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MODULES WHOSE PRIMARY-LIKE SUBMODULES ARE INTERSECTION OF MAXIMAL SUBMODULES

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Abstract: Let R be a commutative ring with identity and M be an unitary R-module. A ring R in which every prime ideal is an intersection of maximal ideals is called Hilbert (or Jacobson) ring. We propose to define modules by the property that primary-like submodules are intersections of maximal submodules which are said to be $\mathcal{PH}$ modules. It is shown that every co-semisimple module is a $\mathcal{PH}$ module. Also, it is shown that an R-module M is a $\mathcal{PH}$ module if and only if every non-maximal primary-like submodule of M is an intersection of properly larger primary-like submodules.

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1. Introduction

In this paper, all rings are commutative with identity and all modules are unitary. Let N be a submodule of M. Then $(N : M)$ denote the ideal $\{r \in R \mid rM \subseteq N\}$. A proper submodule P of an R-module M is said to be p-prime submodule, if $rm \in P$ for $r \in R$ and $m \in M$, then either $m \in P$ or $r \in p := (P : M)$. The set of all prime submodules of M is denoted by $Spec(M)$ [13]. Note that the $Spec(M)$ may be empty for some module M. Let N be a submodule of M. Then the intersection of all prime submodules of M containing N is called the radical of N and denoted by $radN$. If there is no prime submodule containing N, then we define $radN = M$. A proper submod-

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ule Q of M is said to be p-primary-like if $rm \in Q$ implies $r \in p := (Q : M)$ or $m \in radQ$ [3]. Let N be a submodule of a nonzero R-module M. We say that N satisfies the primeful property, if for each prime ideal p of R with $(N : M) \subseteq p$, there exists a prime submodule P of M containing N such that $(P : M) = p$. If N is a submodule of M satisfying the primeful property, then $(radN : M) = \sqrt{(N : M)}$ [11, Proposition 5.3]. An R-module M is called primeful, if either $M = 0$ or the zero submodule of M satisfies the primeful property. For instance, finitely generated modules, projective modules over integral domains, and vector spaces are primeful [11]. The primary-like spectrum $Spec_L(M)$ for an R-module M is defined to be the set of all primary-like submodules of M satisfying the primeful property [4]. If $Q \in Spec_L(M)$, since Q satisfies the primeful property, then there exists a maximal ideal m of R and a prime submodule P of M containing Q such that $(P : M) = m$. Then $P \in V(Q)$, and so $Q \neq M$. In particular, $radQ$ satisfies the primeful property. Moreover, it is proved that if $Q \in Spec_L(M)$, then $p := \sqrt{(Q : M)}$ is a prime ideal of R [4]. If R is a commutative ring and each prime ideal of R is an intersection of maximal ideals, then R is called a Hilbert ring. If R is a Hilbert ring, then the polynomial ring $R[x_1, \ldots, x_2]$ is also a Hilbert ring [2, 5, 6, 7]. Hilbert rings were extended to noncommutative rings in [9].

A generalization of commutative Hilbert rings to modules was extended in [1] and [12]. In this paper, we extend the notion of commutative Hilbert rings to modules via primary-like submodules and study some properties of $\mathcal{PH}$ modules. An R-module M is a $\mathcal{PH}$ module if every primary-like submodule of M is an intersection of maximal submodules. It is clear that any $\mathcal{PH}$ module is a Hilbert module. We show that if $\oplus_{i\in I} M_i$ is a $\mathcal{PH}$ module, then each M_i is a $\mathcal{PH}$ module (Corollary 6). Also, it is shown that M is a $\mathcal{PH}$ module if and only if every non-maximal primary-like submodule of M is an intersection of properly larger primary-like submodules (Theorem 7). Finally, it is shown that if R be a domain and M be a $\mathcal{PH}$ module over R such that every homomorphic image M/N of M is a torsion-free R-module, then N is a $\mathcal{PH}$ module (Theorem 12).

2. On $\mathcal{PH}$ modules

Let M be an R-module. Since each prime submodule of M is a primary-like submodule, each $\mathcal{PH}$ module is a Hilbert module. The following example shows that the converse of this fact is not true in general.

Example 1. Let $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$. Then M is not a multiplication Z-module and $Spec(M) = pM = \mathbb{Z}(p^\infty) \oplus 0$ [13, Example 3.7]. By an easy verification for submodule $N = 0 \oplus \mathbb{Z}_p$ of M, we have $radN = M$ and $(N : M) = 0$, and so N is a primary-like submodule of M that is not a prime submodule of M. It is easy to see that M is a Hilbert $\mathbb{Z}$-module, but it is not a $\mathcal{PH}$ module.

Let M and M' be R-modules. Then M is called M'-injective if for any submodule N' of M' each homomorphism $N' \longrightarrow M$ can be extended to $M' \longrightarrow M$. An R-module M is called co-semisimple if every simple module is M-injective [14, Chap. 4, Sec. 23].

Proposition 2. *Let M be a co-semisimple R-module. Then M is a $\mathcal{PH}$ module.*

Proof. Suppose that M is a co-semisimple R-module. Hence every proper submodule of M is an intersection of maximal submodules by [14, Proposition 23.1]. Therefore M is a $\mathcal{PH}$ module. □

We recall that an R-module M is called semisimple if M is a direct sum of simple submodules [14].

Proposition 3. *Let M be a semisimple R-module. Then M is a $\mathcal{PH}$ module.*

Proof. Suppose that M is a semisimple R-module. Hence M is co-semisimple, by [14, Proposition 23.1]. Therefore M is a $\mathcal{PH}$ module by Proposition 2. □

Proposition 4. *Any homomorphic image of a $\mathcal{PH}$ module is a $\mathcal{PH}$ module.*

Proof. Since a proper submodule Q of M with $N \subseteq Q$ is a primary-like (resp., maximal) submodule of M if and only if Q/N is a primay-like (resp., maximal) submodule of the factor module M/N [3, Corollary 3.5], the assertion is clear. □

Corollary 5. *Let M be an R-module. Then the following statements are equivalent.*

(1) M is a $\mathcal{PH}$ module.

(2) If N is a submodule of M, then M/N is a $\mathcal{PH}$ module.

(3) If N is a minimal primary-like submodule of M, then M/N is a $\mathcal{PH}$ module.

Proof. (1) $\Longrightarrow$ (2) follows from Proposition 4.
(2) $\Longrightarrow$ (3) is clear.
(3) $\Longrightarrow$ (1) Assume that Q is a primary-like submodule of M and $\{Q_i\}_{i\in I}$ is a chain of primary-like submodules of M. Thus $\bigcap_{i\in I} Q_i$ is a primary-like submodule. Therefore Q contains a minimal primary-like submodule Q_0 of M, by Zorn′s lemma. Thus Q/Q_0 is an intersection of maximal submodules of M/Q_0. Then M is a $\mathcal{PH}$ module. □

Corollary 6. *Let $\{M_i\}_{i\in I}$ be a family of R-modules and $\oplus_{i\in I} M_i$ be a $\mathcal{PH}$ module. Then every M_i is a $\mathcal{PH}$ module.*

Proof. Follows from Proposition 4. □

Theorem 7. *An R-module M is a $\mathcal{PH}$ module if and only if every non-maximal primary-like submodule of M is an intersection of properly larger primary-like submodules.*

Proof. Suppose that M is a $\mathcal{PH}$ module. Since any maximal submodule is primary-like, the assertion holds. Conversely, assume that N is a primary-like submodule which is not a maximal submodule and $x \in M\backslash N$. Suppose that $\sum = \{Q \in Spec_L(M) \mid N \subseteq Q, x \notin Q\}$. Since $N \in \sum$, $\sum \neq \emptyset$. Hence $\sum$ has a maximal element, by Zorn's Lemma. Let Q' be a maximal element in $\sum$. Then Q' must be a maximal submodule. If Q' is not a maximal submodule, Q' is the intersection of properly larger primary-like submodules. Since Q' is maximal element in $\sum$, all properly larger primary-like submodules containing Q' must contain x, and so $x \in Q'$, a contradiction. Thus Q' is a maximal submodule. Hence N is the intersection of all maximal submodules of M containing N. □

Lemma 8. *Let M be an R-module and let I be an ideal of R such that $I \subseteq Ann(M)$. Then M is a $\mathcal{PH}$ as an R-module if and only if M is a $\mathcal{PH}$ as an R/I-module.*

Proof. It is clear. □

Proposition 9. *Let M be an R-module and $Q \in Spec_L(M)$. Then $(Q : M)$ is a primary ideal of R, and so $\sqrt{(Q : M)}$ is a prime ideal of R.*

Proof. Suppose that $rs \in (Q : M)$ and $r \notin (Q : M)$ for some $r, s \in R$. Hence $s \in (\operatorname{rad} Q : M)$, since Q is a primary-like submodule of M. Now, $s \in \sqrt{(Q : M)}$, by [11, Proposition 5.3]. □

Proposition 10. *Let M be an R-module. Consider the following statements:*

(1) M is a $\mathcal{PH}$ R-module.

(2) $M/(Nil(R)M)$ is a $\mathcal{PH}$ R-module.

(3) $M/(Nil(R)M)$ is a $\mathcal{PH}$ $R/Nil(R)$-module.

Then (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). Furtheremore, if all primary-like submodules of M satisfy the primeful property and the ideal $P := (Q : M)$ is a radical ideal of R for each $Q \in Spec_L(M)$, then the above statements are equivalent.

Proof. (1) $\Longrightarrow$ (2) It follows from Corollary 4.
(2) $\Longleftrightarrow$ (3) It follows from Lemma 8.
(3) $\Longrightarrow$ (1) Assume that $Q \in Spec_L(M)$ and the ideal $P := (Q : M)$ is a radical ideal of R. Then $(Q : M) = P$ is a prime ideal of R by Proposition 9. Therefore $PM \subseteq Q$. So $Nil(R)M \subseteq Q$. Since $Q/Nil(R)M$ is a primary-like submodule of $M/Nil(R)M$, $Q/Nil(R)M = \bigcap_{i \in I}(M_i/Nil(R)M)$ where each $M_i/Nil(R)M$ is a maximal submodule of $M/Nil(R)M$. Hence $Q = \bigcap_{i \in I} M_i$, and so M is a $\mathcal{PH}$ module.. □

Let M be an R-module. Then $J_R(M)$ is the intersection of all maximal submodules of M.

Proposition 11. *Let M be an R-module. Then the following statements are equivalent:*

(1) M is a $\mathcal{PH}$ module as an R-module.

(2) If all primary-like submodules of M satisfy the primeful property and the ideal $P := (Q : M)$ is a radical ideal for each $Q \in Spec_L(M)$, then M/Q is a $\mathcal{PH}$ module as an R/P-module for every $Q \in Spec_L(M)$.

(3) *If all primary-like submodules of M satisfying the primeful property and the ideal $P := (Q : M)$ is a radical ideal for each $Q \in Spec_L(M)$, then $J_R(M/Q) = 0$ for every $Q \in Spec_L(M)$.*

Proof. (1) $\Longrightarrow$ (2) Let $Q \in Spec_L(M)$ and let $P := (Q : M)$. Then by Corollary 5, M/Q is a $\mathcal{PH}$ module. Since $P = Ann(M/Q)$, the assertion holds, by Lemma 8.
(2) $\Longrightarrow$ (3) Let $Q \in Spec_L(M)$ such that $P = (Q : M)$. Since the zero submodule of the R/P-module M/Q is a primary-like submodule, $J_{R/P}(M/Q) = 0$. Since $J_{R/P}(M/Q) = J_R(M/Q)$, $J_R(M/Q) = 0$.
(3) $\Longrightarrow$ (1) is obvious. □

Theorem 12. *Let R be an integral domain and M be a $\mathcal{PH}$ module as an R-module. If N is a submodule of M such that M/N is torsion-free, then N is also a $\mathcal{PH}$ module.*

Proof. Suppose that Q is a primary-like submodule of N. Suppose that $rm \in Q$ for some $r \in R$ and $m \in M$. If $m \in N$, then since Q is a primary-like submodule of N, $r \in (Q : N)$ or $m \in radQ$. Suppose that $x \notin N$. Since M/N is torsion-free and $m \notin N$, $r = 0$. Therefore $r \in (Q : M)$, and so Q is a primary-like submodule of M. Since M is a $\mathcal{PH}$ module, there exists a family of maximal submodule $\{M_i\}_{i\in I}$ of M such that $Q = \bigcap_{i\in I} M_i$. Let $Q_i := M_i \cap N$ for each $i \in I$. Since Q is a submodule of N, it is easy to see that $Q = \bigcap_{i\in I} Q_i$. Now, assume that $x \in N \setminus Q_i$. We will show that $\langle Q_i, x\rangle = N$. Since $x \notin Q_i$, $x \notin M_i$ and M_i is a maximal submodule of M, $\langle M_i, x\rangle = M$. Let $y \in N$. Since $\langle M_i, x\rangle = M$, $y = x_i + rx$ for some $x_i \in M_i$ and $r \in R$. Since $y \in N$ and $x \in N$, $x_i \in N$. Thus $x_i \in Q_i$, and so $y \in \langle Q_i, x\rangle$. Therefore Q_i is a maximal submodule of N for each i. □

Recall that a submodule N of an R-module M is called pure if $IN = N \cap IM$, for every ideal I of R. The torsion submodule of a module M over an integral domain R, denoted by $T(M)$, is the submodule $\{m \in M : Ann(m) \neq 0\}$ of M. An R-module M is said to be torsion (resp. torsion-free), if $T(M) = M$ (resp. $T(M) = 0$).

Corollary 13. *Let R be an integral domain and M be a $\mathcal{PH}$ module. Then the following statements hold:*

(1) *$T(M)$ is a $\mathcal{PH}$ module.*

(2) If M is torsion-free and N is a pure submodule of M, then N is also a $\mathcal{PH}$ module.

Proof. (1) It is clear by Theorem 12.
(2) Assume that M is torsion-free and N is a pure submodule of M. Suppose that $x \in M \setminus N$ and $rx \in N$. Since N is pure, $rM \cap N = rN$. Thus $rx \in rN$, and so there is $y \in N$ such that $rx = ry$. But then $r(x - y) = 0$. Since $x \notin N$, $x - y \neq 0$. As M is torsion-free, we conclude that $r = 0$. Therefore M/N is a torsion-free R-module, and so the assertion holds by Theorem 12. □

Lemma 14. *Let M be an R-module and $Q \in Spec_L(M)$. Then $\operatorname{rad} Q \in Spec(M)$ if and only if $T(\frac{M}{\operatorname{rad} Q}) = 0$ as an $\frac{R}{\sqrt{(Q:M)}}$-module.*

Proof. Suppose $Q \in Spec_L(M)$. Hence by Proposition 9,

$$\sqrt{(Q : M)} = (\operatorname{rad} Q : M)$$

is a prime ideal of R, and so the proof follows from [8, Lemma 1]. □

Proposition 15. *Let M be a module over a Dedekind domain R and $Q \in Spec_L(M)$ such that $M = radQ \oplus N$ for some torsion-free submodule N of M. Then $\operatorname{rad} Q$ is a prime submodule of M.*

Proof. Suppose that $M = \operatorname{rad} Q \bigoplus N$ for some torsion-free submodule N of M. Then $\frac{M}{\operatorname{rad} Q} \cong N$, and so $\frac{M}{\operatorname{rad} Q}$ is torsion-free. Hence $\operatorname{rad} Q$ a prime submodule of M, by [8, Lemma 1]. □

Proposition 16. *Let R be a Dedekind domain which freely generated by a set of indeterminates $\{x_i\}$ over a division ring K and M be a finitely generated R-module. Let $Q \in Spec_L(M)$ such that $M = radQ \oplus N$ for some torsion-free submodule N of M. Then $radQ$ is an intersection of maximal submodules.*

Proof. Since R freely generated by a set of indeterminates $\{x_i\}$ over a division ring k, $J(R) = (0)$ by [10, Corollary 4.16]. Hence R is a Hilbert ring. Since M is a finitely generated R-module, M is a Hilbert module, by [12, Proposition 2.9]. Since R is a Dedekind domain and $Q \in Spec_L(M)$ such that $M = radQ \oplus N$ for some torsion-free submodule N of M, $radQ$ is a prime submodule of M by Proposition 15. Therefore $radQ$ is an intersection of maximal submodules. □

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