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ON RANDOM MAPS CORRELATED WITH RANDOM DENSITIES

Mohamed Hmissi[1 §], Farida Mokchaha[2]

[1]College of Science, Department of Mathematics and Statistics
Imam Muhammad Ibn Saud Islamic University
Riyadh-11623, SAUDI ARABIA

[2]College of Science, Department of Mathematics and Statistics
Imam Muhammad Ibn Saud Islamic University
Riyadh-11623, SAUDI ARABIA

Abstract: Let $q = (q_{ij}) : 1 \leq i \leq I, 1 \leq j \leq J$ be a bivariate probability vector, let $T = (T_1, \cdots, T_I)$ be a sequence of ρ-nonsingular transformations defined on a probability space $(E, \mathcal{B}, \rho)$ and let $\mathtt{f"} = (f_1, \cdots, f_J)$ be a sequence of densities in $L^1(\rho)$. In this paper, we construct in a natural way, a discrete random dynamical system (with skew product Φ) generated by T and the first marginal of q and a random density ξ generated by $\mathtt{f"}$ and the second marginal of q. Moreover, we characterize the Φ-invariance of ξ.

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1. Introduction

Let $\mathbb{I} = \{1, \cdots, I\}; \mathbb{J} = \{1, \cdots, J\}$ and $q = (q_{ij}) : i \in \mathbb{I}, j \in \mathbb{J}$ be an associated bivariate probability vector. Define $\Omega := (\mathbb{I} \times \mathbb{J})^{\mathbb{N}}$, $\pi_n : \Omega \to \mathbb{I} \times \mathbb{J}$ be the canonical projection of index n, $\mathcal{F} := \sigma(\pi_n : n \in \mathbb{N})$ the σ-algebra on Ω generated by

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§Correspondence author

all canonical projections, and $\theta : \Omega \to \Omega$ be the left shift on Ω. Using a classical extension Theorem of Kolmogorov, we construct first (Proposition 1) a probability measure $\mathbf{P}$ on the measurable space $(\Omega, \mathcal{F})$ such that the left shift θ is $\mathbf{P}$-preserving.

Let $(E, \mathcal{B}, \rho)$ be a probability space and let $T = (T_i) : i \in \mathbb{I}$ be a sequence of ρ-nonsingular transformations defined on $(E, \mathcal{B}, \rho)$. By putting $a_i = \sum_{j=1}^{J} q_{ij}$ (the first marginal of q) and

$$\phi(\omega, x) \;=\; T_i(x) \quad \text{with probability} \quad a_i; \quad i \in \mathbb{I}, x \in E,$$

we define a random dynamical system (θ, ϕ) such that the associated skew product Φ is $(\mathbf{P} \otimes \rho)$-nonsingular (Propositions 4, 6).

Let $\mathtt{f} = (f_j) : j \in \mathbb{J}$ be a sequence of densities in $L^1(\rho)$. By putting $b_j = \sum_{i=1}^{I} q_{ij}$ (the second marginal of q) and

$$\xi(\omega, x) \;=\; T_j(x) \quad \text{with probability} \quad b_j; \quad j \in \mathbb{J}, x \in E,$$

we obtain a random density on the product space $(\Omega \times E, \mathcal{F} \otimes \mathcal{B}, \mathbf{P} \otimes \rho)$ (Propositions 7, 9).

Finally, we prove (Theorem 11) that ξ is Φ-invariant, that is

$$\int\int_{\Phi^{-1}(F\times B)} \xi d\rho d\mathbf{P} = \int\int_{F\times B} \xi d\rho d\mathbf{P}; \quad F \in \mathcal{F}, B \in \mathcal{B},$$

if and only if $f_1, \cdots, f_J$ are identical (to some $f \in L^1(\rho)$) and

$$\sum_{i=1}^{I} a_i P_{T_i} f \;=\; f,$$

where P_{T_i} is the Frobeinus-Perron operator of the deterministic map T_i.

Notice that if the marginals (a_i) and (b_j) are identical and independent, the preceding result is proved (Theorem 7) in our paper [6].

The paper is organized as follows: In the second section, we give the background required to establish our main results according to [1],[2],[3],[4],[12]. In the third section, we set and prove the results cited above.

2. Preliminaries

2.1. Densities for deterministic maps

For the following classical concepts, we refer the reader to the monographs [3],[4], [8], [12]. In fact, since the present paper is a generalization of our article [6], all preliminaries are taken from this paper.

Let (Y, Γ, μ) be a probability space and $L^1(\mu)$ be the Banach space of all (classes of) μ-integrable real-valued functions defined on Y with the L^1-norm $\|f\|_1 = \int_Y |f| d\mu$. A function $f \in L^1(\mu)$ is called density if $f \geq 0$ and $\int_Y f d\mu = 1$. If $\psi : Y \to Y$ is a measurable transformation, the triple (Y, Γ, ψ) is called a discrete dynamical system (DS). Denote by $\psi\mu$ the image measure of μ by ψ, that is, $\psi\mu(A) := \mu(\psi^{-1}(A)); A \in \Gamma$. If $\psi\mu = \mu$, ψ is said to be *μ-preserving*. If $\psi\mu$ is absolutely continuous with respect to μ (i.e. $\psi\mu << \mu$), ψ is said to be *μ-nonsingular*.

Let ψ be a μ-nonsingular transformation of the probability space (Y, Γ, μ). The associated *Frobenius-Perron (F-P) operator* $P_\psi : L^1(\mu) \to L^1(\mu)$ is implicitly defined by the formula

$$\int_A P_\psi f \; d\mu \;=\; \int_{\psi^{-1}(A)} f \; d\mu; \qquad A \in \Gamma, f \in L^1(\mu). \tag{1}$$

It is well known that P_ψ is a positive linear contraction on $L^1(\mu)$.

A density $f \in L^1(\mu)$ is a fixed point of P_ψ (i.e. $P_\psi f = f$) if and only if the measure μ_f defined by $\mu_f(A) := \int_A f(y) d\mu(y); (A \in \Gamma)$, is invariant under ψ, that is, $\psi\mu_f = \mu_f$. In this case, f is called *invariant density for ψ*. In view of formula (1), f is an invariant density for ψ if and only if f is solution of the functional equation

$$\int_{\psi^{-1}(A)} f \; d\mu \;=\; \int_A f \; d\mu; \qquad A \in \Gamma. \tag{2}$$

Many authors [3], [4], [8], [12], [15] have shown the existence of invariant densities for deterministic maps in a variety of settings.

2.2. Random densities for random dynamical systems

For the following notions, we will refer to [1], [6], [11].

Let $(\Omega, \mathcal{F}, \mathbf{P})$ and $(E, \mathcal{B}, \rho)$ be two probability spaces. Ω is always an infinite dimensional space while E is a finite dimensional space endowed with its Borel σ-algebra $\mathcal{B}$. Next, we will consider the product space $Y := \Omega \times E$ endowed with the product σ-algebra $\mathcal{F}\otimes\mathcal{B}$ and the probability measure $\mu=\mathbf{P}\otimes\rho$. Any function $\xi\in L^1(\mathbf{P}\otimes\rho)$ is called a *random function*. A nonnegative random function satisfying $\int_{\Omega\times E}\xi\,d\mathbf{P}d\rho=1$, is called a *random density*. If $\xi(\omega,x)=f(x);\omega\in\Omega,x\in E$, for some density $f\in L^1(\rho)$, ξ is called a *deterministic density*.

A (discrete time) random dynamical system (RDS) defined over $(\Omega,\mathcal{F},\mathbf{P})$ and with state space E, is a pair (θ,ϕ) such that:
(i) $(\Omega,\mathcal{F},\mathbf{P},\theta)$ is a *metric* DS, i.e. $(\Omega,\mathcal{F},\theta)$ is a measurable DS and θ is $\mathbf{P}$-preserving.

(ii) $\phi:\Omega\times E\to E,(\omega,x)\mapsto\phi(\omega,x)$ is a measurable map.

In particular, $\{\phi(\omega):=\phi(\omega,.):\omega\in\Omega\}$ is a family of measurable transformations on E, called *fiber maps*. The associated *skew product* is the measurable map $\Phi:\Omega\times E\to\Omega\times E$ defined by

$$\Phi(\omega,x):=(\theta\omega,\phi(\omega)x);\quad \omega\in\Omega,x\in E. \tag{3}$$

It can be easily verified that for $F\in\mathcal{F}$ and $B\in\mathcal{B}$

$$\Phi^{-1}(F\times B)=\{(\omega,x)\in\Omega\times E:\omega\in\theta^{-1}(F),x\in\phi(\omega)^{-1}(B).\} \tag{4}$$

If Φ is $(\mathbf{P}\otimes\rho)$-nonsingular then, in view of equations (1) and (4), the associated F-P operator is implicitly defined by the formula

$$\int_F\int_B P_\Phi\xi\;d\rho\,d\mathbf{P}=\int_{\theta^{-1}(F)}\left(\int_{\phi(\omega)^{-1}(B)}\xi(\omega,x)d\rho(x)\right)d\mathbf{P}(\omega) \tag{5}$$

for all $F\in\mathcal{F},B\in\mathcal{B}$ and $\xi\in L^1(\mathbf{P}\otimes\rho))$. Therefore, for a general RDS (θ,ϕ), an invariant random density β is solution of the functional equation

$$\int_{\theta^{-1}(F)}\left(\int_{\phi(\omega)^{-1}(B)}\beta(\omega,x)d\rho(x)\right)d\mathbf{P}(\omega)\;=\;\int_F\int_B\beta\;d\rho d\mathbf{P} \tag{6}$$

for all $F\in\mathcal{F},B\in\mathcal{B}$. Equation (6) which is the random version of (2) was introduced in [11]. However, this equation seems to be very complicated to handle in general settings and there are only some restrictive results on this subject (cf. [2], [5], [9], [11]).

3. Random maps correlated with random densities

In this paper, we start from:
(i) A probability space $(E, \mathcal{B}, \rho)$.
(ii) A bivariate probability vector $q = (q_{ij}) : 1 \le i \le J, 1 \le j \le J$.
(iii) A finite sequence $T = (T_1, \cdots, T_I)$ of ρ-nonsingular transformations defined on $(E, \mathcal{B}, \rho)$.
(iv) A finite sequence $\mathtt{f} = (f_1, \cdots f_J)$ of deterministic densities in $L^1(\rho)$.

We then construct in a natural way, a discrete random dynamical system such that, the random map generated by T and the random density defined by $\mathtt{f}$, are correlated by the bivariate probability vector q.

3.1. DS generated by a bivariate distribution

Denote by $\mathbb{N} := \{0, 1, 2, \cdots\}$ and for $I \ge 2, J \ge 2$, let $\mathbb{I} := \{1, \cdots, I\}$ and $\mathbb{J} := \{1, \cdots, J\}$. Let $q := (q_{ij}); i \in \mathbb{I}, j \in \mathbb{J}$ be a *bivariate probability vector*, that is $q_{ij} \in (0, 1)$ for all i, j and $\sum_{i=1}^{I} \sum_{j=1}^{J} q_{ij} = 1$.

Define $\Omega := \{\omega : \mathbb{N} \to \mathbb{I} \times \mathbb{J}, n \mapsto \omega(n)\}$ and $\pi_n : \Omega \to \mathbb{I} \times \mathbb{J}$ the canonical projection of index $n \in \mathbb{N}$ (i.e. $\pi_n(\omega) = \omega(n)$). Let $\mathcal{F} := \sigma(\pi_n : n \in \mathbb{N})$ be the σ-algebra on Ω generated by all canonical projections. It is known that $\mathcal{F}$ is generated by the sets of the form

$$\{(\pi_0 = (i_0, j_0), \cdots, \pi_n = (i_n, j_n)) : n \in \mathbb{N}; \tag{7}$$
$$i_0, \cdots, i_n \in \mathbb{I}; j_0, \cdots, j_n \in \mathbb{J}.\} \tag{8}$$

Now let $\theta : \Omega \to \Omega$ be the left shift on Ω, that is $(\theta\omega)(n) = \omega(n+1)$. In others words

$$\theta^{-1}\{(\pi_0 = (i_0, j_0), \cdots, \pi_n = (i_n, j_n))\} = \tag{9}$$
$$\{(\pi_1 = (i_0, j_0), \cdots, \pi_{n+1} = (i_n, j_n))\} \tag{10}$$

for all $n \in \mathbb{N}; i_0, \cdots, i_n \in \mathbb{I}; j_0, \cdots, j_n \in \mathbb{J}$.

Proposition 1. *There exists a unique probability measure* $\mathbf{P}$ *on the measurable space* $(\Omega, \mathcal{F})$ *such that for all* $n \in \mathbb{N}; i_0, \cdots, i_n \in \mathbb{I}$ *and* $j_0, \cdots, j_n \in \mathbb{J}$

$$\mathbf{P}(\pi_0 = (i_0, j_0), \cdots, \pi_n = (i_n, j_n)) \;=\; q_{i_0 j_0} \cdots q_{i_n j_n} \tag{11}$$

In particular, the random variables $\{\pi_n : n \in \mathbb{N}\}$ *are independent.*

Moreover, the left shift θ *is* $\mathbf{P}$*-preserving.*

Proof. The family of finite distributions $P_n : n \geq 1$ defined on $(\mathbb{I} \times \mathbb{J})^n$ by

$$P_n((i_0, j_0) \cdots, (i_n, j_n)) \;=\; q_{i_0 j_0} \cdots q_{i_n j_n}$$

is trivially consistent, that is $P_{n+1}(A_n \times (\mathbb{I} \times \mathbb{J})) = P_n(A_n)$ for any $n \geq 1$ and $A_n \subset (\mathbb{I} \times \mathbb{J})^n$. Hence by Kolmogorov's extension Theorem ([10], Fundamental Theorem), there exists a unique probability measure $\mathbf{P}$ on the measurable space $(\Omega, \mathcal{F})$ such that (11) holds. Moreover, by taking F of the form (7) and by using (9) and (11) we get

$$\begin{aligned}
(\theta\mathbf{P})(F) &= (\theta\mathbf{P})(\pi_0 = (i_0, j_0), \cdots, \pi_n = (i_n, j_n)) \\
&= \mathbf{P}(\theta^{-1}\{\pi_0 = (i_0, j_0)\}, \cdots, \theta^{-1}\{\pi_n = (i_n, j_n)\}) \\
&= \mathbf{P}(\pi_1 = (i_0, j_0)\}, \cdots, \pi_{n+1} = (i_n, j_n)) \\
&= q_{i_0 j_0} \cdots q_{i_n j_n} = \mathbf{P}(\pi_0 = (i_0, j_0) \cdots, \pi_n = (i_n, j_n))
\end{aligned}$$

for all $n \in \mathbb{N}, i_0, \cdots, i_n \in \mathbb{I}$ and $j_0, \cdots, j_n \in \mathbb{J}$. Hence, in view of (7), θ is $\mathbf{P}$-preserving. □

Definition 2. The metric DS $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ defined above, is called *canonical* dynamical system generated by the bivariate probability vector $q = (q_{ij}); i \in \mathbb{I}, j \in \mathbb{J}$.

Before we continue, let us fix some notations:
1- Since the canonical projection $\pi_n : \Omega \to \mathbb{I} \times \mathbb{J}$, we write $\pi_n := (\pi_n^{(1)}, \pi_n^{(2)})$ where $\pi_n^{(1)} : \Omega \to \mathbb{I}$ and $\pi_n^{(2)} : \Omega \to \mathbb{J}$, in a natural way.
2- Since $q = (q_{ij}) : 1 \leq i \leq I, 1 \leq j \leq J$ is a bivariate probability vector, we may associate its marginals

$$a_i \;=\; \sum_{j=1}^{J} q_{ij}; \quad 1 \leq i \leq I \text{ and } b_j \;=\; \sum_{i=1}^{I} q_{ij}; \quad 1 \leq j \leq J \tag{12}$$

Obviously $(a_1, \cdots, a_I)$ (resp. $(b_1, \cdots, b_J)$) is a probability vector on I (resp. J).

Since Ω is the union of the disjoint sets $\{\{\pi_0 = (i, j)\} : 1 \leq i \leq I, 1 \leq j \leq J\}$, or $\{\{\pi_0^{(1)} = i\} : 1 \leq i \leq I\}$ or $\{\{\pi_0^{(2)} = j\} : 1 \leq j \leq J\}$ we obtain the following useful formula:

Lemma 3. *For any random variable $X \in L^1(\mathbf{P})$*

$$\int_\Omega X d\mathbf{P} \;=\; \sum_{i=1}^{I} \sum_{j=1}^{J} \int_{\{\pi_0 = (i,j)\}} X d\mathbf{P} \tag{13}$$

$$= \sum_{i=1}^{I} \int_{\{\pi_0^{(1)}=i\}} X d\mathbf{P} = \sum_{j=1}^{J} \int_{\{\pi_0^{(2)}=j\}} X d\mathbf{P}. \tag{14}$$

Next we give us a probability space $(E, \mathcal{B}, \rho)$ where E is always a finite dimensional space endowed with its Borel σ-algebra $\mathcal{B}$. We combine it with the canonical metric space $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ given by Definition 2. Recall that the product space $\Omega \times E$ is endowed with the product σ-algebra $\mathcal{F} \otimes \mathcal{B}$ and the product probability measure $\mathbf{P} \otimes \rho$.

3.2. Random maps and associated RDS

Proposition 4. *Let $T = (T_1, \cdots, T_I)$ be a finite sequence of ρ-nonsingular transformations defined on $(E, \mathcal{B}, \rho)$. Define the fiber maps $\phi : \Omega \times E \to E, (\omega, x) \mapsto \phi(\omega)(x)$ by*

$$\phi(\omega)(x) \ := \ T_{\pi_0^{(1)}(\omega)}(x); \qquad \omega \in \Omega,\, x \in E. \tag{15}$$

Then (θ, ϕ) is a RDS and

$$\mathbf{P}(\omega \in \Omega : \phi(\omega) = T_i) = a_i, \quad 1 \leq i \leq I, \tag{16}$$

where $a_i; 1 \leq i \leq I$ is the marginal probability vector defined by (12).

Proof. In Proposition 1, we have already proved that $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ is a metric DS.

The map ϕ defined by (15) is measurable as composition of two measurable maps $\Omega \times E \to \mathbb{I} \times E; (\omega, x) \mapsto (\pi_0^{(1)}(\omega), x)$ and $\mathbb{I} \times E \to E; (i, x) \mapsto T_i(x)$.

For any $1 \leq i \leq I$, we have

$$\{\phi(.) = T_i\} = \{\pi_0^{(1)} = i\} = \{\pi_0^{(1)} = i, \pi_0^{(2)} \in \mathbb{J}\} = \bigcup_{j=1}^{J} \{\pi_0 = (i, j)\} \tag{17}$$

Hence, using (17) and (12) we get

$$\mathbf{P}(\{\phi(.) = T_i\}) \ = \ \mathbf{P}(\bigcup_{j=1}^{J} \{\pi_0 = (i, j)\}) \ = \ \sum_{j=1}^{J} q_{ij} \ = \ a_i$$

□

Remarks 5. 1- Formula (15) means that $\phi(\omega) = T_{\omega_0^{(1)}}$ if $\omega = (\omega_0, \omega_1, \cdots)$.
2- Formula (16) means that

$$\phi(\omega) \ = \ T_i \quad \text{with probability} \quad a_i; \quad 1 \leq i \leq I. \tag{18}$$

In particular $(T_i, a_i) : 1 \leq i \leq I$ is a *random map* as defined in [13].
3- Let Φ be the skew product associated to the RDS defined by Proposition 4. From (3) and (15) we have

$$\Phi(\omega, x) \ = \ (\theta(\omega), T_{\pi_0^{(1)}(\omega)}(x)); \quad \omega \in \Omega,\, x \in E. \tag{19}$$

Proposition 6. *For all $B \in \mathcal{B}, F \in \mathcal{F}$*

$$(\mathbf{P} \otimes \rho)(\Phi^{-1}(F \times B)) = \sum_{i=1}^{I} \mathbf{P}(F \cap \{\pi_0^{(1)} = i\}) \cdot \rho(T_i^{-1}(B)). \tag{20}$$

In particular Φ is $(\mathbf{P} \otimes \rho)$-nonsingular.

Proof. For $F \in \mathcal{F}$ and $B \in \mathcal{B}$. By putting $\Upsilon := \Phi^{-1}(F \times B)$, the relation (4) becomes

$$\Upsilon = \{(\omega, x) \in \Omega \times E : \omega \in \theta^{-1}(F) \text{ and } x \in T^{-1}_{\pi_0(\omega)^{(1)}}(B)\}. \tag{21}$$

Hence, by putting $\mu := \mathbf{P} \otimes \rho$ and by using (13) we get

$$\begin{aligned}
\mu(\Upsilon) \ &= \ \int \int_{\Phi^{-1}(F \times B)} \rho(dx)\mathbf{P}(d\omega) \\
&= \ \sum_{i=1}^{I} \int_{\theta^{-1}(F \cap \{\pi_0^{(1)} = i\})} \left(\int_{T^{-1}_{\pi_0^{(1)}(\omega)}(B)} \rho(dx) \right) \mathbf{P}(d\omega) \\
&= \ \sum_{i=1}^{I} \int_{\theta^{-1}(F \cap \{\pi_0^{(1)} = i\})} \left(\int_{T_i^{-1}(B)} \rho(dx) \right) \mathbf{P}(d\omega) \\
&= \ \sum_{i=1}^{I} \left(\int_{\theta^{-1}(F \cap \{\pi_0^{(1)} = i\})} \mathbf{P}(d\omega) \right) \cdot \left(\int_{T_i^{-1}(B)} \rho(dx) \right) \\
&= \ \sum_{i=1}^{I} \mathbf{P}(F \cap \{\pi_0^{(1)} = i\}) \cdot \rho(T_i^{-1}(B)).
\end{aligned}$$

Suppose now that $(\mathbf{P} \otimes \rho)(F \times B) = \mathbf{P}(F) \cdot \rho(B) = 0$.

If $\rho(B) = 0$ then $\rho(T_i^{-1}(B)) = 0$ since T_i is ρ-nonsingular for each $1 \le i \le I$ and therefore $(\mathbf{P} \otimes \rho)(\Phi^{-1}(F \times B)) = 0$ in view of (20). If $\mathbf{P}(F) = 0$ then $\mathbf{P}(\theta^{-1}(F)) = 0$ since θ is $\mathbf{P}$-preserving. Hence $\mathbf{P}(\theta^{-1}(F) \cap \{\pi_0^{(1)} = i\}) = 0$ and therefore $(\mathbf{P} \otimes \rho)(\Phi^{-1}(F \times B)) = 0$ once again according to (20). □

Proposition 6 allows to define the Frobenius-Perron operator P_Φ associated with the skew product Φ.
Next, we will associate, in a natural way, some random densities to the metric DS $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ generated by the bivariate probability vector $q = (q_{ij}); i \in \mathbb{I}, j \in \mathbb{J}$ by considering its second marginal.

3.3. An associated class of random densities

The proof of the following result is similar to the proof of Proposition 4 and is omitted.

Proposition 7. *Let $\mathtt{f} = (f_1, \cdots f_J)$ be a finite sequence of functions in $L^1(\rho)$. Define the random function $\xi : \Omega \times E \to \mathbb{R}$ by*

$$\xi(\omega, x) \;:=\; f_{\pi_0^{(2)}(\omega)}(x); \qquad \omega \in \Omega,\, x \in E. \tag{22}$$

Then

$$\mathbf{P}(\omega \in \Omega : \xi(\omega, .) = f_j) = b_j, \quad 1 \le j \le J, \tag{23}$$

where $b_j; 1 \le j \le J$ is the marginal probability vector defined by (12).

Remark 8. As for random maps, formula (23) means that

$$\xi(\omega, .) \;=\; f_j \quad \text{with probability} \quad b_j; \quad j = 1, 2, \cdots J.$$

In particular, if $f_1, \cdots, f_J$ are identical (to some $f \in L^1(\rho)$) then $\xi(\omega, x) = f(x)$ for all $\omega \in \Omega$ and $x \in E$. In other words, ξ is a deterministic density.

The proof of the following result is adapted from the proof of Proposition 4 of our paper [6].

Proposition 9. *Let ξ be the random function defined by (22). Then, for all $F \in \mathcal{F}$ and $B \in \mathcal{B}$,*

$$\int\int_{F \times B} \xi d\rho d\mathbf{P} = \sum_{i=1}^{I} \sum_{j=1}^{J} \mathbf{P}(F \cap \{\pi_0 = (i, j)\}) \cdot (\int_B f_j \, d\rho). \tag{24}$$

In particular, if each $f_j, j \in \mathbb{J}$ is a deterministic density, then ξ is a random density.

Proof. Let $\mu := \mathbf{P} \otimes \rho$ and for $F \in \mathcal{F}, B \in \mathcal{B}$ let $O := F \times B$. Using Formula (13) of Lemma 3, we get

$$\begin{aligned}
\int_O \xi d\mu &= \int_F \left(\int_B \xi(\omega, x)\rho(dx)\right) \mathbf{P}(d\omega) \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J} \int_{F\cap\{\pi_0=(i,j)\}} \left(\int_B \xi(\omega, x)\rho(dx)\right) \mathbf{P}(d\omega) \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J} \int_{F\cap\{\pi_0^{(1)}=i\}\cap\{\pi_0^{(2)}=j\}} \left(\int_B f_{\pi_0^{(2)}(\omega)}(x)\rho(dx)\right) \mathbf{P}(d\omega) \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J} \int_{F\cap\{\pi_0=(i,j)\}} \left(\int_B f_j(x)\,\rho(dx)\right) \mathbf{P}(d\omega) \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J} \mathbf{P}(F \cap \{\pi_0 = (i,j)\})(\int_B f_j d\rho).
\end{aligned}$$

By taking $F = \Omega$ and $B = E$ in equation (24), $|\xi|$ instead of ξ, and by using (11), we obtain

$$\begin{aligned}
\|\xi\|_1 &= \int\int_{\Omega\times E} |\xi|\; d\mathbf{P}d\rho \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J} \mathbf{P}(\Omega \cap \{\pi_0 = (i,j)\})(\int_E |f_j| d\rho) \\
&= \sum_{i=1}^{I}\sum_{j=1}^{J} q_{ij} \cdot \|f_j\|_1 \\
&= \sum_{j=1}^{J}(\sum_{i=1}^{I} q_{ij})\|f_j\|_1 = \sum_{j=1}^{J} b_j \|f_j\|_1.
\end{aligned}$$

Therefore $\|\xi\|_1 = 1$ whenever $\|f_1\|_1 = \cdots = \|f_J\|_1 = 1$.

□

3.4. Invariant random densities

For any $1 \leq i \leq I$ let $P_{T_i}, 1 \leq i \leq I$ be the Frobenius-Perron operator of the single map T_i, defined implicitly by equation (1).
Let ξ be the random function defined by (22) for a given sequence $\mathtt{f} = (f_1, \cdots f_J)$ of functions in $L^1(\rho)$.

Proposition 10. *For all $F \in \mathcal{F}$ and $B \in \mathcal{B}$,*

$$\int\int_{\Phi^{-1}(F\times B)} \xi d\rho d\mathbf{P} = \mathbf{P}(F) \cdot \int_B \left(\sum_{i=1}^{I} \sum_{j=1}^{J} q_{ij} P_{T_i} f_j \right) d\rho. \tag{25}$$

Proof. Denote by U the left term of the equality (25). Starting from the definitions (15) and (22) and using formulas (13), (21) we get

$$\begin{aligned} U &= \int\int_{\Phi^{-1}(F\times B)} \xi(\omega, x)\rho(dx)\mathbf{P}(d\omega) \\ &= \int_{\theta^{-1}(F)} \left(\int_{\phi(\omega)^{-1}(B)} \xi(\omega, x)\rho(dx) \right) \mathbf{P}(d\omega) \\ &= \int_{\theta^{-1}(F)} \left(\int_{T^{-1}_{\pi_0^{(1)}(\omega)}(B)} f_{\pi_0^{(2)}(\omega)}(x)\rho(dx) \right) \mathbf{P}(d\omega) \\ &= \sum_{i=1}^{I} \sum_{j=1}^{J} \int_{\theta^{-1}(F)\cap\{\pi_0=(i,j)\}} \left(\int_{T_i^{-1}(B)} f_j(x)\rho(dx) \right) \mathbf{P}(d\omega) \\ &= \sum_{i=1}^{I} \sum_{j=1}^{J} \mathbf{P}\left(\theta^{-1}(F) \cap \{\pi_0 = (i,j)\}\right) \cdot \left(\int_B P_{T_i} f_j d\rho \right). \end{aligned}$$

Let $\Gamma := \Gamma_{ij} := \theta^{-1}(F) \cap \{\pi_0 = (i,j)\}$. By taking F of the form (7) and by using (9), we get

$$\begin{aligned} \mathbf{P}(\Gamma) &= \mathbf{P}\left(\theta^{-1}(F) \cap \{\pi_0 = (i,j)\}\right) \\ &= \mathbf{P}\left(\theta^{-1}\{\pi_0 = (i_0, j_0)\}, ..., \theta^{-1}\{\pi_n = (i_n, j_n)\}, \{\pi_0 = (i,j)\}\right) \\ &= \mathbf{P}(\pi_1 = (i_0, j_0), \cdots, \pi_{n+1} = (i_n, j_n), \pi_0 = (i,j)) \\ &= q_{i_0 j_0} ... q_{i_n j_n} \cdot q_{ij} \;=\; \mathbf{P}(F) \cdot q_{ij}. \end{aligned}$$

Thus formula (25) follows immediately. □

Now, we come to the main result of this paper.

Theorem 11. *The random function ξ defined by (22) is invariant by the skew product given by (19), if and only if $f_1, \cdots, f_K$ are identical to some $f \in L^1(\rho)$ and*

$$\sum_{i=1}^{I} a_i P_{T_i} f \;=\; f, \tag{26}$$

where $(a_i), i \in \mathbb{I}$ is the first marginal probability vector defined by (12) and P_{T_i} is the F-P operator associated to the single map $T_i, i \in \mathbb{I}$.

Proof. Suppose that the random function ξ defined by (22) is invariant with respect to the skew production Φ defined by (19), that is

$$\int\int_{\Phi^{-1}(F\times B)} \xi d\rho d\mathbf{P} = \int\int_{F\times B} \xi d\rho d\mathbf{P}; \quad F \in \mathcal{F}, B \in \mathcal{B} \tag{27}$$

Now, by taking $F_{kl} = \{\pi_0 = (k,l)\}$ for $k \in \mathbb{I}, l \in \mathbb{J}$ and by using (11), Formula (25) becomes

$$\int\int_{\Phi^{-1}(F_{kl}\times B)} \xi d\rho d\mathbf{P} = q_{kl} \cdot \int_B \left(\sum_{i=1}^{I} \sum_{j=1}^{J} q_{ij} P_{T_i} f_j \right) d\rho, \tag{28}$$

while Formula (24) becomes (using the fact that $\mathbf{P}(F_{kl} \cap \{\pi_0 = (i,j)\}) = \mathbf{P}(\pi_0 = (k,l) \cap \{\pi_0 = (i,j)\}))$

$$\int\int_{F_{kl}\times B} \xi d\rho d\mathbf{P} \;=\; q_{kl} \int_B f_l \, d\rho. \tag{29}$$

Now, by combining (27), (28), and (29) we deduce that

$$\int_B \left(\sum_{i=1}^{I} \sum_{j=1}^{J} q_{ij} P_{T_i} f_j \right) d\rho = \int_B f_l \, d\rho \tag{30}$$

for all $l \in \mathbb{J}$ and $B \in \mathcal{B}$. Since the left hand (30) does not depend on l, we deduce that

$$\int_B f_l \, d\rho \;=\; \int_B f_m \, d\rho$$

for all $1 \le l, m \le J$ and any $B \in \mathcal{B}$, which means that all deterministic densities f_l are identical.

Let $f := f_1 = \cdots = f_J$. Notice first that by linearity of P_{T_i}

$$\begin{aligned} \sum_{i=1}^{I}\sum_{j=1}^{J} q_{ij} P_{T_i} f_j &= \sum_{i=1}^{I}\sum_{j=1}^{J} q_{ij} P_{T_i} f = \sum_{i=1}^{I} P_{T_i} (\sum_{j=1}^{J} q_{ij}) f \\ &= \sum_{i=1}^{I} P_{T_i} a_i f = \sum_{i=1}^{I} a_i P_{T_i} f. \end{aligned}$$

Therefore, (30) is reduced to

$$\int_B \left(\sum_{i=1}^{I} a_i P_{T_i} f \right) d\rho \ = \ \int_B f \, d\rho; \quad B \in \mathcal{B} \tag{31}$$

which is equivalent to (26).

Conversely, if $f := f_1 = \cdots = f_J$, then following Remark 8, $\xi(\omega, x) = f(x)$ for all $\omega \in \Omega$ and $x \in E$. Hence (25) will be equivalent to

$$\int \int_{\Phi^{-1}(F \times B)} \xi d\rho d\mathbf{P} = \mathbf{P}(F) \int_B \left(\sum_{i=1}^{I} a_i P_{T_i} f \right) d\rho \tag{32}$$

while (24) will be the same as

$$\int \int_{F \times B} \xi d\rho d\mathbf{P} = \mathbf{P}(F) \int_B f \, d\rho. \tag{33}$$

Finally, combining (31), (32), and (33), we deduce that (27) holds. □

Remarks 12. 1- The main result of this paper (Theorem 11) extends a previous result (Theorem 7) in our paper [6] where we have considered the particular case $I = J$ and $q_{ij} = p_i p_j$ for all $1 \le i, j \le I$, that is the marginals are identical and independent.

2- In general, a bivariate probability vector may have some zero entries (i.e. $q_{kl} = 0$ for some k, l). In this case, it is always supposed that $a_i \ne 0, b_j \ne 0$ for all i, j. It can be easily verified that Theorem 11 remains true in this case.

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