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SMALL LIPSCHITZ PERTURBATION OF SCALAR MAPS AND LYAPUNOV EXPONENT FOR LIPSCHITZ MAPS

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Abstract: In this paper we consider small Lipschitz perturbations for Lipschitz maps. We obtain conditions to ensure the permanence of fixed points (sink and source) for scalar Lipschitz maps without requiring differentiability, in a step norm weaker than the C^1-norm and stronger than the C^0-norm. Moreover, we also propose conditions in order to guarantee the permanence of periodic points. Additionally, we propose a new definition of Lyapunov exponent for Lipschitz maps which extends, in a natural way, the definition of Lyapunov exponent for differentiable maps.

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1. Introduction

The Theory of Dynamical Systems is widely investigated from the point of view of C^1 framework, that is, usually the maps considered are diffeomorphisms in which it provides smooth dynamical systems [7, 8, 9]. The differentiability condition enables to ensure, under generic assumptions, the permanence of hyperbolic fixed points [7, 12]. Moreover, the notions of Lyapunov exponents is essential to characterize chaotic behavior in a neighborhood of a periodic orbit [1, 6]. In both concepts, the notion of differentiability is required. However,

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dynamical systems generated by Lischitz functions, without differentiability requirement, have interesting qualitative properties as we can see in the works [2, 3, 4, 5, 10, 11, 14].

In this paper we propose a framework of small Lipschitz perturbations for Lipschitz maps. We show that some results which are valid to discrete standard smooth dynamical systems also hold when considering a class of Lipschitz maps instead of considering differentiable ones. Moreover, since a Lipschitz map is not necessarily differentiable (recall that a Lipschitz map $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies that following condition: for all $x, y \in \mathbb{R}$ with $x \neq y$, one has $\frac{|f(x)-f(y)|}{|x-y|} \leq c$, for some $c \in \mathbb{R}, c > 0$; the existence of the limit is not guaranteed), this approach aims to point out some results that lie in the small gap between C^0 and C^1 theory of discrete dynamical systems.

Although the Lipschitz condition does not guarantee differentiability it is known that it guarantees differentiability almost everywhere with respect to the Lebesgue measure. This fact is shown in Rademacher's Theorem.

Theorem 1. *[13, Thm.3.1] Let $\Omega \subset \mathbb{R}$ be an open set, and let $f : \Omega \longrightarrow \mathbb{R}$ be a Lipschitz map. Then f is differentiable at almost every point in Ω.*

Hence, a Lipschitz map which is not differentiable should produce interesting dynamics even if we start at a point of non-differentiability or if a fixed point is a point in which the differentially fails. This approach has been proposed in [5] for maps in finite dimension and in [2] for semigroups in infinity dimension. In order to state our contributions we consider the following simple preliminary examples. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ maps given by

$$f(x) = \begin{cases} 2x & x < 0, \\ x^2, & 0.1 \leq x \leq 1, \\ 0.5x + 0.5, & x > 1, \end{cases} \qquad g(x) = x^3 - x$$

and

$$h(x) = \begin{cases} 0.2x, & x < 0, \\ x^2, & 0.1 \leq x \leq 1, \\ 2.1x - 1.1, & x > 1. \end{cases}$$

We can see in Figure 1 that f, g and h has two fixed points. The map g is smooth while f and h are not differentiable at the two fixed points. Thus, f and h do not belong to the general theory of smooth dynamical systems, which implies that the study of both hyperbolicity and permanence of fixed points are not possible in this context.

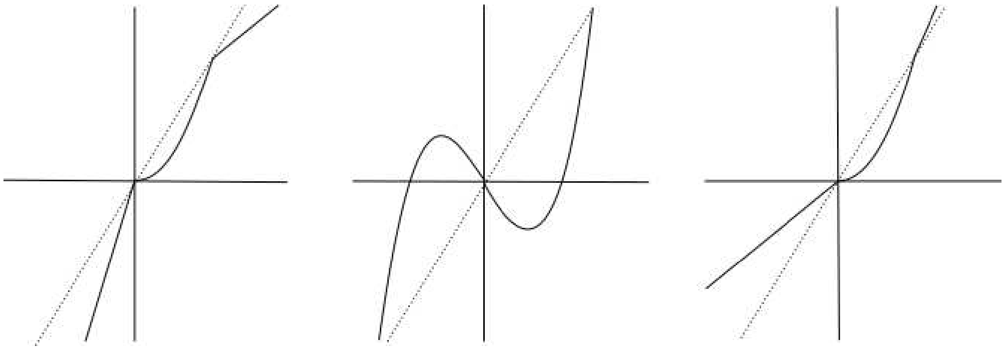

Figure 1: Locally Lipschitz maps f, g, h respectively

Informally, if we move the graphics smoothly we can see that the numbers of fixed points of f can increase or disappear, so f does not behave well under small perturbations. Such a bad behavior is not due to the lack of differentiability. In fact, the map h is not differentiable at the fixed points and, even if we move its graphic smoothly, we can see that the behavior of h and g are similar, that is, the fixed points are preserved.

The main aim of this work is to find classes of Lipschitz functions whose dynamics are preserved under small Lipschitz perturbations. In the next section we state precisely what we mean by small Lipschitz perturbation and, in Theorem 11, we exhibit a class of locally Lipschitz maps which is stable under this notion. The results are in agreement with the existing works related to permanence of hyperbolic fixed points in the C^1-topology and Lipschitz dynamical systems [7, 10].

The second aim of this work is to define Lyapunov exponent for Lipschitz maps. In Ref. [5], the authors defined sink and source for Lipschitz maps (without differentiability). In this sense, the fixed point $p = 0$ of h is a sink and we can see that $(-\infty, 0) \cup (0, 1)$ is its basin of attraction; since h is differentiable in this set, for each initial data $x_1 \in (-\infty, 0) \cup (0, 1)$, the Lyapunov number (and exponent) is well-defined on the orbit of x_1. In other words, considering $\{x_1, x_2, x_3, \ldots\}$ the orbit of x_1, then $x_n \to p = 0$ when $n \to \infty$; moreover, it is well-defined the limit

$$L(x_1) = \lim_{n\to\infty} (|f'(x_1)||f'(x_2)| \ldots |f'(x_n)|)^{\frac{1}{n}} \quad \text{and} \quad h(x_1) = \ln(L(x_1)).$$

Theorem 3.4 in Ref. [1] states that if h is differentiable at $p = 0$ then the Lyapunov number $L(p) = |h'(p)|$ equals the limit $L(x_1)$. But h is not differentiable, then such a theory does not apply. Note that we cannot apply Definition 3.2. (of Lyapunov number) presented in [5], where from Rademacher's Theorem the authors excluded the zero Lebesgue measure set in which h is not differentiable. We also present an analogous of Theorem 3.4 in Ref. [1].

The paper is arranged as follows. In Section 2 we establish conditions in order to prove the stability of fixed points. In Section 3 we extend the results to periodic orbits. In Section 4, we propose a definition of Lyapunov exponent for locally Lipschitz maps. Finally, in Section 5, a summary of the paper is presented.

2. Permanence of fixed points

In this section we recall some known concepts on discrete dynamical systems and after this, we present some new generalizations in the context of Lipschitz maps.

As usual, a function whose domain is equal to its range is called *map*. Let $f : A \longrightarrow A$ be a map and $x \in A$. The *orbit* $\mathcal{O}_x$ of x under f is the set of points $\mathcal{O}_x = \{x, f(x), f^2(x), \ldots, \}$, where $f^2(x) = f(f(x))$ and so on. The point x is said to be the *initial value* of the orbit. If there exists a point p in the domain of f such that $f(p) = p$ then p is called a *fixed point* of f.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map. Recall that f is said to be *Lipschitz* if there exists a constant $c \in \mathbb{R}$, $c > 0$ (called Lipschitz constant of f), such that $\forall\ x, y \in \mathbb{R} \Longrightarrow |f(x) - f(y)| \leq c|x - y|$, where $|\cdot|$ denotes the absolute value function on $\mathbb{R}$. In other words, if $x \neq y$ then $\frac{|f(x)-f(y)|}{|x-y|} \leq c$, i.e., the quotient is bounded. If $\forall x, y \in \mathbb{R} \Longrightarrow |f(x) - f(y)| < c|x - y|$, then f is called *strictly Lipschitz.*

Given $x \in \mathbb{R}$, the *delta neighborhood* $N_\delta(x)$ of x is defined as $N_\delta(x) = \{y \in \mathbb{R} : |x - y| < \delta\}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and $x \in \mathbb{R}$. We say that f is *locally Lipschitz at* x if, for each $\delta > 0$, there exists an δ-neighborhood $N_\delta(x)$ of x such that f restricted to $N_\delta(x)$ is Lipschitz.

In Ref. [5] the authors introduced the concept of *reverse Lipschitz map.* This concept was utilized in order to characterize sources for locally Lipschitz maps (without requiring differentiability).

Definition 2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map. We say that f is reverse Lipschitz (RL) if there exists a constant $r \in \mathbb{R}$, $r > 0$ (called reverse Lipschitz constant of f) such that, $\forall\ x, y \in \mathbb{R} \Longrightarrow |f(x) - f(y)| \geq r|x - y|$. Similarly, f is called locally reverse Lipschitz at x if, for each $\delta > 0$, there exists an

δ-neighborhood $N_\delta(x)$ of x such that f restricted to $N_\delta(x)$ is reverse Lipschitz.

Remark 3. Note that the local Lipschitz constant and reverse constant depends of the neighborhood of the point, that is, in general these constants may change from increasing or decreasing δ. It is clear that if c is a Lipschitz constant then also is $\bar{c} > c$; thus, we always consider the smallest constant in this neighborhood. An analogous convention is made for reverse Lipschitz constant.

We next define sink and source for Lipschitz maps. We do not use differentiability and, since we have considering the scalar situation, we do not have the presence of saddle points which requires differentiability in its definition.

Definition 4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and p be a fixed point of f. One says that p is a *sink* (or *attracting fixed point*) if there exists an $\delta > 0$ such that, for all $x \in N_\delta(p)$, $\lim_{k\to\infty} f^k(x) = p$. On the other hand, if all points sufficiently close to p are repelled from p, then p is called a *source*. In other words, p is a source if there exists a delta neighborhood $N_\delta(p)$ such that, for every $x \in N_\delta(p)$, $x \neq p$, there exists a positive integer k with $|f^k(x) - p| > \delta$.

The first main result of Ref. [5] was to characterize sinks and sources of locally Lipschitz and reverse Lipschitz maps, respectively, based on Lipschitz and reverse Lipschitz constants.

Theorem 5. *[5, Thm.3.2] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and $p \in \mathbb{R}$ a fixed point of f.*

1- *If f is strictly locally Lipschitz map at p, with Lipschitz constant $c < 1$, then p is a sink.*

2- *If f is locally reverse Lipschitz map at p, with constant $r > 1$, then p is a source.*

The next result improves Theorem 5, i.e., it shows that sinks and sources are isolated fixed points which are stable by small Lipschitz perturbations.

Theorem 6. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and p a fixed point of f.*

1- *If f is locally strictly Lipschitz, with constant $c < 1$ in a neighborhood of p, then p is the unique fixed point in this neighborhood, and p is a sink.*

2- *If f is reverse Lipschitz with constant $r > 1$ in a neighborhood of p, then p is the unique fixed point in such a neighborhood, and p is a source.*

Proof. To prove Item 1 note that from Theorem 5 p is a sink; then there exists a neighborhood $N_\delta(p)$ of p such that $f(\overline{N_\delta(p)}) \subset \overline{N_\delta(p)}$. The result follows from Banach Contraction Theorem.

To show Item 2, it follows from Theorem 5 that p is a source. If $q \in N_\delta(p)$ is a source, $q \neq p$, then there exists a positive integer k_0 such that $f^{k_0}(q) \notin N_\delta(p)$, a contradiction, since q is a fixed point of f. □

We next explain what we mean by Lipschitz perturbation. Let f and g be locally Lipschitz maps and $p \in \mathbb{R}$. We denote

$$\|f-g\|_{N_\delta(p)} = \sup_{\substack{x,y\in N_\delta(p)\\ x\neq y}} \left| \frac{f(x)-f(y)}{x-y} - \frac{g(x)-g(y)}{x-y} \right| + \sup_{x\in N_\delta(p)} |f(x)-g(x)|. \quad (1)$$

Note that if f is continuously differentiable then f is locally Lipschitz and Eq. (1) is well-defined.

Lemma 7. *Let $f:\mathbb{R}\to\mathbb{R}$ be a map and let $p\in\mathbb{R}$. If $g:\mathbb{R}\to\mathbb{R}$ is a map such that $\|f-g\|_{N_\delta(p)} < \epsilon$ (that is, Eq. (1) is well-defined and smaller than ϵ) for some $\delta > 0$ then, for sufficiently small ϵ, we have:*

1- *if f is locally Lipschitz with locally Lipschitz constant $c_{f,p} < 1$ in $N_\delta(p)$ then g is also locally Lipschitz with locally Lipschitz constant less than one;*

2- *if f is locally reverse Lipschitz with locally reverse Lipschitz constant $r_{f,p} > 1$ in $N_\delta(p)$ then g is also reverse locally Lipschitz with locally reverse Lipschitz constant greater than one.*

Proof. Let $x, y \in N_\delta(p)$, $x \neq y$. From definition of $c_{f,p}$ we have

$$\begin{aligned}|g(x)-g(y)| &\leq |g(x)-g(y)-f(x)+f(y)| + |f(x)-f(y)|\\ &\leq \varepsilon|x-y| + c_{f,p}|x-y| \leq (\varepsilon + c_{f,p})|x-y|.\end{aligned}$$

Since $c_{f,p} < 1$, Item (1) follows by taking $\varepsilon < 1 - c_{f,p}$.

To show Item (2), note that

$$\epsilon \geq \left| \frac{f(x)-f(y)}{|x-y|} - \frac{g(x)-g(y)}{|x-y|} \right| \geq \left| \frac{f(x)-f(y)}{x-y} \right| - \left| \frac{g(x)-g(y)}{x-y} \right|,$$

which implies

$$\left|\frac{g(x)-g(y)}{x-y}\right| \geq r_{fp} - \epsilon.$$

Since $r_{f,p} > 1$, it is sufficient to take ϵ sufficiently small and obtains the desired result. $\square$

Remark 8. Note that if we assume in Lemma 7 that p is a fixed point of f, a sink for instance, then g is locally Lipschitz with locally Lipschitz constant less than 1 in $N_\delta(p)$. However, this estimate cannot be transferred to all $\mathbb{R}$; furthermore, we cannot even ensure that $g(N_\delta(p)) \subset N_\delta(p)$. Therefore, we cannot establish until now that g has a fixed point (sink) in $N_\delta(p)$. Fortunately, we can circumvent this problem assuming sufficient differentiability for f.

Theorem 9. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and p a fixed point of f such that f is sufficiently differentiable in $\mathbb{R}$ and $|f'(p)| \neq 1$. If g is a locally Lipschitz function such that $\|f-g\|_{N_\delta(p)} < \epsilon$, then for δ and ϵ sufficiently small, there exists a unique fixed point q of g in $N_\delta(p)$. Moreover, if $|f'(p)| < 1$ then q is a sink and if $|f'(p)| > 1$, q is a source.*

Remark 10. Note that we do not require differentiability in g. Therefore, Theorem 9 extends the permanence of equilibrium points when f and g are both continuously differentiable and the perturbation is performed w.r.t. the C^1-norm.

Proof. of Theorem 9. We start denoting $L = f'(p)$ and defining the auxiliary function $h(x) = g(x+p) - p$. Note that $h(x-p) = g(x) - p$; hence, $g(x) = x$ if only if $h(x-p) = x - p$. We have:

$$\begin{aligned} h(x-p) = x-p &\Leftrightarrow h(x-p) - L(x-p) = x - p - L(x-p) \\ &\Leftrightarrow (1-L)^{-1}[h(x-p) - L(x-p)] = x - p. \end{aligned}$$

If we denote $z = x - p$ and $\psi(z) = (1-L)^{-1}[h(z) - Lz]$, then x is a fixed point of g if and only if z is a fixed point of ψ. In the sequence we prove that, for ϵ and δ sufficiently small, ψ is a strict contraction in $N_\delta(p)$. In fact, for $|x-p| \leq \delta$, we have

$$\begin{aligned} |\psi(x-p)| &\leq |(1-L)^{-1}[g(x) - p - L(x-p)]| \\ &\leq |(1-L)^{-1}[g(x) - g(p) - f(x) + f(p)]| + |(1-L)^{-1}[g(p) - f(p)]| \\ &\quad + |(1-L)^{-1}[f(x) - f(p) - L(x-p)]|. \end{aligned}$$

By the Lipschitz closeness (1), one obtains

$$\begin{aligned}&|(1-L)^{-1}[g(x)-g(p)-f(x)+f(p)]|+|(1-L)^{-1}[g(p)-f(p)]|\\&\qquad\leq|1-L|^{-1}\epsilon|x-p|+|1-L|^{-1}\varepsilon\leq|1-L|^{-1}\epsilon\delta+|1-L|^{-1}\epsilon.\end{aligned}$$

Since f is differentiable we can take the remainder in the definition of differentiability such that

$$|(1-L)^{-1}[f(x)-f(p)-L(x-p)]|\leq|1-L|^{-1}\epsilon|x-p|\leq|1-L|^{-1}\epsilon\delta.$$

Thus, taking $\epsilon\leq\min\{|1-L|\delta/3,|1-L|/3\}$, it follows that $|\psi(z)|\leq\delta$.

We next prove that ψ is a contraction. We have:

$$\begin{aligned}|\psi(z)-\psi(\bar{z})|&\leq|(1-L)^{-1}[g(x)-p-L(x-p)-g(\bar{x})+p+L(\bar{x}-p)]|\\&=|(1-L)^{-1}[g(x)-g(\bar{x})-L(x-\bar{x})|\\&\leq|(1-L)^{-1}[g(x)-g(\bar{x})-f(x)+f(\bar{x})]|\\&\qquad+|(1-L)^{-1}[f(x)-f(\bar{x})-L(x-\bar{x})]|.\end{aligned}$$

Proceeding as above, we obtain

$$|(1-L)^{-1}[g(x)-g(\bar{x})-f(x)+f(\bar{x})]|\leq|1-L|^{-1}\epsilon|x-\bar{x}|;$$

since f is sufficiently differentiable, for δ sufficiently small,

$$\begin{aligned}|(1-L)^{-1}[f(x)-f(\bar{x})-L(x-\bar{x})]|&\leq|(1-L)^{-1}[f(x)-f(\bar{x})-f'(\bar{x})(x-\bar{x})]|\\&\quad+|(1-L)^{-1}[(f'(\bar{x})-f'(p))(x-\bar{x})]|\\&\leq|(1-L)|^{-1}\epsilon|x-\bar{x}|+|(1-L)|^{-1}\epsilon|x-\bar{x}|.\end{aligned}$$

Taking $\epsilon<|1-L|/3$ we obtain

$$|\psi(z)-\psi(\bar{z})|\leq3|1-L|^{-1}\epsilon|x-\bar{x}|<|x-\bar{x}|=|z-\bar{z}|.$$

Hence, there exists a unique $q\in\overline{N_\delta(p)}$ such that $g(q)=q$.

Since f is continuously differentiable, f is locally Lipschitz. Because $|f'(p)|<1$, we can take $\delta,\lambda>0$ such that $|f'(x)|<\lambda<1$ for all $x\in N_\delta(p)$; from the Mean Value Theorem, the locally Lipschitz constant of f in $N_\delta(p)$ equals $\sup_{x\in N_\delta(p)}|f'(x)|<1$. The result now follows from Lemma 7 and Theorem 6.

Analogously we obtain that if $|f'(p)|>1$ then, for δ sufficiently small, q is a source for g. □

In Theorem 11, removing the differentiability of f, we exhibit a class of locally Lipschitz maps in which it is possible to ensure the permanence and stability of fixed points. This class involves maps like h exhibited in Figure 1 and exclude maps such as f.

Theorem 11. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and p a fixed point of f such that f is locally Lipschitz with constant $C \neq 1$ in the neighborhood $N_\delta(p)$, for some $\delta > 0$. Assume that f satisfies the inequality*

$$|(1-C)^{-1}[f(x)-f(y)-C(x-y)| \leq \frac{1}{3}|x-y|, \quad \text{for all } x, y \in N_\delta(p). \tag{2}$$

If g is a locally Lipschitz map such that $\|f-g\|_{N_\delta(p)} < \epsilon$, then for sufficiently small ε, there exists a unique fixed point q of g in $N_\delta(p)$ which is a sink if $C < 1$ and which is a source if $C > 1$.

Proof. Utilizing the same argument of the proof of Theorem 9, we obtain

$$\begin{aligned}|\psi(x-p)| \leq{}& |(1-C)^{-1}[g(x)-g(p)-f(x)+f(p)]| + |(1-C)^{-1}[g(p)-f(p)]| \\ &+ |(1-C)^{-1}[f(x)-f(p)-C(x-p)]|.\end{aligned}$$

Considering $\epsilon < \min\{|1-C|\delta/3, |1-C|/3\}$, it follows from (2) that ψ takes $\overline{N_\delta(p)}$ into itself. For $z = x-p$ and $\bar{z} = \bar{x} - p$, we have

$$\begin{aligned}|\psi(z)-\psi(\bar{z})| \leq{}& |(1-C)^{-1}[g(x)-g(\bar{x})-f(x)+f(\bar{x})]| \\ &+ |(1-C)^{-1}[f(x)-f(\bar{x})-L(x-\bar{x})]|.\end{aligned}$$

If we take $\epsilon < |1-C|/2$, it follows from (2) that ψ is a strict contraction in $\overline{N_\delta(p)}$, hence g has a unique fixed point q in $\overline{N_\delta(p)}$. The stability of q follows from Lemma 7 and Theorem 6. □

Remark 12. Note that if f is sufficiently differentiable then the inequality (2) is always true in a neighborhood of $p = f(p)$. Moreover, if f is locally Lipschitz, we can rewrite (2) in the form

$$\left|\frac{f(x)-f(y)}{x-y} - C\right| \leq \frac{|1-C|}{3}, \quad x \neq y.$$

It is easy to see that h and g in Figure 1 satisfy the inequality (2) although f does not.

Remark 13. We finish this section by observing that Theorem 11 is applicable to a class of locally Lipschitz maps satisfying (2). This class is not an optimal class. In fact, the map

$$\begin{cases} 0.5x, \quad x < 0, \\ x^2, \quad 0 \leq x \leq 1, \\ 1.48x - 0.48, \quad x > 1, \end{cases}$$

has apparently stable equilibria for small Lipschitz perturbation (the graphic looks like the graph of h) but it does not satisfy (2) in $p = 0$.

3. Permanence of Periodic Points

In this section we investigate permanence of periodic points. We first recall some known results concerning such a topic.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and $p \in \mathbb{R}$. Recall that p is a *periodic point* of period k (or k-periodic point) if $f^k(p) = p$ and if k is the smallest such a positive integer. If we cannot ensure that k is the smallest positive integer we say that p is a *preperiodic point*. The orbit of p (which consists of k points) is called a *periodic orbit* of period k (or k-periodic orbit). We denote the k-periodic orbit of p by $\mathcal{O}_p^k$.

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a map and if p is a k-periodic point or a prepriodic point, then the orbit $\mathcal{O}_p^k$ of p is called a *periodic sink* if p is a sink of map f^k. Analogously, $\mathcal{O}_p^k$ is a *periodic source* if p is a source of f^k.

The following result is a version of Theorem 5 for periodic points of (reverse) Lipschitz maps. The differentiable version can be found in Ref. [1].

Theorem 14. *[5, Thm.3.5] Let $g = f^k : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and $p \in \mathbb{R}$ a fixed point of g.*

1- If g is strictly locally Lipschitz map at p, with Lipschitz constant $c < 1$, then $\mathcal{O}_p^k$ is a periodic sink.

2- If g is locally reverse Lipschitz map at p, with constant $r > 1$, then $\mathcal{O}_p^k$ is a periodic source.

Since a k-periodic point is a fixed point of f^k we can apply Theorems 9 and 11 in order to obtain the following results.

Theorem 15. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and p a fixed point of f^k, $k > 1$, such that f is continuously differentiable in $\mathbb{R}$ and $|(f^k)'(p)| \neq 1$. If g is a locally Lipschitz function such that $\|f^k - g^k\|_{N_\delta(p)} < \epsilon$, then, for δ and ϵ sufficiently small, there exists a unique fixed point q of g^k in $N_\delta(p)$ which is a preperiodic point of g. Moreover, if $|(f^k)'(p)| < 1$, then the orbit of q by g is a periodic sink; if $|(f^k)'(p)| > 1$, the orbit of q by g is a periodic source.*

The Lipschitz version of Theorem 15 without requiring differentiability in f can be stated as follows.

Theorem 16. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a map and p a fixed point of f^k, $k > 1$, such that f^k locally Lipschitz in $\mathbb{R}$ with locally Lipschitz constant $C \neq 1$ in the neighborhood $N_\delta(p)$, for some $\delta > 0$. Assume that f^k satisfies*

$$|(1-C)^{-1}[f^k(x) - f^k(y) - C(x-y)| \leq \frac{1}{3}|x-y|, \quad \text{for all } x, y \in N_\delta(p). \quad (3)$$

If g is a locally Lipschitz function such that $\|f^k - g^k\|_{N_\delta(p)} < \epsilon$, then, for ϵ sufficiently small, there exists a unique fixed point q of g^k in $N_\delta(p)$. Moreover, if $C < 1$ then the orbit of q by g is a periodic sink; if $C > 1$, the orbit of q by g is a periodic source.

Example 17 (Small Lipschitz perturbation of a Lipschitz Logistic map)**.** Let us consider the logistic map $g(x) = 3.3x(1-x)$. Then g has 2-periodic sink orbit $\{0.4794, 08236\}$ (see [1]), considering four decimal places accuracy. We can now remove the differentiability of g into 0.4794 and make a small perturbation. In other words, we define the locally Lipschitz maps

$$f(x) = \begin{cases} 0.15x + 0.75, \ x < 0.4794, \\ 3.3x(1-x), \ x \geq 0.4794, \end{cases} \quad \text{and } h_\epsilon(x) = \begin{cases} 0.15x + 0.75 + \epsilon, \ x < 0.4794, \\ 3.3x(1-x) + \epsilon, \ x \geq 0.4794. \end{cases}$$

Note that f is not differentiable at 0.4794 but $\{0.4794, 08236\}$ is also a periodic orbit for f. Furthermore, the product of the locally Lipschitz constant in a neighborhood of 0.4794 and 0.8236 is less than one; hence, it is a periodic sink. Moreover, h_ε is a Lipschitz perturbation of h, thus, for sufficiently small ϵ, it follows that h_ε has also a periodic orbit with the same stability of h.

4. Lyapunov Exponent

In this section we introduce in the literature the Lyapunov number and the Lyapunov exponent for Lipschitz maps. We only consider the case of maps defined over $\mathbb{R}$ (or over any subset of $\mathbb{R}$), since the procedure for maps on $\mathbb{R}^n$ (or over any subset of $\mathbb{R}$) is quite similar.

We denote by $\mathcal{O}_{x_1} = \{x_1, x_2, x_3, \ldots\}$ an arbitrary orbit with initial point $x_1 \in \mathbb{R}$, where $x_2 = f(x_1), x_3 = f^2(x_1), x_4 = f^3(x_1), \ldots$. Assume that f is a smooth map on $\mathbb{R}$ and $x_1 \in \mathbb{R}$. Recall that the Lyapunov number $L(x_1)$ of the orbit $\mathcal{O}_{x_1} = \{x_1, x_2, x_3, \ldots\}$ is defined as $L(x_1) = \lim\limits_{n\to\infty} (|f^{'}(x_1)| \cdots |f^{'}(x_n)|)^{1/n}$, if the limit exists. The Lyapunov exponent $h(x_1)$ is defined as $h(x_1) = \lim\limits_{n\to\infty} (1/n)[\ln|f^{'}(x_1)| + \cdots + \ln|f^{'}(x_n)|]$, if the limit exists. We say that the orbit $\mathcal{O}_{x_1}$ is *asymptotically periodic* if it converges to a periodic orbit $\mathcal{O}_{y_1}^k$ for some integer $k \geq 1$ and $y_1 \in \mathbb{R}$, when $n \longrightarrow \infty$. In other words, there exists a periodic orbit $\{y_1, y_2, \ldots, y_k\} = \{y_1, y_2, \ldots, y_k, y_1, y_2, \ldots, y_k, \ldots\}$ such that $\lim\limits_{n\to\infty} |x_n - y_n| = 0$.

We next define Lyapunov number and Lyapunov exponent for Lipschitz maps, which are not necessarily differentiable.

Definition 18. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz map and $\delta > 0$. Denote by $C_{x_i,\delta}$ the locally Lipschitz constant of f in $N_\delta(x_i)$, $i = 1, 2, \ldots$. Then the δ-Lyapunov number $L_\delta(x_1)$ of the orbit $\mathcal{O}_{x_1} = \{x_1, x_2, x_3, \ldots\}$ is defined as

$$L_\delta(x_1) = \lim_{n\to\infty} (C_{x_1,\delta} \cdots C_{x_n,\delta})^{1/n}, \tag{4}$$

if the limit exists.

The δ-Lyapunov exponent $h_\delta(x_1)$ is defined as

$$h_\delta(x_1) = \lim_{n\to\infty} (1/n)[\ln C_{x_1,\delta} + \cdots + \ln C_{x_n,\delta}], \tag{5}$$

if the limit exists.

Remark 19. Note that since f is locally Lipschitz, Eq. (4) and (5) are well-defined for $\delta > 0$. Moreover, if f is continuously differentiable with nonzero derivative, then for all $x_1 \in \mathbb{R}$, we have

$$\lim_{\delta\to 0} L_\delta(x_1) = L(x_1).$$

The following result is a variant of [1, Theorem 3.4] in the context of Lipschitz maps.

Theorem 20. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally Lipschitz map and $\delta > 0$. Assume that $\mathcal{O}_{x_1} = \{x_1, x_2, ...\}$ is asymptotically periodic to the periodic orbit $\mathcal{O}_{y_1} = \{y_1, y_2, \ldots\}$. Then $h_\delta(x_1) \leq h_\delta(y_1)$, if both Lyapunov exponent exist.*

Proof. Assume that $\mathcal{O}_{y_1} = \{y_1, y_2, \ldots\} = \{y\}$, i.e., y is a fixed point of f. Then $x_n \to y$ as $n \to \infty$ and

$$\lim_{n\to\infty} \left| \frac{f(x_n) - f(z)}{x_n - z} \right| = \left| \frac{f(y) - f(z)}{y - z} \right| \leq C_{y,\delta}, \quad z \neq x_n, z \neq y, z \in N_\delta(y),$$

where $C_{y,\delta}$ denotes the locally Lipschitz constant of f in $N_\delta(y)$. Thus, for n_δ sufficiently large, we have $x_k \in N_\delta(y)$ for $k > n_\delta$ and

$$\left| \frac{f(x_k) - f(z)}{x_k - z} \right| \leq C_{y,\delta},$$

which implies $C_{x_k,\delta} \leq C_{y,\delta}$ for $k > n_\delta$; hence $h_\delta(x_1) \leq h_\delta(y)$.

If $k > 1$, we know that y_1 is a fixed point of f^k (which is also locally Lipschitz) and $\mathcal{O}_{x_1}$ is asymptotically periodic under f^k to $\mathcal{O}_{y_1}$. Applying the same reasoning above to x_1 and f^k, it follows that $h_\delta^k(x_1) \leq h_\delta^k(y_1)$. The result follows by observing that $h_\delta(x_1) = \frac{1}{k} h_\delta^k(x_1)$. □

Example 21. Let us consider the locally Lipschitz maps

$$f(x) = \begin{cases} 0.15x + 0.75, \ x < 0.4794, \\ 3.3x(1-x), \ x \geq 0.4794. \end{cases} \quad \text{and}$$

We compute the δ-Lyapunov exponent of the periodic orbit $\{y_1, y_2\} = \{0.4794, 08236\}$. It is easy to see that for each $\delta > 0$,

$$C_{y_1,\delta} C_{y_2,\delta} \leq 0.15 \cdot 2.1357 \cdot 2\delta. \tag{6}$$

Since this orbit is a sink, every orbit $\mathcal{O}_{x_1} = \{x_1, x_2, ...\}$ which converges asymptotically to $\{x_1, x_2\}$ have the δ-Lyapunov exponent bounded by the inequality (6).

5. Final Remarks

We have obtained conditions to ensure the permanence of fixed points (sink and source) for scalar Lipschitz maps without requiring differentiability in a step norm weaker than the C^1-norm and stronger than the C^0-norm. Moreover, we also derived conditions to guarantee the permanence of periodic points. We have also proposed a new definition of Lyapunov exponent for Lipschitz maps which extends, in a natural way, the definition of Lyapunov exponent for differentiable maps. Dynamical systems based on Lipschitz maps seem to be an interesting area of research due to the fact that it is not necessary to require differentiability of a map, which is a strong condition to be satisfied.

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