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SOLVING FREDHOLM INTEGRAL EQUATIONS USING BEES ALGORITHM BASED ON CHEBYSHEV POLYNOMIALS

Azzam S.Y. Aladool[1], Mohammed Abdulrazaq Kahya[2 §]

[1]Department of Mathematics
College of Education for Pure Sciences, University of Mosul
Mosul - Nineveh, IRAQ

[2] Department of Computer Sciences
College of Education for Pure Sciences, University of Mosul
Mosul - Nineveh, IRAQ

Abstract: An approximate solution of the Fredholm integral equations (FIEs) of the second kind is obtained. The equations are converted into an unconstrained optimization problem. In this paper, an algorithm (BAC) for solving the Fredholm integral equations (FIEs), with a combination of Bees algorithm and Chebyshev polynomials, is presented. Chebyshev polynomials are first formulated, with undetermined coefficients, as an approximate solution of FIEs. These polynomials are replaced by the unknown function in the given Fredholm equation. The algorithm is in turn calculating the coefficients. Numerical examples are employed to approve the validity and the applicability of the proposed algorithm and the results are compared to the exact solution. The results show the efficiency and accuracy of the proposed algorithm to solve Fredholm integral equations (FIEs).

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1. Introduction

Fredholm Integral Equations (FIEs) play a crucial role in solving numerous

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§Correspondence author

problems applied mathematics and sciences [17]. Many analytical methods have been used for solving (FIEs). However, The explicit method for solving range of these equations are relatively restricted. Therefore, in many cases, numerical methods is more applicable to find Fredholm integral equations solution.

Numerical methods have been used to handle different types of Fredholm Integral Equations such as the Adomian decomposition method (ADM), the variational iteration method, and the successive approximations method [17, 3, 5]. Moreover, the numerical methods is often depends on finding an approximation solution for unknown function based on an approximate expansion. Taylor expansion is used to approximate a solution of the Fredholm equation of second kind [15, 19]. Furthermore, Chebyshev polynomials are also applied to find an approximation solution for unknown function by converting Fredholm integral equation to a system of linear equations [18].

In general, truncated expansion for unknown function is replaced by the unknown function in the given Fredholm equation. Then, obtaining polynomials corresponded to the approximate solution by formulating an unknown function in terms of Chebyshev polynomials with undetermined coefficients [2]. Therefore, calculating the undetermined coefficients of Chebyshev polynomials for a function then truncates the expansion at the certain degree leads to obtain approximated function $f(x)$.

However, Swarm optimization techniques can be employed to optimize a problem to improve a solution quality, these techniques is successfully applied for solving Mathematical problems [8, 9, 11]. In this paper, Fredholm Integral Equations are formulated as an unconstrained optimization problems to find their approximate solutions using optimization techniques. Therefore, Bees algorithm (BSO) which is swarm optimization technique is applied to solve the Fredholm integral equation based on Chebyshev polynomials. In this algorithm, appropriate number of terms of Chebyshev polynomials, with undetermined coefficients, are chosen. These coefficients are determined by intelligent processes. Consequently, the Bees algorithm is utilized to obtain the coefficients.

This paper is organized as follows: starting with above introduction, it is followed by the form of Fredholm integral equations (FIEs) of the second kind. The Bee algorithm is then presented. The methodology is also detailed included the proposed algorithm, followed by results and discussion of this work and main conclusions.

2. Preliminary

2.1. Fredholm Integral Equation of the 2nd Kind

The Fredholm integral equations have the form

$$y(x) = f(x) + \int_a^b K(x,t)y(t)dt, \tag{1}$$

with the unknown function $y(x)$ and the kernel function $K(x,t)$, which are bounded in the square $a \leq x \leq b$ and $a \leq t \leq b$. They can be characterized by upper and lower constant bounds a and b respectively, in the integration [16, 17].

2.2. Bees Algorithm

The Bees algorithm (BA) is an optimization algorithm based on the honey bee intelligent behavior for searching food foraging [6, 14, 4]. The processes of this algorithm depends on Waggle Dance mechanism which is utilized to simulate bees communication. Best bees have more chance to perform waggle dance, consequently, they are able to attract more bees to approach to their aim and location. This assist the algorithm to consider the promising fields in the search space.

For more details, the Bees algorithm (BA) required number of setting parameters namely the number of: scout bees (s), elite sites (el), best sites (b_1) out of (s) points, bees recruited for elite el sites (s_1), bees recruited for best (b_1) sites (s_2), bees recruited around other visited sites (r_1),the dimension of the search space ($nVar$) and stopping criteria.

The above can be illustrated as follows [12, 1]:

step (1). Initializing scout bees (s): Scout bees (s) are randomly placed on the search space.

step (2). Fitness evaluating:The loop is started by evaluating the fitness of scout bee.

step (3). Selecting elite sites el from scout bees (s): Elite bees el that have best fitness are chosen and sites they visit saved for neighbourhood search.

step (4). Recruit bees (s_1) starting neighbourhood dance search: The algorithm starts by searching in the neighbourhood of chosen sites, assigning further recruit bees (s_1) to dance close to elite sites. Recruit bees (s_1) can be selected depends on the fitness and their dance sites.

step (5). Best sites (b_1) are selected from scout bees: contrarily, selecting Elite bees (el) with the best fitness as the best bees, and selecting the sites they visit for neighbourhood search.

step (6). Recruiting bees (s_2) start neighbourhood dancing search: The algorithm starts by searching in the neighbourhood of chosen sites, assigning further recruit bees (s_2) to dance close to best sites. Recruit bees (s_2) can be selected based on the fitness and their dance sites.

step (7). Random bees (r_1) is recruited for different visited sites: The remaining bees (r_1) within the populace are assigned randomly in neighbourhood space to discover the new possible solutions.

step (8). The convergence: In the present iteration, the bees with the best fitness is chosen for the later iteration

3. Methodology

In the methodology section, an approach for finding approximate solution of Fredholm integral equation is presented. the solution is expressed as Chebyshev polynomials with unknown coefficients, Moreover, proposed algorithm is used to calculate the Chebyshev polynomials coefficients.

3.1. Chebyshev Polynomials (CHP)

Chebyshev polynomials are suitable to approximate functions. The first several polynomials can be expressed as follows [10, 7]: $T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x$. They can be calculated from the recurrence relation $T_{(n+1)}(x) = 2xT_n(x) - T_{(n-1)}(x)$ for $n \geq 1$.

3.2. Function Approximation

The Chebyshev polynomials is used to approximate the unknown function in the Fredholm integral equation. In other words, a given function $y(x)$ is approximated in terms of Chebyshev polynomials with undetermined coefficients

α_i, and it can be defined as:

$$y(x) \sim Y(x) = \sum_{i=0}^{\infty} \alpha_i T_i(x), \tag{2}$$

when $T_i(x)$ denote the Chebyshev polynomials and α_i are Chebyshev coefficients.

3.3. The Fitness Function (FFN)

Suppose the exact solution of the Fredholm integral equation is $y(x)$, and $Y(x)$ satisfies the Eq(1) then it can be determined the error function as follows:

$$Y(x) - f(x) - \int_a^b K(x,t)Y(t)dt = EF(x). \tag{3}$$

The aim is to minimize the value of the error function ($EF(x)$) that is included into a quantitative measure which is represented by fitness function. The fitness function is defined as follows: the interval $I = [a, b]$ is divided into N points such as $x_1 = a, x_2, ..., x_N = b$ where $x_k = x_1 + hk$,$h > 0$ then FFN is calculated as:

$$FFN = \frac{1}{N}\sqrt{\sum_{k=1}^{N}\left(Y(x_k) - f(x_k) - \int_a^b K(x_k,t)Y(t)dt\right)^2}. \tag{4}$$

3.4. Proposed Algorithm

In this section, the proposed algorithm (BAC), which is a combination of Bees algorithm and Chebyshev polynomials for solving Fredholm integral equation (FIEs), and it is illustrated in the following steps:

step (1). Approximating the unknown $y(x)$ using Chebyshev polynomials:

$$y(x) \sim \sum_{i=0}^{nVar-1} \alpha_i T_i(x). \tag{5}$$

step (2). Determine the error function: see Eq (3).

step (3). Determine the fitness function (FFN): see Eq (6)

step (4). Applying the BA algorithm 2.2 to minimize the fitness function.

step (5). Evaluating the fitness function FFN and MEA, MEA is calculated as follows:

$$MEA = \frac{1}{N} \sum_{k=1}^{N} |y(x_x) - Y_{approx}(x_k)|. \tag{6}$$

step (6). if FFN $<$ TOL STOP and give the CHP , else GOTO step(4).

4. Results and Discussions

In this section, numerical experiments are conducted in order to approve the validity and the applicability of the proposed algorithm (BAC).

The algorithm is executed 10 times to verify the reliability of the results, and it is implemented using MATLAB 2020a in Laptop HP Intel (R) Core (TM) i7-4500U CPU @ 1.80GHz 2.40GHz with 16GB RAM 64bit system.

Fredholm integral equation examples with their exact solution are provided in Table 1.

Table 1: shows the examples of testing [13, 17]

Names	Equations	Solution
FIE1	$y(x) = 1 + \int_{-1}^{1} y(t)(xt + x^2t^2)dt$	$y = 1 + \frac{10}{9}x^2$
FIE2	$y(x) = e^x - x + x \int_0^1 ty(t)dt$	$y = e^x$
FIE3	$y(x) = \sin(x) + x - x \int_0^{\pi/2} y(t)dt$	$y = \sin(x)$

Table 2: Parameters and inputs values

Tol	VarMin	VarMax	Maxit
1e-03	-1	1	100
nVar	Scout Bee(s)	elite sites (el)	Selected site (b)
5	20	4	10

Parameters and inputs for all examples are given in Table 2. These values are adopted in this work in order to obtain satisfactory results. The results evaluation criteria are time consuming, number of iterations, and the accuracy comparison to the exact solution which can be evaluated depends on the values of the mean absolute error (MEA). Number of variables (nVar) is the number of Chebyshev coefficients, increasing the value of this parameter leads to achieve more accurate results, while it is computationally expensive. Also, it can be observed that increasing the number of variables requires reducing the search space to obtain more accurate results. Moreover, expanding the search space affect the algorithm speed with unacceptable accuracy of the results with high value of (MEA), and also requires more number of iterations and time.

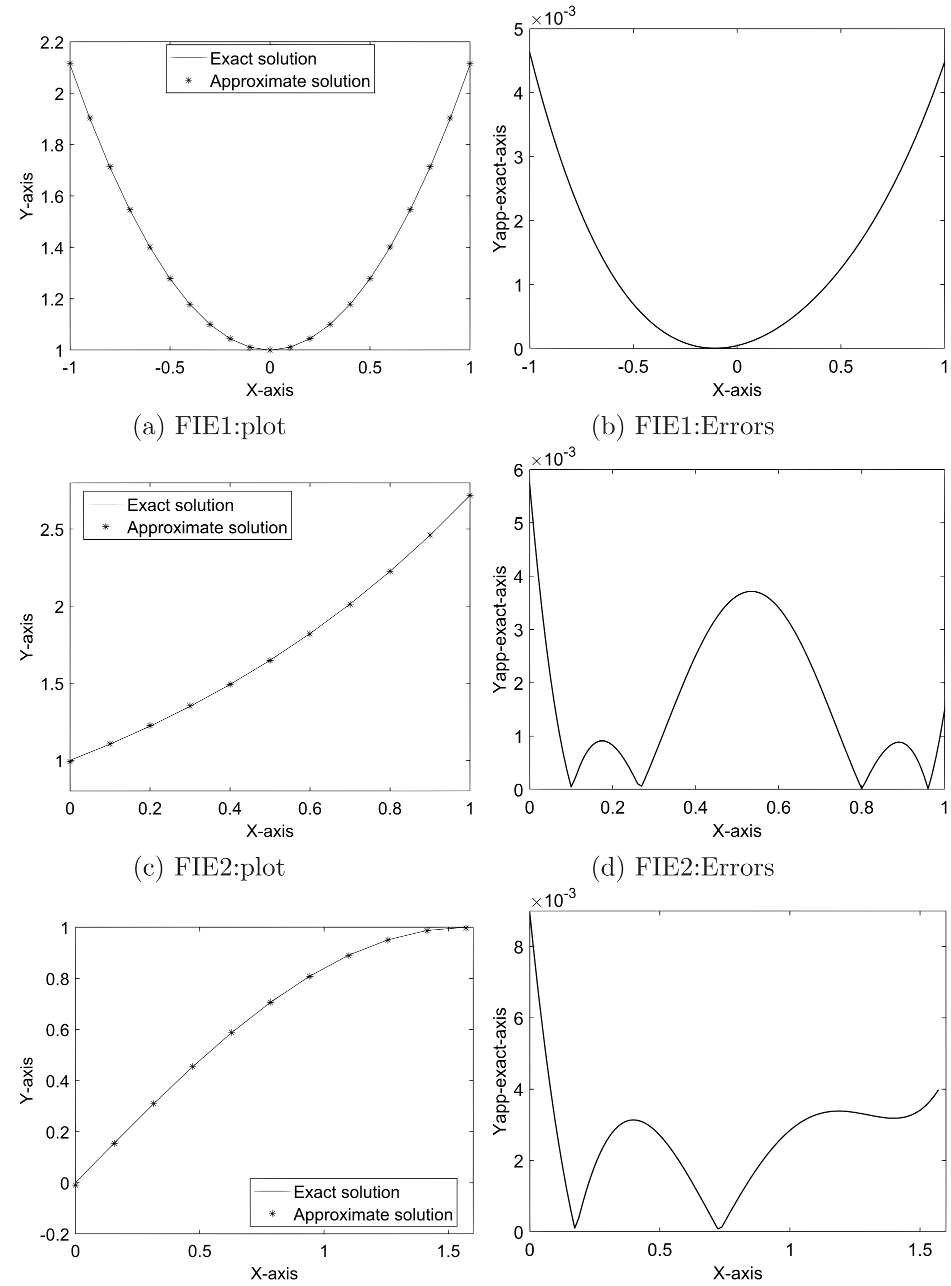
Figure 1 represents the approximate solution corresponding to the exact solution with errors arising from the proposed algorithm in all examples. The approximate solutions show an perfect agreement compared to exact solution. The convergence of the proposed algorithm is also included and confirmed in all examples with maximum 47 iterations (See Figure 2). Finally, the output of the proposed algorithm, which are the Chebyshev polynomials coefficients and the values of the mean absolute error (MEA), the number of iterations, and the consumed time to reach the approximate solution are given in Table 3. Although, the results are convergent, the values in Table 3 varies for each time of the implementation of the algorithm due to its stochastic nature.

Table 3: presents the approximate solution generated by proposed algorithm, the consumed time (time (s)),number of iteration(NOI) and the values of MEA

Approximate solution	time (s)	NOI	MEA
FIE1: $Y(x) = 1.5577\ T_0(x)$ $+0.0001\ T_1(x) + 0.5578\ T_2(x)$ $-0.0002\ T_3(x) + 0.0001\ T_4(x)$	79.53	30	1.4e-03
FIE2: $Y(x) = 0.9683\ T_0(x)$ $+1.6430\ T_1(x) - 0.0523\ T_2(x)$ $+0.1840\ T_3(x) - 0.0263\ T_4(x)$	63.88	23	1.7e-03
FIE3: $Y(x) = -0.0771\ T_0(x)$ $+1.0060\ T_1(x) - 0.0681\ T_2(x)$ $-0.0221\ T_3(x)$	93.7	47	2.7e-03

5. Conclusion

The algorithm proposed in this work performs well and it is an alternative to classical numerical methods for solving Fredholm integral equations. The equations were formulated as unconstrained optimization problem. This was appli-

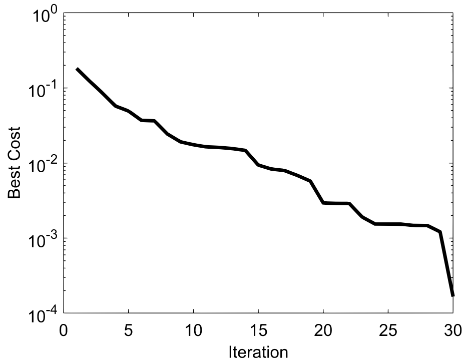


(a) FIE1:plot (b) FIE1:Errors

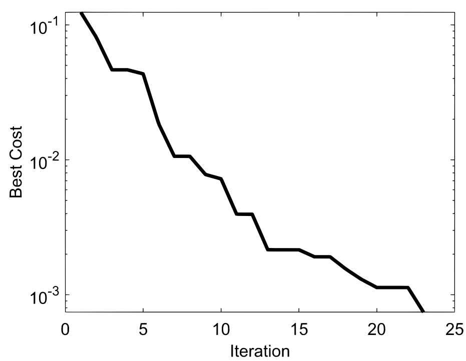
(c) FIE2:plot (d) FIE2:Errors

(e) FIE3:plot (f) FIE3:Errors

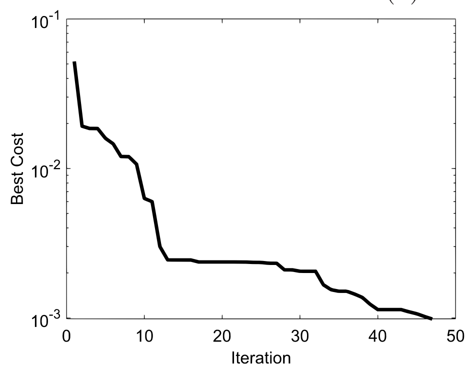
Figure 1: (A)-(C)-(E) show a comparison between exact and approximate solutions. (B)-(D-(F) show errors of the resulting approximate solution with the corresponding exact solutions.



(a) FIE1



(b) FIE2



(c) FIE3

Figure 2: shows the convergence of the proposed algorithm in all examples.

cable due to the possibility of including Chebyshev polynomials, with unknown coefficients, in the Bees algorithm. The coefficients of Chebyshev polynomials then optimized to approximate the solution. Numerical examples were solved and the results show an excellent agreement compare to the exact solution. Convergence of the proposed algorithm was also confirmed.

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