

3. Interaction Morawetz identities and inequalities

We introduce the following further notations. Given a complex valued function $f(t,x,z) \in H^2(\mathbb{R}^d \times \mathbb{T})$, for $t \in \mathbb{R}$, we denote the mass density and the momentum density, respectively by

$$m_f(t,x,z) = |f(t,x,z)|^2, \quad j_f(t,x,z) := \mathrm{Im}\left[\overline{f}(t,x,z)\nabla_x f(t,x,z)\right].$$

Then, our first achievement is the following.

Lemma 3. *Let $d \geq 1$ and $u \in \mathcal{C}(\mathbb{R}; H^2_{x,z})$ be a global solution to* (1) *with radial initial datum $u_0 \in H^2_{x,z}$ and such that* (2) *is satisfied. Let $\phi = \phi(x) : \mathbb{R}^d \to \mathbb{R}$ be a sufficiently regular and decaying function and introduce the action given by*

$$I(t) = 2\int_{\mathbb{R}^d \times \mathbb{T}} j_u(t,x,z) \cdot \nabla_x \phi(x)\, dxdz.$$

The following identity holds:

$$\begin{aligned}
\dot{I}(t) = & \int_{\mathbb{R}^d \times \mathbb{T}} -\Delta_x^3 \phi(x) m_u(t,x,z)\, dxdz \\
& +2\int_{\mathbb{R}^d \times \mathbb{T}} \Delta_x^2 \phi(x) m_{\nabla_x u}(t,x,z)\, dxdz \\
& +4\mathrm{Re}\int_{\mathbb{R}^d \times \mathbb{T}} \nabla_x u(t,x,z) D_x^2 \Delta_x \phi(x) \cdot \nabla_x \overline{u}(t,x,z)\, dxdz \\
& -8\mathrm{Re}\int_{\mathbb{R}^d \times \mathbb{T}} D_x^2 u(t,x,z) D_x^2 \phi(x) D_x^2 \overline{u}(t,x,z)\, dxdz \\
& -8\mathrm{Re}\int_{\mathbb{R}^d \times \mathbb{T}} D_{x,z}^2 u(t,x,z) D_x^2 \phi(x) D_{x,z}^2 \overline{u}(t,x,z)\, dxdz \\
& +2\int_{\mathbb{R}^d \times \mathbb{T}} \Delta_x^2 \phi(x) m_{\partial_z u}(t,x,z)\, dxdz \\
& -\frac{2\alpha}{\alpha+2}\int_{\mathbb{R}^d \times \mathbb{T}} |u(t,x,z)|^{\alpha+2} \Delta_x \phi(x)\, dxdz,
\end{aligned} \tag{5}$$

where $D_{x,z}^2 = \nabla_x \partial_z$, $D_x^2\phi, D_x^2 u \in \mathcal{M}_{d\times d}(\mathbb{R}^d)$ are the Hessian matrices of ϕ and u, $\Delta_x^2\phi = \Delta_x(\Delta_x\phi)$ and $\Delta_x^3\phi = \Delta_x(\Delta_x(\Delta_x\phi))$ are the second and the third power of the Laplacian operator, respectively.

Proof. We prove this identity for a smooth rapidly decreasing solution $u = u(t,x,z)$, while the general case $u \in \mathcal{C}(\mathbb{R}; H^2_{x,z})$ can be obtained by a final standard density argument. We will provide more details on deriving (5).

An integration by parts and (1) give

$$\begin{aligned}
&2\partial_t \int_{\mathbb{R}^d\times\mathbb{T}} j_u(t,x,z)\cdot\nabla_x\phi(x)\,dxdz\\
&= -2\mathrm{Im}\int_{\mathbb{R}^d\times\mathbb{T}} \partial_t u(t,x,z)\Xi(t,x,z)\,dxdz\\
&= 2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} i\partial_t u(t,x,z)\Xi(t,x,z)\,dxdz\\
&= -2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \Big(\Delta^2_{x,z}u(t,x,z)+|u(t,x,z)|^\alpha u(t,x,z)\Big)\Xi(t,x,z)\,dxdz
\end{aligned} \tag{6}$$

with

$$\Xi(t,x,z)=\Delta_x\phi(x)\bar u(t,x,z)+2\nabla_x\phi(x)\cdot\nabla_x\bar u(t,x,z).$$

We set $\tilde x=(x,z)$, $\nabla_{\tilde x}=(\nabla_x,\partial_z)$ and have the identity (see [13])

$$\begin{aligned}
&2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} -\Delta^2_{x,z}u(t,x,z)\Xi(t,x,z)\,dxdz\\
&= 2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} -\Delta^2_{x,z}u(t,x,z)\Delta_{x,z}\phi(x)\bar u(t,x,z)\,dxdz\\
&\quad+4\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} -\Delta^2_{x,z}u(t,x,z)\nabla_{x,z}\phi(x)\cdot\nabla_{x,z}\bar u(t,x,z)\,dxdz\\
&= -\int_{\mathbb{R}^d\times\mathbb{T}} \Delta^2_{\tilde x}\phi(x)\Delta_{\tilde x}|u(t,\tilde x)|^2\,d\tilde x+2\int_{\mathbb{R}^d\times\mathbb{T}}\Delta_{\tilde x}\phi(x)|\nabla_{\tilde x}u(t,\tilde x)|^2\,d\tilde x\\
&\quad-2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \Delta_{\tilde x}u(t,\tilde x)\nabla_{\tilde x}\Delta_{\tilde x}\phi(x)\cdot\nabla_{\tilde x}\bar u(t,\tilde x)\,d\tilde x\\
&\quad+2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \nabla_{\tilde x}\Delta_{\tilde x}u(t,\tilde x)\Delta_{\tilde x}\phi(x)\cdot\nabla_{\tilde x}\bar u(t,\tilde x)\,d\tilde x\\
&\quad-8\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} D^2_{\tilde x}u(t,\tilde x)D^2_{\tilde x}\phi(x)D^2_{\tilde x}\bar u(t,\tilde x)\,d\tilde x\\
&\quad+2\int_{\mathbb{R}^d\times\mathbb{T}} \Delta_{\tilde x}\phi(x)|D^2_{\tilde x}u(t,\tilde x)|^2\,d\tilde x\\
&\quad-4\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} D^2_{\tilde x}u(t,\tilde x)\nabla_{\tilde x}D^2_{\tilde x}\phi(x)\cdot\nabla_{\tilde x}\bar u(t,\tilde x)\,d\tilde x.
\end{aligned} \tag{7}$$

By applying now the following (see again [13])

$$-\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \Delta_{\tilde x}u(t,\tilde x)\nabla_{\tilde x}\Delta_{\tilde x}\phi(x)\cdot\nabla_{\tilde x}\bar u(t,\tilde x)\,d\tilde x$$

$$
\begin{aligned}
&+\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \nabla_{\tilde{x}}\Delta_{\tilde{x}}u(t,\tilde{x})\Delta_{\tilde{x}}\phi(x)\cdot\nabla_{\tilde{x}}\bar{u}(t,\tilde{x})\,d\tilde{x}\\
&= \mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \nabla_{\tilde{x}}u(t,\tilde{x})D^2_{\tilde{x}}\Delta_{\tilde{x}}\phi(x)\cdot\nabla_{\tilde{x}}\bar{u}(t,\tilde{x})\,d\tilde{x}\\
&\qquad -\int_{\mathbb{R}^d\times\mathbb{T}} \Delta_{\tilde{x}}\phi(x)|D^2_{\tilde{x}}u(t,\tilde{x})|^2\,d\tilde{x}
\end{aligned}
$$

and

$$
\begin{aligned}
&2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} D^2_{\tilde{x}}u(t,\tilde{x})\nabla_{\tilde{x}}D^2_{\tilde{x}}\phi(x)\cdot\nabla_{\tilde{x}}\bar{u}(t,\tilde{x})\,d\tilde{x}\\
&= -\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \nabla_{\tilde{x}}u(t,\tilde{x})D^2_{\tilde{x}}\Delta_{\tilde{x}}\phi(x)\cdot\nabla_{\tilde{x}}\bar{u}(t,\tilde{x})\,d\tilde{x},
\end{aligned}
$$

one can see that the r.h.s. of the above identity (7) is equivalent to

$$
\begin{aligned}
&-\int_{\mathbb{R}^d\times\mathbb{T}} \Delta^3_{\tilde{x}}\phi(x)\,|u(t,\tilde{x})|^2\;d\tilde{x} + 2\int_{\mathbb{R}^d\times\mathbb{T}} \Delta_{\tilde{x}}\phi(x)|\nabla_{\tilde{x}}u(t,\tilde{x})|^2\,d\tilde{x}\\
&\quad +4\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \nabla_{\tilde{x}}u(t,\tilde{x})D^2_{\tilde{x}}\Delta_{\tilde{x}}\phi(x)\cdot\nabla_{\tilde{x}}\bar{u}(t,\tilde{x})\,d\tilde{x}\\
&\quad -8\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} D^2_{\tilde{x}}u(t,\tilde{x})D^2_{\tilde{x}}\phi(x)D^2_{\tilde{x}}\bar{u}(t,\tilde{x})\,d\tilde{x}\\
&= \int_{\mathbb{R}^d\times\mathbb{T}} -\Delta^3_{x}\phi(x)\,|u(t,x,z)|^2\;dxdz\\
&\quad +2\int_{\mathbb{R}^d\times\mathbb{T}} \Delta^2_{x}\phi(x)|\nabla_{x}u(t,x,z)|^2\,dxdz\\
&\quad +4\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} \nabla_{x}u(t,x,z)D^2_{x}\Delta_{x}\phi(x)\cdot\nabla_{x}\overline{u}(t,x,z)\,dxdz\\
&\quad -8\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} D^2_{x}u(t,x,z)D^2_{x}\phi(x)D^2_{x}\overline{u}(t,x,z)\,dxdz
\end{aligned}
$$

$$
\begin{aligned}
&-8\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} D^2_{x,z}u(t,x,z)D^2_{x,z}\overline{u}(t,x,z)D^2_{x}\phi(x)\,dxdz\\
&\quad +2\int_{\mathbb{R}^d\times\mathbb{T}} \Delta^2_{x}\phi(x)\,|\partial_z u(x,z)|^2\;dxdz.
\end{aligned}
\tag{8}
$$

Moreover, one gets for the nonlinear terms

$$
\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}} |u(t,x,z)|^\alpha\,u(t,x,z)\cdot\Delta_x\phi(x)\bar{u}(t,x,z)\,dxdz
$$

$$+2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}}|u(t,x,z)|^{\alpha}\,u(t,x,z)\cdot\nabla_x\phi(x)\cdot\nabla_x\bar{u}(t,x,z)\,dxdz$$

$$= \mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}}|u(t,x,z)|^{\alpha+2}\,\Delta_x\phi(x)\,dxdz$$
$$+2\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}}\nabla_x\phi(x)\cdot\frac{\nabla_x|u(t,x,z)|^{\alpha+2}}{\alpha+2}\,dxdz$$
$$=2\left(1-\frac{2}{\alpha+2}\right)\mathrm{Re}\int_{\mathbb{R}^d\times\mathbb{T}}|u(t,x,z)|^{\alpha+2}\,\Delta_x\phi(x)\,dxdz.$$

Taking into account this, (6), (7), (8), we complete the proof. □

Inspired by [13], we have also the following.

Lemma 4. *Assume $d\geq 1$ and let $u\in\mathcal{C}(\mathbb{R};H^2_{x,z})$ be a global solution to (1) with radial initial datum $u_0\in H^2_{x,z}$ and such that (2) is satisfied. Let $\phi=\phi(|x|):\mathbb{R}^d\to\mathbb{R}$ be a convex radial sufficiently regular and decaying function and so that, for any $f\in H^2_{x,y}$,*

$$D^2_x f D^2_x\phi(|x-y|)D^2_x\overline{f}\geq C_1\rho_1(|x-y|)\left|(v\cdot\nabla_x f)\frac{v}{|v|^2}\right|^2 \tag{9}$$

and

$$\nabla_x f D^2_x\Delta_x\phi(|x-y|)\nabla_x\overline{f} \tag{10}$$
$$\leq -C_2\left(\rho_2(|x-y|)|\nabla^{\perp}_v f|^2-\rho_3(|x-y|)\left|(v\cdot\nabla_x f)\frac{v}{|v|^2}\right|^2\right),$$

for any $y\in\mathbb{R}^d$ with $v=x-y$, $\rho_1(|x|),\rho_2(|x|),\rho_3(|x|)>0$, $C_1,C_2>0$, such that $C_1\rho_2(|x|)-C_2\rho_3(|x|)\geq 0$, where $\nabla^{\perp}_v f=\nabla_x f-\frac{(v\cdot\nabla_x f)v}{|v|^2}$. Moreover, let us denote by $\psi^{x,y}=\psi(x,y):=\phi(|x-y|):\mathbb{R}^{2d}\to\mathbb{R}$ and introduce the action

$$\mathcal{M}(t)=2\int_{(\mathbb{R}^d\times\mathbb{T})^2}j_u(x,z_1)\cdot\nabla_x\psi^{x,y}\,m(y,z_2)\,dV_1dV_2, \tag{11}$$

where $(\mathbb{R}^d\times\mathbb{T})^2=(\mathbb{R}^d\times\mathbb{T})\times(\mathbb{R}^d\times\mathbb{T})$, $dV_1=dxdz_1$ and $dV_2=dydz_2$. Then the following holds:

$$\dot{\mathcal{M}}(t)\leq 2\int_{(\mathbb{R}^d\times\mathbb{T})^2}\Delta^2_x\psi^{x,y}K(t,x,y,z_1,z_2)\,dV_1dV_2$$

$$-\frac{4\alpha}{\alpha+2}\int_{(\mathbb{R}^d\times\mathbb{T})^2}|u(t,x,z_1)|^{\alpha+2}m_u(t,y,z_2)\Delta_x\psi^{x,y}\,dV_1dV_2$$
$$-16\int_{(\mathbb{R}^d\times\mathbb{T})^2}D^2_{x,z_1}u(t,x,z_1)D^2_x\psi^{x,y}D^2_{x,z_1}\overline{u}(t,x,z_1)m_u(t,y,z_2)\,dV_1dV_2 \qquad (12)$$

with

$$K(t,x,y,z_1,z_2)=\nabla_x m_u(t,x,z_1)\cdot\nabla_y m_u(t,y,z_2) \qquad (13)$$
$$+2m_{\nabla_x u}(t,x,z_1)m_u(t,y,z_2)+2m_{\partial_{z_1}u}(t,x,z_1)m_u(t,y,z_2).$$

Proof. As in the previous theorem, we prove the identity for a smooth, rapidly decreasing solution $u=u(t,x,z)$, treating the general case $u\in C(\mathbb{R};H^2_{x,z})$ by a standard density argument. Hence, we differentiate (11) with respect to the time variable and get

$$\dot{\mathcal{M}}(t)=$$
$$-2\mathrm{Re}\int_{(\mathbb{R}^d\times\mathbb{T})^2}m_u(t,y,z_2)i\partial_t(\overline{u}(t,x,z_1)\nabla_x u(t,x,z_1))\cdot\nabla_x\psi^{x,y}\,dV_1dV_2$$
$$-2\mathrm{Re}\int_{(\mathbb{R}^d\times\mathbb{T})^2}m_u(t,x,z_1)i\partial_t(\overline{u}(t,x,z_2)\nabla_y u(t,x,z_2))\cdot\nabla_y\psi^{x,y}\,dV_1dV_2$$
$$-2\mathrm{Re}\int_{(\mathbb{R}^d\times\mathbb{T})^2}i\partial_t m_u(t,x,z_1)\overline{u}(t,x,z_1)\nabla_y u(t,x,z_1)\cdot\nabla_y\psi^{x,y}\,dV_1dV_2$$
$$-2\mathrm{Re}\int_{(\mathbb{R}^d\times\mathbb{T})^2}i\partial_t m_u(t,y,z_2)\overline{u}(t,x,z_1)\nabla_x u(t,x,z_1)\cdot\nabla_x\psi^{x,y}\,dV_1dV_2. \qquad (14)$$

□

We evoke now the following lemma.

Lemma 5 (Tensor Morawetz)**.** *Let be* u,v *two solutions of* (1), *such that* (2) *is satisfied. Then one has the following identity,*

$$\int_{(\mathbb{R}^d\times\mathbb{T})^2}j_u(t,y,z_2)D^2_{xy}\psi^{x,y}j_v(t,x,z_1)\,dxdz_1dydz_2=0. \qquad (15)$$

We refer to [15] for details on the proof of this lemma. It ensures, by picking $u=v$, that

$$2\mathrm{Re}\int_{(\mathbb{R}^d\times\mathbb{T})^2}i\partial_t m_u(t,x,z_1)\overline{u}(t,x,z_2)\nabla_y u(t,x,z_2)\cdot\nabla_y\psi^{x,y}\,dV_1dV_2$$

$$-2\mathrm{Re}\int_{(\mathbb{R}^d\times\mathbb{T})^2} i\partial_t m_u(t,y,z_2)\overline{u}(t,x,z_1)\nabla_x u(t,x,z_1)\cdot\nabla_x\psi^{x,y}\,dV_1dV_2,$$

will vanish. Let us write $\dot{\mathcal{M}}(t) := \mathcal{I}+\mathcal{II}$. Then, by using (5), Fubini's Theorem and exploiting the symmetry of $\psi(x,y)$, we can write

$$\begin{aligned}\mathcal{I} = -2\int_{(\mathbb{R}^d\times\mathbb{T})^2} & m_u(t,x,z_1)m_u(t,y,z_2)\Delta_x^3\psi^{x,y}\,dV_1dV_2\\ +4\int_{(\mathbb{R}^d\times\mathbb{T})^2} & m_{\partial_{z_1}u}(t,x,z_1)m_u(t,y,z_2)\Delta_x^2\psi^{x,y}\,dV_1dV_2\\ +4\int_{(\mathbb{R}^d\times\mathbb{T})^2} & m_{\nabla_x u}(t,x,z_1)m_u(t,y,z_2)\Delta_x^2\psi^{x,y}\,dV_1dV_2\\ -\frac{4p}{\alpha+1}\int_{(\mathbb{R}^d\times\mathbb{T})^2} & |u(t,x,z_1)|^{\alpha+2}m_u(t,y,z_2)\Delta_x\psi^{x,y}\,dV_1dV_2\\ -16\int_{(\mathbb{R}^d\times\mathbb{T})^2} D^2_{x,z_1}u(t,x,z_1)&D_x^2\psi^{x,y}D^2_{x,z_1}\overline{u}(t,x,z_1)m_u(t,y,z_2)\,dV_1dV_2.\end{aligned}$$

We reshape the linear term in the previous identity by performing an integration by parts (with no boundary terms) and using the property $\partial_{x_k}\psi^{x,y} = -\partial_{y_k}\psi^{x,y}$.

$$\begin{aligned}&-\int_{(\mathbb{R}^d\times\mathbb{T})^2} m_u(t,x,z_1)m_u(t,y,z_2)\Delta_x^3\psi^{x,y}\,dV_1dV_2\\ &=\int_{(\mathbb{R}^d\times\mathbb{T})^2} m_u(t,x,z_1)m_u(t,y,z_2)\partial_{x_i}\partial_{y_i}\Delta_x^2\psi^{x,y}\,dV_1dV_2\\ &=\int_{(\mathbb{R}^d\times\mathbb{T})^2}\nabla_x m_u(t,x,z_1)\nabla_y m_u(t,y,z_2)\Delta_x^2\psi^{x,y}\,dV_1dV_2.\end{aligned}\tag{16}$$

In conclusion, we arrive to

$$\begin{aligned}\mathcal{I} = 2\int_{(\mathbb{R}^d\times\mathbb{T})^2} & \Delta_x^2\psi^{x,y}K(t,x,y,z_1,z_2)\,dV_1dV_2\\ -\frac{4\alpha}{\alpha+2}\int_{(\mathbb{R}^d\times\mathbb{T})^2} & |u(t,x,z_1)|^{\alpha+2}m_u(t,y,z_2)\Delta_x\psi^{x,y}\,dV_1dV_2\\ -16\int_{(\mathbb{R}^d\times\mathbb{T})^2} D^2_{x,z_1}u(t,x,z_1)&D_x^2\psi^{x,y}D^2_{x,z_1}\overline{u}(t,x,z_1)m_u(t,y,z_2)\,dV_1dV_2.\end{aligned}\tag{17}$$

In addition, by (5) and the symmetry of $\psi^{x,y}$, we can further set

$$\mathcal{II}$$

$$
\begin{aligned}
&= 8 \int_{(\mathbb{R}^d\times\mathbb{T})^2} \nabla_x u(t,x,z_1) D_x^2 \Delta_x \psi^{x,y} \nabla_x \overline{u}(t,x,z_1) m_u(t,y,z_2)\, dV_1 dV_2 \\
&+8 \int_{(\mathbb{R}^d\times\mathbb{T})^2} m_u(t,x,z_1) \nabla_x u(t,y,z_2) D_y^2 \Delta_y \psi^{x,y} \nabla_x \overline{u}(t,y,z_2)\, dV_1 dV_2 \\
&-16 \int_{(\mathbb{R}^d\times\mathbb{T})^2} D_x^2 u(t,x,z_1) D_x^2 \psi^{x,y} D_x^2 \overline{u}(t,x,z_1) m_u(t,y,z_2)\, dV_1 dV_2.
\end{aligned} \tag{18}
$$

Utilizing the assumptions (9) and (10) in combination with Fubini's Theorem, one drives the bound

$$
\begin{aligned}
&\mathcal{II} \\
&\lesssim \int_{(\mathbb{R}^d\times\mathbb{T})^2} \nabla_x u(x,z_1) D_x^2 \Delta_x \psi^{x,y} \nabla_x \overline{u}(t,x,z_1) m_u(t,y,z_2)\, dV_1 dV_2 \\
&\quad - \int_{(\mathbb{R}^d\times\mathbb{T})^2} D_x^2 u(t,x,z_1) D_x^2 \psi^{x,y} D_x^2 \overline{u}(t,x,z_1) m_u(t,y,z_2)\, dV_1 dV_2 \\
&\leq \int_{(\mathbb{R}^d\times\mathbb{T})^2} -C_1 \rho_2(|x-y|) \left| v \cdot \nabla_v u(t,x,z_1) \frac{v}{|v|^2} \right|^2 m_u(t,y,z_2)\, dV_1 dV_2 \\
&\leq \int_{(\mathbb{R}^d\times\mathbb{T})^2} C_2 \rho_3(|x-y|) \left| v \cdot \nabla_v u(t,x,z_1) \frac{v}{|v|^2} \right|^2 m_u(t,y,z_2)\, dV_1 dV_2 \\
&\quad - C_2 \int_{(\mathbb{R}^d\times\mathbb{T})^2} \rho_1(|x-y|) |\nabla_{x-y}^{\perp} u(t,x,z_1)|^2 m_u(t,y,z_2)\, dV_1 dV_2 \leq 0
\end{aligned}
$$

with $v = x - y$. Hence, $\mathcal{II} \leq 0$. The previous step together with (14), (16) and (17) imply (12) with $K(x,y,z_1,z_2)$ as in (13).

Remark 6. The results displayed in Lemmas 3 and 4 remain valid in a more general setting. Namely, let u be a sufficiently smooth and decaying solution to (1), defined on $\mathbb{R}^d \times \mathcal{M}^k$, where $\mathcal{M}^k$ is a k-dimensional compact manifold with metric tensor $g_{hj}(z)$, $d\eta$ is the volume element of $\mathcal{M}^k$, which reads in local coordinates as $\sqrt{|g(z)|}dz$ with $|g(z)| = \det(g_{hj}(z))$. Then the following holds

$$
\begin{aligned}
\dot{\mathcal{M}}(t) &\lesssim \int_{(\mathbb{R}^d\times\mathcal{M}^k)^2} \Delta_x^2 \psi^{x,y} K(t,x,y,z_1,z_2)\, dx d\eta_1 dy d\eta_2 \\
&\quad - \int_{(\mathbb{R}^d\times\mathcal{M}^k)^2} \Delta_x \psi^{x,y} |u(x,z_1)|^{\alpha+2} m_u(y,z_2)\, dx d\eta_1 dy d\eta_2 \\
&\quad - \int_{(\mathbb{R}^d\times\mathcal{M}^k)^2} D_{x,z_1}^2 u(x,z_1) D_x^2 \psi^{x,y} D_{x,z_1}^2 \overline{u}(x,z_1) m_u(y,z_2)\, dx d\eta_1 dy d\eta_2,
\end{aligned}
$$

where $\psi^{x,y}$ and $K(t,x,y,z_1,z_2)$ are as in (13).

We are now in position to prove the following proposition involving new nonlinear correlation inequalities.

Proposition 7. *Assume $d \geq 3$ and let $u \in \mathcal{C}(\mathbb{R}; H^2_{x,z})$ be a global solution to* (1) *with radial initial datum $u_0 \in H^2_{x,z}$ and such that* (2) *is satisfied. Then, one has*

- *For $d = 3$:*

$$\int_{\mathbb{R}} \int_{(\mathbb{R}^3 \times \mathbb{T})^2} |u(t,x,z)|^{\alpha+4}\, dtdxdz \leq C\|u_0\|^4_{H^2_{x,z}}. \tag{19}$$

- *For $d \geq 4$:*

$$\int_{\mathbb{R}} \int_{(\mathbb{R}^d \times \mathbb{T})^2} \frac{|u(t,x,z_1)|^{\alpha+2}|u(t,y,z_2)|^2}{|x-y|}\, dxdz_1dydz_2dt \leq C\|u_0\|^4_{H^2_{x,z}}. \tag{20}$$

Proof. We choose $\psi(x,y) = |x-y|$ and a straightforward calculation leads to

$$\Delta_x|x-y| = \frac{(d-1)}{|x-y|} \tag{21}$$

for $d > 1$,

$$\Delta_x^2|x-y| = \begin{cases} -\frac{(d-1)(d-3)}{|x-y|^3}, & \text{if} \quad d \geq 4, \\ \\ -4\pi\delta_{x=y}, & \text{if} \quad d = 3 \end{cases} \tag{22}$$

and

$$D_x^2 u(t,x,z) D_x^2|x-y| D_x^2 \bar{u}(t,x,z) \geq \frac{(d-1)}{|x-y|^3}|\nabla^{\perp}_{x-y} u(t,x,z)|^2,$$

if $d \geq 2$, so that the assumption (9) is fulfilled. Additionally,

$$\nabla_x u(t,x,z) D_x^2 \Delta_x^2|x-y| \nabla_x \bar{u}(t,x,z) = \\ -\frac{(d-1)}{|x-y|^3}\Big(|\nabla^{\perp}_{x-y}u(t,x,z)|^2 - 2|\nabla_x u(t,x,z) - \nabla^{\perp}_{x-y}u(t,x,z)|^2\Big)$$

is satisfied if $d \geq 2$, that is condition (10). From (12) with $\Delta_x^2 \psi^{x,y} \leq 0$ as in (22), discarding some non-positive terms, one obtains

$$\begin{aligned} \dot{\mathcal{M}}(t) \leq 2 \int_{(\mathbb{R}^d \times \mathbb{T})^2} \nabla_x m_u(t,x,z_1) \cdot \nabla_y m_u(t,y,z_1) \Delta_x^2 \psi^{x,y} \, dV_1 dV_2 \\ - \frac{4\alpha}{\alpha+2} \int_{(\mathbb{R}^d \times \mathbb{T})^2} |u(t,x,z_1)|^{\alpha+2} m_u(t,y,z_2) \Delta_x \psi^{x,y} \, dV_1 dV_2 \\ - \int_{(\mathbb{R}^d \times \mathbb{T})^2} D^2_{x,z_1} u(t,x,z_1) D_x^2 \psi^{x,y} D^2_{x,z_1} \overline{u}(t,x,z_1) m_u(t,y,z_2) \, dV_1 dV_2. \end{aligned} \tag{23}$$

The last term of the above inequality is non-positive, because of the next chain of identities,

$$\begin{aligned} & D^2_{x,z} u(t,x,z) D_x^2 \psi^{x,y} D^2_{x,z} \overline{u}(t,x,z) \\ &= \sum_{i,j=1}^{d} \partial_{x_i} \partial_z u(t,x,z) \partial_{x_j} \partial_z \bar{u}(t,x,z) \left(\frac{\delta_{ij}}{|x-y|} - \frac{(x-y)_i (x-y)_j}{|x-y|^3} \right) \\ &= \frac{|\nabla_x \partial_z u(t,x,z)|^2}{|x-y|} - \frac{|\nabla_{x-y} \partial_z u(t,x,z)|^2}{|x-y|} = \frac{\left|\nabla^{\perp}_{x-y} \partial_z u(t,x,z)\right|^2}{|x-y|} \geq 0. \end{aligned}$$

Next, we directly check by using (22), the Fourier transform and Plancherel's identity that

$$\begin{aligned} & \int_{(\mathbb{R}^d \times \mathbb{T})^2} \nabla_x m_u(t,x,z_1) \cdot \nabla_y m_u(t,y,z_2) \Delta_x^2 \psi^{x,y} \, dx dz_1 dy dz_2 \\ & \quad = -C \int_{\mathbb{T}^2} \langle \nabla m_u(t,\cdot), (-\Delta)^{3-d} \nabla m_u(t,\cdot) \rangle dz_1 dz_2 \leq 0, \end{aligned}$$

for some $C > 0$, where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}^d)$. This means that the terms on the right hand side of inequality (23) are all non-positive. Integrating (23) in $[T_1, T_2]$, for any $T_1, T_2 \in \mathbb{R}$, one obtains by (11) the following

$$\begin{aligned} & \sup_{t \in [T_1, T_2]} |\mathcal{M}(t)| \\ & \geq \frac{4\alpha}{(\alpha+2)} \int_{T_1}^{T_2} \int_{(\mathbb{R}^d \times \mathbb{T})^2} |u(t,x,z_1)|^{\alpha+2} m_u(t,y,z_2) \Delta_x \psi^{x,y} \, dx dz_1 dy dz_2 dt. \end{aligned} \tag{24}$$

By an application of Hölder's inequality, we have also the following fundamental estimate

$$\sup_{t \in [T_1, T_2]} \left| \int_{(\mathbb{R}^d \times \mathbb{T})^2} j_u(t,x,z_1) \cdot \nabla_x \psi^{x,y} m_u(t,y,z_2) \, dx dz_1 dy dz_2 \right|$$

$$\lesssim \sup_{t\in[T_1,T_2]} \|u(t)\|^4_{H^2_{x,z}} \lesssim \|u_0\|^4_{H^2_{x,z}} < \infty, \tag{25}$$

since the $H^2_{x,z}$-norm is conserved. Then (19) and (20) follow by (21), (24) and (25), letting $T_2 \to \infty, T_1 \to -\infty$. $\square$

4. The decay of solutions

This section is devoted to proving the main Theorem 1. Namely, we have:

Proof. We shed light only on the case $t \to \infty$, as the case $t \to -\infty$ can be handled analogously. We will split the proof into two parts: first we treat the case $d \geq 4$ and then $d = 3$. It is sufficient to prove the property (3) for a suitable $2 < r < \frac{2(d+1)}{d-3}$, since the general outcomes can be then obtained by (4) and interpolation. In order to do this, we shall prove that

$$\|u(t)\|_{L^{2+\frac{4}{d+1}}_{x,z}} = 0.$$

For this purpose, we assume by contradiction that there exists $\{t_n\}$ such that $\lim_{t\to\infty} t_n = \infty$ and

$$\inf_n \|u(t_n)\|_{L^{2+\frac{4}{d+1}}_{x,z}} = \epsilon_0, \tag{26}$$

for some $\epsilon_0 > 0$. Next, we employ the localized Gagliardo-Nirenberg inequality (see [16])

$$\|\varphi\|^{2+\frac{2}{d+1}}_{L^{2+\frac{2}{d+1}}_{x,z}} \leq C \left(\sup_{x\in\mathbb{R}^d} \|\varphi\|_{L^2(Q_x\times\mathbb{T})} \right)^{\frac{4}{d+1}} \|\varphi\|^2_{H^2_{x,z}}, \tag{27}$$

where $Q_x = x + [-1,1]^d$ is the unit cube in $\mathbb{R}^d$, centered in x. By combining (26), (27), where we choose $\varphi = u(t_n, x)$, with the bound $\|u(t_n)\|_{H^2_{x,z}} < +\infty$, we conclude that there exists $x_n \in \mathbb{R}^d$ such that

$$\|u(t_n)\|_{L^2(Q_{x_n}\times\mathbb{T})} = \delta_0$$

for some $\delta_0 > 0$. Therefore, we claim that there exists a $\bar{t} > 0$ such that

$$\|u(t)\|^2_{L^2(\tilde{Q}_{x_n}\times\mathbb{T})} \gtrsim \delta_0^2 \tag{28}$$

for any $t \in (t_n, t_n + \bar{t})$, where the intervals are selected to be disjoint and $\tilde{Q}_x = x + [-2,2]^d$ denotes the cube in $\mathbb{R}^d$ of side length 2, centered in x.

The bound (28) contradicts the Morawetz estimate (20). Indeed, by Hölder's inequality, we arrive at

$$\|u(t)\|^{\bar{p}}_{L^{\bar{p}}_x(\tilde{Q}_{x_n}\times\mathbb{T})} \gtrsim \delta_0^{\bar{p}}, \tag{29}$$

for any $\bar{p} \geq 2$ and with $t \in (t_n, t_n + \bar{t})$ and $\bar{t}$ as above. As a result, we get that

$$\begin{aligned}\int_{\mathbb{R}}\int_{(\mathbb{R}^d\times\mathbb{T})^2} &\frac{|u(t,x,z_1)|^{\alpha+2}|u(t,y,z_2)|^2}{|x-y|}\, dV_1 dV_2 dt \\ \gtrsim \sum_n \int_{t_n}^{t_n+\bar{t}} \int_{(\tilde{Q}_{x_n}\times\mathbb{T})^2} &|u(t,x,z_1)|^{\alpha+2}|u(t,y,z_2)|^2\, dV_1 dV_2\, dt \\ &\gtrsim \sum_n \int_{t_n}^{t_n+\bar{t}} \delta_0^{\alpha+4}\, dt = \infty,\end{aligned}$$

where in the last inequality we used (28) in combination with (29) and Fubini's Theorem. This leads to a contradiction with (20).

For $d = 3$ we can proceed as in the previous case just using (19) instead of (20). More precisely,

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}^3\times\mathbb{T}} |u(t,x,z)|^{\alpha+4}\, dt\, dxdz \\ &\gtrsim \sum_n \int_{t_n}^{t_n+\bar{t}} \int_{\tilde{Q}_{x_n}\times\mathbb{T}} |u(t,x,z)|^{\alpha+4}\, dt\, dxdz = \infty,\end{aligned} \tag{30}$$

but this contradicts the interaction estimate (19). Finally, we prove the claim (28), that concerns the existence of $\bar{t} > 0$ such that

$$\inf_n \left(\inf_{t\in(t_n,t_n+\bar{t})} \|u(t,x,z)\|^2_{L^2_{(\tilde{Q}_{x_n}\times\mathbb{T})}} \right) \gtrsim \delta_0^2. \tag{31}$$

To show (31) we fix a cut–off function $\widetilde{\chi}(x) \in C_0^\infty(\mathbb{R}^d)$, such that $\widetilde{\chi}(x) = 1$ for $x \in Q_x$ and $\widetilde{\chi}(x) = 0$ for $x \notin \tilde{Q}_x$. Then, one has

$$\begin{aligned}&\left| \frac{d}{dt} \int_{\mathbb{R}^d\times\mathbb{T}} \widetilde{\chi}(x-x_n)|u(t,x,z)|^2\, dxdz \right| \\ &\lesssim \left| \int_{\mathbb{R}^d\times\mathbb{T}} \Delta_{x,z}\widetilde{\chi}(x-x_n)\mathrm{Im}\left(\Delta_{x,z}u(t,x,z)\overline{u}(t,x,z)\right) dxdz \right| \\ &+ \left| \int_{\mathbb{R}^d\times\mathbb{T}} \nabla_{x,z}\widetilde{\chi}(x-x_n)\mathrm{Im}\left(\Delta_{x,z}u(t,x,z)\nabla_{x,z}\overline{u}(t,x,z)\right) dxdz \right| \\ &\lesssim \sup_t \|u(t,x,z)\|^2_{H^2_{x,z}}.\end{aligned}$$

The $H^2_{x,z}$-norm of the solution is bounded, thus we can apply the fundamental theorem of calculus as

$$\left| \int_{\mathbb{R}^d \times \mathbb{T}} \widetilde{\chi}(x - x_n)(|u(s,x,z)|^2 - |u(t,x,z)|^2)\, dxdz \right| \leq C|t-s|,$$

for some $C > 0$ that does not depend on n. Consequently, picking $t = t_n$ leads to the inequality

$$\int_{\mathbb{R}^d \times \mathbb{T}} \widetilde{\chi}(x - x_n)|u(s,x,z)|^2\, dxdz + C|t_n - s| \\ \geq \int_{\mathbb{R}^d} \widetilde{\chi}(x - x_n)|u(t_n,x,z)|^2\, dxdz.$$

Thanks to the nature of the function $\widetilde{\chi}$, the upper yields

$$\int_{\widetilde{Q}_{x_n} \times \mathbb{T}} |u(s,x,z)|^2\, dxdz \geq \int_{Q_{x_n} \times \mathbb{T}} |u(t_n,x,z)|^2\, dxdx - C|t_n - s|.$$

Then (31) follows by choosing $s = t_n + \bar{t}$, with $\bar{t} > 0$ suitably small. This gives (31) and the proof of the thorem is now completed. □

Remark 8. We underline the fact that in the proofs of Lemmas 3 and 4 and Theorem 1 we do not use any radial assumption on the initial data. This is required in the proof of the unconditional well-posedness Proposition 2, based on the Strichartz estimates obtained in [6] and valid in the radial case only. We confide not only to remove such assumption, but to extend also our result to $\mathbb{T}^k$, with $k = 2, 3$. This will be carried out in our forthcoming papers.

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