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AN APPROACH TO SOLVE FULLY FUZZY MULTI-OBJECTIVE LINEAR PROGRAMMING PROBLEMS WITH n-POLYGONAL FUZZY NUMBERS

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Abstract: Fully Fuzzy Multi-Objective Linear Programming and its applications are getting more interest in the last few years. In this problem, the variables, coefficients, and indices are represented by fuzzy numbers, many types of fuzzy numbers have been used. In this paper, a more practical type of fuzzy number is used to represent fuzziness and approximated fuzzy numbers by n-polygonal fuzzy number, a more generalized form to some of the fuzzy number types that are mostly used in the literature. Based on the max-min operator and binary operations of n-polygonal fuzzy number that have been defined recently, we develop an algorithm that provides a fuzzy Pareto optimal solution for the given problem. The proposed method is implemented on numerical examples, the numerical results indicate that the proposed method gives better solutions compared with the previous methods.

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Key Words: multi-objective linear programming; n-polygonal fuzzy numbers; Pareto optimal solution

1. Introduction

Fully fuzzy linear programming (FFLP) problem at which all indices of the variables and the coefficients are fuzzy numbers has gained attention due to its importance in real-world applications. However, these problems may have many

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objectives to optimize to consider many standards in the decision-making process, for example, if we consider agricultural production planning, the optimal model should have the objectives of maximizing the profit and minimizing the inputs and cost of cultivation. Thus, multi-objective problems are used more than single-objective problems in such cases. At the same time, it is important to recognize that multi-objectives are most of the time hard to measure and conflict with each other's in multi-objective programming problems. Due to this observation, the Pareto optimal solution has been proposed instead of the optimal solution which is a solution that is non-dominated by any other solution. This observation motivates multi-objective optimization under imprecision or fuzziness. If the indices of the variables and the coefficients are fuzzy numbers in the multi-objective linear programming problems, the problem is known as a fully fuzzy multi-objective linear programming (FFMOLP) problem.

After the principles of the fuzzy set were introduced [1], Bellman and Zadeh developed the concept of a fuzzy decision-making process [2] and Zimmermann was the first to use fuzzy linear programming problems [3] and applied that approach to multi-objective linear programming problems [4]. He assumed that each one of the objective functions has a linear membership function. The original multi-objective linear programming problem can be simplified to one objective function through linear membership functions. Some researchers, like Chen and Tsai [5], Lin [6], Yaghoobi and Tamiz [7], have used the fuzzy goal programming approach to solve multi-objective programming with fuzzy parameters. Cheng et al. [8] solved a fuzzy multi-objective linear programming problem with deviation degree measures and the weighted max-min approach. Sharma and Aggarwal [9] solved fully fuzzy multi-objective linear programming problem using nearest interval approximation of fuzzy number and interval programming.

Many types of fuzzy numbers were used to represent fuzziness, including the triangular fuzzy number, the trapezoidal fuzzy number, and others that were widely used fuzzy numbers. Since each fuzzy number has a distinct membership function, it may be difficult to convert from one fuzzy number to another in order to examine the results correctly and accurately in any system. Based on this observation, the n-polygonal fuzzy number (n-PFN), which generalizes the triangular and trapezoidal fuzzy numbers, is a more general type of fuzzy number and recently it got a lot of interest. Tuffaha and Alrefaei [10] studied the piecewise linear fuzzy number of order n (PLFN-n), which is an n-polygonal fuzzy number with equal distance knots. In [11], convenient arithmetic operations on the PLFN-n were introduced and demonstrated to satisfy properties such as commutativity, associativity, having an identity, and conserving the ranking value. Furthermore, the operations were proven to generalize standard binary operations on real numbers. These new definitions were used for a fully fuzzy transportation problem (FFTP) [12]. Tuffaha and Alrefaei [13] developed an approach of the well-known simplex method for a fully fuzzy linear programming problem, representing the fuzziness for the first time using the n-PFN.

The main aim of this paper is to develop an algorithm to solve the FFMOLP problems with fuzziness represented by n-polygonal fuzzy numbers. To get that, we define a new linear membership functions for the objectives with n-polygonal fuzzy numbers and solve the problem to get a fuzzy Pareto-optimal solution. We then implement the proposed method to solve real-life examples.

The paper is organized as follows. In Section 2, the definition of a n-PFN, as well as its arithmetic operations and its ranking value are presented. The mathematical formulation of the FFMOLP problems with n-PFN is presented in Section 3. The solution algorithm is proposed together with numerical applications in Section 4. Finally, concluding remarks are presented in Section 5.

2. Preliminaries

In this section we first preview the principles and the definition of the n-polygonal fuzzy numbers (n-PFN), as well as their arithmetic operations and their ranking value that is presented by Tuffaha and Alrefaei [10, 11].

Definition 1. $\tilde{A}$ is a **polygonal fuzzy number of order n** (n-PFN) if $\tilde{A}$ is a fuzzy set whose membership function is provided by:

$$f_{\tilde{A}}(x) = \begin{cases} \frac{1}{n}\left[\frac{x-p_i}{p_{i+1}-p_i}\right] + \frac{i}{n} & ;\ p_i \leq x \leq p_i + 1, i = 0, ..., n-1 \\ 1 & ; p_n \leq x \leq q_0 \\ \frac{-1}{n}\left[\frac{x-q_i}{q_{i+1}-q_i}\right] + \frac{n-i}{n} & ;\ q_i \leq x \leq q_i + 1, i = 0, ..., n-1 \\ 0 & ;\ \text{otherwise}, \end{cases} \quad (1)$$

$\tilde{A}$ is represented by its knots: $(p_0, p_1, ..., p_n; q_0, q_1, ..., q_n)$.

An example of a flat 3-PFN is given in Figure 1.

Remark 2. A special case of n-PFN when $n = 1$ is the triangular and trapezoidal fuzzy numbers.

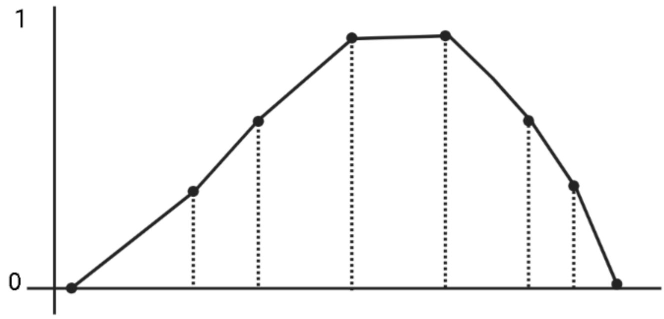

Figure 1: Example of 3-PFN

Remark 3. A crisp real number c (unfuzzy) can be represented in the n-PFN form as $c = (c, c, ..., c; c, c, ..., c)$.

Definition 4. ([10]) Let $\tilde{P} = (p_0, p_1, ..., p_n; q_0, q_1, ..., q_n)$ be an n-PFN then its ranking is given by:

$$\mathcal{R}(\tilde{P}) = \frac{1}{4n}[p_0 + 2p_1 + 2p_2 + ... + 2p_{n-1} + p_n + q_0 + 2q_1 + 2q_2 + ... + 2q_{n-1} + q_n]. \quad (2)$$

Remark 5. The ranking value satisfies the equivalent relation between any two n-PFN, which means $\tilde{p}$ and $\tilde{q}$ are said to be equivalent if $\mathcal{R}(\tilde{p}) = \mathcal{R}(\tilde{q})$. Furthermore, the maximum(minimum) of n-PFNs is the one having the maximum(minimum) ranking value.

Definition 6. ([10]) Let $\tilde{P} = (p_0, p_1, .., p_n; q_0, q_1, ., q_n), \tilde{Q} = (r_0, r_1, .., r_n, s_0, s_1, .., s_n)$ be n-PFNs. The binary operations are defined as follows:

1. $\tilde{P} \oplus \tilde{Q} = (p_0 + r_0, p_1 + r_1, ..., p_n + r_n; q_0 + s_0, q_1 + s_1, .., q_n + s_n)$.

2. $\tilde{P} \ominus \tilde{Q} = \tilde{P} \oplus (-\tilde{Q}) = (p_0 - s_n, p_1 - s_{n-1}, ..., p_n - s_0; q_0 - r_n, q_1 - r_{n-1}, ..., q_n - r_0)$.

3. $\tilde{P} \otimes \tilde{Q} = (t_0, t_1, .., t_n, u_0, u_1, ..., u_n)$, where

$$u_n = \frac{1}{4n}[1 + \sum_{i=1}^{n}(2n-1)X_i + 2nX_{n+1} + \sum_{i=1}^{n}(2(n+i)-1)X_{n+1+i}], \quad (3)$$

$$u_{i-1} = u_i - X_{n+1+i} \;\; ; \text{ for } i = n, n-1, ..., 1,$$
$$t_n = u_0 - X_{n+1},$$
$$t_{i-1} = t_i - X_i \; ; \text{ for } i = n, n-1, ..., 1,$$

$$\text{and } I = \frac{1}{4n}[(p_0 + 2p_1 + ... + 2p_{n-1} + p_n + q_0 + 2q_1 + .. + 2q_{n-1} + q_n) * (r_0 + 2r1_+ ... + 2r_{n-1} + r_n + s_0 + 2s_1 + ... + 2s_{n-1} + s_n)],$$

$$X_i = (p_i - p_{i-1}) + (r_i - r_{i-1}), \text{ for } i = n, n-1, ..., 1,$$
$$X_{n+1} = (q_0 - p_n) + (s_0 - r_n),$$
$$X_{n+1+i} = (q_i - q_{i-1}) + (s_i - s_{i-1}) \;, \text{ for } i = n, n-1, ..., 1.$$

Definition 7. ([10]) Let $\tilde{P} = (p_0, p_1, .., p_n, q_0; q_1, .., q_n) \in \mathcal{PL}_n$. If $\mathcal{R}(\tilde{P}) \neq 0$, then the **multiplicative inverse** of $\tilde{P}$ is defined to be $\tilde{P}^{-1} = (t_0, t_1, .., t_n, u_0; u_1, .., u_n)$, where

$$t_0 = \frac{1}{\mathcal{R}(\tilde{P})} + \frac{1}{4n}(p_0 + 2p_1 + .. + 2p_{n-1} + p_n + q_0 + 2q_1 + .. + 2q_{n-1} - (4n-1)q_n),$$
$$t_i = t_{i-1} + (q_{n-i+1} - q_{n-i}) \text{ for all } i = 1, .., n,$$
$$u_0 = t_n + (q_0 - p_n),$$
$$u_i = u_{i-1} + (p_{n-i+1} - p_{n-i}) \text{ for all } i = 1, .., n.$$

3. Fully Fuzzy Multi-Objective Linear Programming with n-PFN

The fully fuzzy multi-objective linear programming (FFMOLP) problem, is a multi-objective linear programming problem at which all the attributes and variables are n-polygonal fuzzy numbers (piecewise linear fuzzy numbers) which are more generalization of other type of fuzzy numbers. The standard form of the FFMOLP problem with n-PFN is given by:

$$\begin{aligned} \max \;\; & \tilde{z} = (\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_k)^T \\ \text{s.t.} \;\; & \sum_{j=1}^{n} \tilde{a}_{lj}\tilde{x}_j \leq \tilde{b}_l, \;\; l = 1, 2, .., m \\ \text{and} \;\; & \tilde{x}_j \geq \tilde{0}, \;\; j = 1, 2, .., n, \end{aligned} \tag{4}$$

where $\tilde{z}_i = \sum_{j=1}^{n} \tilde{c}_{ij}\tilde{x}_j$, $\tilde{c}_{ij} = (\tilde{c}_{i1}, \tilde{c}_{i2}, ..., \tilde{c}_{in})$, for $i = 1, 2, .., k$ and $\tilde{a}_{lj}, \tilde{c}_{ij}, \tilde{x}_j, \tilde{b}_l$ are n-PFN's. In many cases, there is no one solution that solves all the objectives, therefore, a Pareto optimal solution is proposed that cannot be dominated by other solutions.

3.1. A fuzzy Pareto optimal solution

The solution to the FFMOLP problem is chosen from the set of fuzzy feasible solutions which is defined in the following definitions [14]:

Definition 8. Let $\tilde{X}$ be the set of feasible solutions, then $\tilde{x}^* \in \tilde{X}$ is called **a complete optimal** solution for problem (4), if it satisfies $\tilde{z}_i(\tilde{x}^*) \geq \tilde{z}_i(\tilde{x}), i = 1, 2, .., k$ for all $\tilde{x} \in \tilde{X}$.

As mentioned above, a fully fuzzy multi-objective linear programming problem, unlike a fully fuzzy single-objective linear programming problem, does not necessarily have an optimal solution that maximizes all the fuzzy objective functions simultaneously, since the fuzzy objective functions usually conflict with each other. Therefore, a complete optimal solution is not always achievable, this observation presents the concept of a fuzzy Pareto optimal solution to FFMOLP.

Definition 9. $\tilde{x}^*$ is called **a fuzzy Pareto optimal** solution if there is no $\tilde{x} \in \tilde{X}$ such that $\tilde{z}_i(\tilde{x}) \geq \tilde{z}_i(\tilde{x}^*)$, for all $i = 1, 2, .., k$ and $\tilde{z}_j(\tilde{x}) > \tilde{z}_j(\tilde{x}^*)$ for at least one j.

3.2. The proposed method

To solve the FFMOLP problem with n-PFN given in Equation (4) and find the fuzzy Pareto-optimal solution, we first define membership functions for each objective on a n-PFN form. Many types of membership functions such as linear and non-linear have been applied to the objective functions for the fully fuzzy multi-objective linear programming problems. However, a linear membership function is the most used one because of its simplicity and it is defined by known and fixed parameters from the objective functions. Many types of fuzzy numbers were used to present the fuzziness. Triangular and trapezoidal fuzzy numbers were the most used ones. However, in this paper, we consider the n-PFN's to define a new membership function for each fuzzy objective function that is presented by n-PFN fuzzy number.

We first find the individual minimum ($\tilde{z}_i^-$) and the individual maximum ($\tilde{z}_i^+$) for each fuzzy objective function using the generalization of the simplex method with n-PFN that was proposed by Tuffaha and Alrefaei [13] and let the vectors $\tilde{\mathbf{z}}^-$ and $\tilde{\mathbf{z}}^+$ be as follows:

$$\tilde{\mathbf{z}}^- = [\tilde{z}_1^-, \tilde{z}_2^-, ..., \tilde{z}_k^-] = [\min \tilde{z}_1, \min \tilde{z}_2, ..., \min \tilde{z}_k], \tag{5}$$

$$\tilde{\mathbf{z}}^+ = [\tilde{z}_1^+, \tilde{z}_2^+, ..., \tilde{z}_k^+] = [\max \tilde{z}_1, \max \tilde{z}_2, ..., \max \tilde{z}_k]. \tag{6}$$

Therefore, $[\tilde{z}_i^-, \tilde{z}_i^+]$ represents the possible range for the i^{th} fuzzy objective function. In other words, the minimum and maximum individual values $\tilde{z}_i^-, \tilde{z}_i^+$ for each objective function $\tilde{z}_i$ represent the range of each fuzzy objective function, which will be used to define linear membership functions to the objective functions.

Definition 10. For the maximization type fuzzy objective functions, we define an increasing linear membership function as follows:

$$\tilde{\mu}_{\mathbf{i}}(\tilde{\mathbf{x}}) = \begin{cases} \tilde{0} & ,\text{if } \tilde{z}_i(\tilde{x}) \le \tilde{z}_i^- \\ \frac{1}{n}\left[\frac{\tilde{z}_i(\tilde{x}) - \tilde{z}_i^-}{\tilde{z}_i^+ - \tilde{z}_i^-}\right] & ,\text{if } \tilde{z}_i^- \le \tilde{z}_i(\tilde{x}) \le \tilde{z}_i^+ \\ \tilde{1} & ,\text{if } \tilde{z}_i(\tilde{x}) \ge \tilde{z}_i^+. \end{cases} \tag{7}$$

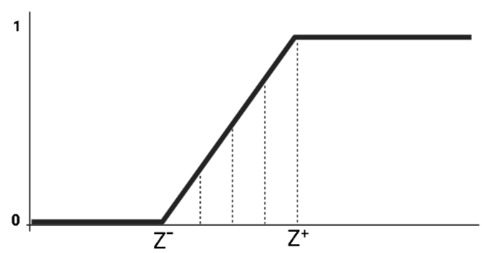

Figure 2: Example of the increasing linear membership function $\tilde{\mu}_i(\tilde{x})$ with 4-PFN.

For the minimization type fuzzy objective function, we define a decreasing linear membership function as follows:

$$\tilde{\mu}_{\mathbf{i}}(\tilde{\mathbf{x}}) = \begin{cases} \tilde{1} & ,\text{if } \tilde{z}_i(\tilde{x}) \leq \tilde{z}_i^- \\ \frac{1}{n}\left[\frac{\tilde{z}_i^+ - \tilde{z}_i(\tilde{x})}{\tilde{z}_i^+ - \tilde{z}_i^-}\right] & ,\text{if } \tilde{z}_i^- \leq \tilde{z}_i(\tilde{x}) \leq \tilde{z}_i^+ \\ \tilde{0} & ,\text{if } \tilde{z}_i(\tilde{x}) \geq \tilde{z}_i^+. \end{cases} \tag{8}$$

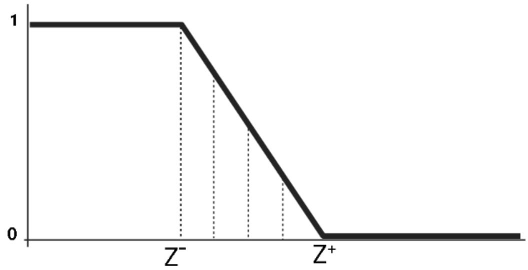

Figure 3: Example of the decreasing linear membership function $\tilde{\mu}_i(\tilde{x})$ with 4-PFN.

After we define a linear membership function for each fuzzy objective function with satisfaction degree (number of knots, n), the original fully fuzzy multi-objective linear programming problem (4) can be interpreted as:

$$\begin{aligned} \max \quad & \{\min \ \mu_i(\tilde{z}_i(\tilde{x}))\},\ i = 1, 2, .., k, \\ \text{s.t.} \quad & \sum_{j=1}^{n} \tilde{a_{lj}}\tilde{x_j} \leq \tilde{b_l}, l = 1, 2, ..., m, \\ & \tilde{x} \geq \tilde{0}. \end{aligned} \tag{9}$$

Formulate a new FFLPP for maximizing the satisfaction degree $\tilde{\lambda}$ of the final decision maker subject to the original constraints as well as the additional constraint $\tilde{\lambda} \leq \tilde{\mu}_{\mathbf{i}}(\tilde{\mathbf{x}}), i = 1, 2, ..., k$, therefore (10) can be reformulated as:

$$\begin{aligned} \max \quad & \tilde{\lambda} \\ \text{s.t.} \quad & \tilde{\lambda} \leq \mu_i(\tilde{z}_i(\tilde{x})),\ i = 0, 1, .., k, \\ & \sum_{j=1}^{n} \tilde{a_{lj}}\tilde{x_j} \leq \tilde{b_l}, l = 1, 2, ..., m, \\ & \tilde{x} \geq \tilde{0},\ \Re(\tilde{\lambda}) \in [0, 1]. \end{aligned} \tag{10}$$

The following theorem show that the solution to this problem is in fact a solution to the original problem (4).

Theorem 11. *Let $\tilde{x}^*$ be an optimal solution for (10), then $\tilde{x}^*$ is a fuzzy Pareto optimal solution to the FFMOLP problem (4).*

Proof. Let $\tilde{x}^*$ be an optimal solution for (10), and suppose that $\tilde{x}^*$ is not a fuzzy Pareto optimal solution for (4), then there exists a feasible solution $\tilde{y}$, such that

$$\tilde{z}_i(\tilde{y}) \geq \tilde{z}_i(\tilde{x}^*) \text{ for all } i = 1, 2, ..., k,$$

$$\text{and } \tilde{z}_s(\tilde{y}) > \tilde{z}_s(\tilde{x}^*) \text{ for some } s,\ 1 \leq s \leq k.$$

Now without loss of generality, we have

$$\mu_i(\tilde{y}) \geq \mu_i(\tilde{x}^*) \text{ and } \mu_s(\tilde{y}) > \mu_s(\tilde{x}^*) \text{ for some } s,\ 1 \leq s \leq k.$$

Thus

$$\begin{aligned} \tilde{\lambda}(\tilde{y}) &= \min\{\mu_1(\tilde{y}), \mu_1(\tilde{y}), ..., \mu_k(\tilde{y})\} \\ > \tilde{\lambda}(\tilde{x}^*) &= \min\{\mu_1(\tilde{x}^*), \mu_2(\tilde{x}^*), ..., \mu_k(\tilde{x}^*)\} \\ &\Rightarrow \tilde{\lambda}(\tilde{y}) > \tilde{\lambda}(\tilde{x}^*). \end{aligned}$$

This implies that $\tilde{x}^*$ is not an optimal solution to problem (10), which is a contradiction. Hence $\tilde{x}^*$ is a fuzzy Pareto optimal solution of (4). □

Remark 12. If the FFMOLP problem needs to maximize some and minimize the others of the fuzzy objective functions then it can be solved directly by defining an increasing linear membership function for the maximization type and a decreasing linear membership function for the minimization type of the fuzzy objective functions.

In the next section, we compare the performance of the proposed method with that of previous methods on some numerical examples.

4. Numerical Applications

In this section, we implement the proposed method on three examples and compare the results with Cheng's method [8] and Sharma and Aggarwal's method [9]. Sharma and Aggarwal solved the FFMOLP problem with LR flat fuzzy numbers using different binary operations than the proposed in this paper.

Cheng used the triangular fuzzy numbers to solve the FFMOLP problem using different binary operations to the triangular fuzzy numbers than the proposed in this paper.

Example 1: Consider the following FFMOLP problem with n-PFN:

$$\max \tilde{z}_1 = (1,2,2;4,5,6) \otimes \tilde{x}_1 \oplus (-7,-5,-4;-2,2,4) \otimes \tilde{x}_2$$

$$\min \tilde{z}_2 = (3,3.5,4;4,4.5,5) \otimes \tilde{x}_1 \oplus (-5,-4.5,-4;-3,2,3) \otimes \tilde{x}_2$$

$$s.t. (-3,-2,-1;2,3,6) \otimes \tilde{x}_1 \oplus (2,3,5;6,7,8) \otimes \tilde{x}_2 \leq (-1,1,2;2,3.5,4)$$

$$(-0.5,3,4;4.4,5,7) \otimes \tilde{x}_1 \oplus (-3,-2.5,-2;-1,0,3) \otimes \tilde{x}_2 \leq (5,6,7;7,10,11)$$

$$\tilde{x}_1, \tilde{x}_2 \geq \tilde{0}.$$

First, note that $\tilde{z}_1$ is a maximization type and $\tilde{z}_2$ is a minimization type so define an increasing membership function to $\tilde{z}_1$ and a decreasing membership function to $\tilde{z}_2$. Thus solve each fuzzy objective function with the same constraints in the original problem to get the $\tilde{z}_i^+$ and $\tilde{z}_i^-$ for each fuzzy objective function.

Thus, $\tilde{z}_i^+$ and $\tilde{z}_i^-$ for each fuzzy objective function are given in the following table:

	$\tilde{z}_i^+$	$\tilde{z}_i^-$
$\tilde{z}_1$	(-1.5,2.5,4.1;6.5,11.5,17)	(-12,-6.7,-3.7;-0.7,6.8,10.3)
RV	6.8	-0.73
$\tilde{z}_2$	(1.1,4.6,6.7;7.1,11.6,16.6)	(-9.8,-6.3,-3.8;-1.8,6.7,9.1)
RV	8	-0.68

Table 1: The values of $\tilde{z}_i^+$ and $\tilde{z}_i^-$ for Example 1

Define the membership function of each fuzzy objective function $\tilde{z}_1$ and $\tilde{z}_2$,

$$\tilde{\mu}_1(\tilde{x}) = \begin{cases} \tilde{0} & ,\text{if } \tilde{z}_1(\tilde{x}) \leq \tilde{z}_1^- \\ \frac{1}{2}\left[\frac{\tilde{z}_1(\tilde{x})-(-12,-6.7,-3.7;-0.7,6.8,10.3)}{(-11.8,-4.3,4.8;10.2,18.2,29)}\right] & ,\text{if } \tilde{z}_1^- \leq \tilde{z}_1(\tilde{x}) \leq \tilde{z}_1^+ \\ \tilde{1} & ,\text{if } \tilde{z}_1(\tilde{x}) \geq \tilde{z}_1^+, \end{cases}$$

$$\tilde{\mu}_2(\tilde{x}) = \begin{cases} \tilde{1} & ,\text{if } \tilde{z}_2(\tilde{x}) \leq \tilde{z}_2^- \\ \frac{1}{2}\left[\frac{(1.1,4.6,6.7;7.1,11.6,16.6)-\tilde{z}_2(\tilde{x})}{(-8,-2.1,4.9;10.9,17.9,26.4)}\right] & ,\text{if } \tilde{z}_2^- \leq \tilde{z}_2(\tilde{x}) \leq \tilde{z}_1^+ \\ \tilde{0} & ,\text{if } \tilde{z}_2(\tilde{x}) \geq \tilde{z}_1^+. \end{cases}$$

Formulate a new FFLPP for maximizing the satisfaction degree $\tilde{\lambda}$ of the final decision maker subject to the original constraints as well as the additional constraint $\tilde{\lambda} \leq \tilde{\mu}_1(\tilde{x})$ and $\tilde{\lambda} \leq \tilde{\mu}_2(\tilde{x})$. Then solve the problem below to find the fuzzy Pareto-optimal solution to the given fully fuzzy multi-objective linear programming problem with n-PFN:

$$\max \tilde{\lambda}$$
$$\text{s.t.} \quad \tilde{\lambda} \leq \mu_1(\tilde{x})$$
$$\tilde{\lambda} \leq \mu_2(\tilde{x})$$
$$(-3,-2,-1;2,3,6) \otimes \tilde{x}_1 \oplus (2,3,5;6,7,8) \otimes \tilde{x}_2 \leq (-1,1,2;2,3.5,4)$$
$$(-0.5,3,4;4.4,5,7) \otimes \tilde{x}_1 \oplus (-3,-2.5,-2;-1,0,3) \otimes \tilde{x}_2 \leq (5,6,7;7,10,11)$$
$$\Re(\tilde{\lambda}) \in [0,1] \text{ and } \tilde{x}_1, \tilde{x}_2 \geq \tilde{0}.$$

After solving the above problem, we get the fuzzy Pareto optimal solution:

Fuzzy variable	Fuzzy Pareto optimal solution	Ranking value
$\tilde{x}_1$	(-95.5,-54.4,-4.55;6.4,55.1,99.5)	0.90
$\tilde{x}_2$	$\tilde{0}$	0
$\tilde{\lambda}$	(-150.3,-86.4,-12;14.6,84.5,153.5)	0.25

Table 2: The Fuzzy Pareto optimal solution for Example 1

The fuzzy objective value is given by Table 3.

Objective	Objective value	Ranking value
$\tilde{z}_1$	(-95.7,-53.6,-3.7;9.2,58.9,104.2)	3.07
$\tilde{z}_2$	(-93.8,-52.1,-1.8;9.1,58.3,103.2)	3.63

Table 3: The objective values of Fuzzy FFMOLP given in Example 1

We see that the largest possible range of objective functions with their ranking values $\tilde{z}_1$ and $\tilde{z}_2$ are [-0.73,6.8] and [-0.683,8.03] respectively, by the proposed method. which lie inside the possible range.

Example 2: This example is given by Sharma and Aggarwal [9], consider the following FFMOLP problem with trapezoidal fuzzy numbers:

$$\max \tilde{z}_1 = (20,21;22,23) \otimes \tilde{x}_1 \oplus (21,23;24,25) \otimes \tilde{x}_2$$
$$\max \tilde{z}_2 = (-11,-10;-9,-8) \otimes \tilde{x}_1 \oplus (12,13;14,15) \otimes \tilde{x}_2$$
$$s.t. (0.1,0.2;0.3,0.4) \otimes \tilde{x}_1 \oplus (0.2,0.3;0.4,0.5) \otimes \tilde{x}_2 \leq (8,9;10,11)$$

$$(0.2,0.3;0.4,0.5)\otimes\tilde{x}_1\oplus(0.1,0.2;0.3,0.4)\otimes\tilde{x}_2\leq(7,8;9,10)$$
$$\tilde{x}_1,\tilde{x}_2\geq\tilde{0}.$$

First, note that $\tilde{z}_1$ and $\tilde{z}_2$ are a maximization type problems, so define an increasing membership function to $\tilde{z}_1$ and $\tilde{z}_2$. The values of $\tilde{z}_i^+$ and $\tilde{z}_i^-$ are given in the following table:

	$\max\tilde{z}_i$ ($\tilde{z}_i^+$)	$\min\tilde{z}_i$ ($\tilde{z}_i^-$)
$\tilde{z}_1$	(669,677;683,691)	$\tilde{0}$
RV	680	
$\tilde{z}_2$	(364,365;367,369)	(-234,-231;-229,-228.8)
RV	366.4	-230.7

Table 4: The values of $\tilde{z}_i^+$ and $\tilde{z}_i^-$ for Example 2

Define the membership function for each objective function $\tilde{z}_1$ and $\tilde{z}_2$ as:

$$\tilde{\mu}_1(\tilde{x})=\begin{cases}\tilde{0} & ,\text{if } \tilde{z}_1(\tilde{x})\leq\tilde{z}_1^-\\ \frac{\tilde{z}_1(\tilde{x})-\tilde{0}}{(669,677;683,691)-\tilde{0}} & ,\text{if } \tilde{z}_1^-\leq\tilde{z}_1(\tilde{x})\leq\tilde{z}_1^+\\ \tilde{1} & ,\text{if } \tilde{z}_1(\tilde{x})\geq\tilde{z}_1^+,\end{cases}$$

$$\tilde{\mu}_2(\tilde{x})=\begin{cases}\tilde{0} & ,\text{if } \tilde{z}_2(\tilde{x})\leq\tilde{z}_2^-\\ \frac{\tilde{z}_2(\tilde{x})-(-234,-231;-229,-228.8)}{(592.8,594;598,6.3)} & ,\text{if } \tilde{z}_2^-\leq\tilde{z}_2(\tilde{x})\leq\tilde{z}_1^+\\ \tilde{1} & ,\text{if } \tilde{z}_2(\tilde{x})\geq\tilde{z}_1^+.\end{cases}$$

Formulate a new FFLPP for maximizing the satisfaction degree $\tilde{\lambda}$ of the final decision maker subject to the original constraints as well as the additional constraint $\tilde{\lambda}\leq\tilde{\mu}_1(\tilde{x})$ and $\tilde{\lambda}\leq\tilde{\mu}_2(\tilde{x})$. Then solve the problem below to find the fuzzy Pareto-optimal solution to the given fully fuzzy multi-objective linear programming problem with n-PFN:

$$\max\tilde{\lambda}$$
$$\text{s.t.}\quad \tilde{\lambda}\leq\mu_1(\tilde{x})$$
$$\tilde{\lambda}\leq\mu_2(\tilde{x})$$
$$(0.1,0.2;0.3,0.4)\otimes\tilde{x}_1\oplus(0.2,0.3;0.4,0.5)\otimes\tilde{x}_2\leq(8,9;10,11)$$
$$(0.2,0.3;0.4,0.5)\otimes\tilde{x}_1\oplus(0.1,0.2;0.3,0.4)\otimes\tilde{x}_2\leq(7,8;9,10)$$

$$\Re(\tilde{\lambda}) \in [0,1] \text{ and } \tilde{x}_1, \tilde{x}_2 \geq \tilde{0}.$$

After solving the above problem, we get the fuzzy Pareto optimal solution, Table 5:

Var.	Proposed Method	Sharma Aggarwal [9]
$\tilde{x}_1$	(-117.82,-38.63;36.05,126.2)	(0,0;0,0)
RV	1.8	0
$\tilde{x}_2$	(-34.6,5.6;43.5,89)	(7.5,7.5;7.5,7.5)
RV	25.8	7.5
$\tilde{\lambda}$	(-156.49,-49.31;46.36,163.23)	
RV	0.947	

Table 5: The Fuzzy Pareto optimal solution for Example 2 compared to Sharma and Aggarwal [9]

Where the fuzzy objective function values are, see Table 6:

Obj.	Proposed Method	Sharma Aggarwal [9]
$\tilde{z}_1$	(457,579;693,831)	(157.5, 172.5; 180, 187.5)
RV	640	174.37
$\tilde{z}_2$	(149,271;385,522)	(90, 97.5; 105, 112.5)
RV	331.75	101.25

Table 6: The objective values of FFMOLP given in Example 2 compared to Sharma and Aggarwal [9]

Remark 13. Using the proposed method, we see that the largest range of ranking values $\tilde{z}_1$ and $\tilde{z}_2$ are [0,680] and [-230.7,366.4] respectively. Using the ranking value of the fuzzy Pareto optimal solution to compare the results with other competent methods, we observe that the fuzzy Pareto optimal solution obtained using the proposed method gives better results than Sharma and Aggarwal's method [9] (see Table 6).

Remark 14. Note that the solution obtained by the proposed method does not fall in the feasible region according to the binary operations used by Sharma and Aggarwal's method [9]. However, the solution obtained by Sharma and Aggarwal's method satisfies the constraints using the binary operations used in this paper.

Example 3: This example is given by Cheng [8], consider the following FFMOLP problem with triangular fuzzy numbers:

$$\max \tilde{z}_1 = (5,7,9) \otimes \tilde{x}_1 \oplus (4,5,6) \otimes \tilde{x}_2 \oplus (1,2,3) \otimes \tilde{x}_3$$

$$\max \tilde{z}_2 = (4,6,7) \otimes \tilde{x}_1 \oplus (2,4,7) \otimes \tilde{x}_2 \oplus (4,5,6) \otimes \tilde{x}_3$$

$$\max \tilde{z}_3 = (2,4,6) \otimes \tilde{x}_1 \oplus (3,4,6) \otimes \tilde{x}_2 \oplus (2,3,5) \otimes \tilde{x}_3$$

$$s.t.(2,5,7) \otimes \tilde{x}_1 \oplus (2,3,4) \otimes \tilde{x}_2 \oplus (1,2,3) \otimes \tilde{x}_3 = (8,16,24)$$

$$(2,3,4) \otimes \tilde{x}_1 \oplus (1,2,4) \otimes \tilde{x}_2 \oplus (2,3,4) \otimes \tilde{x}_3 \geq (12,18,25)$$

$$(1,2,3) \otimes \tilde{x}_1 \oplus (1,2,3) \otimes \tilde{x}_2 \oplus (1,3,4) \otimes \tilde{x}_3 \leq (7,17,22)$$

$$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \geq \tilde{0}.$$

After solving each fuzzy objective function with the same constraints we get the $\tilde{z}_i^+$ and $\tilde{z}_i^-$ for each fuzzy objective function:

	$\max \tilde{z}_i$ ($\tilde{z}_i^+$)	$\min \tilde{z}_i$ ($\tilde{z}_i^-$)
$\tilde{z}_1$	(17.12,19,22)	(17,19,21.4)
RV	19.28	19.10
$\tilde{z}_2$	(30,31.8,32.6)	(30.44,31.35,32.7)
RV	31.55	31.46
$\tilde{z}_3$	(20.05,21.04,22.19)	(20.14,21,21.1)
RV	21.08	20.81

Define the membership function for each objective function $\tilde{z}_1, \tilde{z}_2$, and $\tilde{z}_3$ as:

$$\tilde{\mu}_1(\tilde{x}) = \begin{cases} \tilde{0} & ,\text{if } \tilde{z}_1(\tilde{x}) \leq \tilde{z}_1^- \\ \frac{\tilde{z}_1(\tilde{x})-(17,19,21.4)}{(-4.2,0,5)} & ,\text{if } \tilde{z}_1^- \leq \tilde{z}_1(\tilde{x}) \leq \tilde{z}_1^+ \\ \tilde{1} & ,\text{if } \tilde{z}_1(\tilde{x}) \geq \tilde{z}_1^+, \end{cases}$$

$$\tilde{\mu}_2(\tilde{x}) = \begin{cases} \tilde{0} & ,\text{if } \tilde{z}_2(\tilde{x}) \leq \tilde{z}_2^- \\ \frac{\tilde{z}_2(\tilde{x})-(30.44,31.35,32.7)}{(-2.7,0.45,2.16)} & ,\text{if } \tilde{z}_2^- \leq \tilde{z}_2(\tilde{x}) \leq \tilde{z}_1^+ \\ \tilde{1} & ,\text{if } \tilde{z}_2(\tilde{x}) \geq \tilde{z}_1^+, \end{cases}$$

$$\tilde{\mu}_3(\tilde{x}) = \begin{cases} \tilde{0} & ,\text{if } \tilde{Z}_3(\tilde{x}) \leq \tilde{Z}_3^- \\ \frac{\tilde{Z}_3(\tilde{x})-(20.14,21,21.1)}{(-1.05,0.04,2.05)} & ,\text{if } \tilde{Z}_3^- \leq \tilde{Z}_3(\tilde{x}) \leq \tilde{Z}_3^+ \\ \tilde{1} & ,\text{if } \tilde{Z}_3(\tilde{x}) \geq \tilde{Z}_3^+. \end{cases}$$

Formulate a new FFLPP for maximizing the satisfaction degree $\tilde{\lambda}$ of the final decision maker subject to the original constraints as well as the additional constraint $\tilde{\lambda} \leq \tilde{\mu}_1(\tilde{x}), \tilde{\lambda} \leq \tilde{\mu}_2(\tilde{x}),$ and $\tilde{\lambda} \leq \tilde{\mu}_3(\tilde{x})$. Then solve the problem below to find the fuzzy

Pareto-optimal solution to the given fully fuzzy multi-objective linear programming problem with n-PFN:

$$\begin{aligned} &\max \tilde{\lambda} \\ \text{s.t.} \quad &\tilde{\lambda} \leq \mu_1(\tilde{x}) \\ &\tilde{\lambda} \leq \mu_2(\tilde{x}) \\ &\tilde{\lambda} \leq \mu_3(\tilde{x}) \\ &(2,5,7) \otimes \tilde{x}_1 \oplus (2,3,4) \otimes \tilde{x}_2 \oplus (1,2,3) \otimes \tilde{x}_3 = (8,16,24) \\ &(2,3,4) \otimes \tilde{x}_1 \oplus (1,2,4) \otimes \tilde{x}_2 \oplus (2,3,4) \otimes \tilde{x}_3 \geq (12,18,25) \\ &(1,2,3) \otimes \tilde{x}_1 \oplus (1,2,3) \otimes \tilde{x}_2 \oplus (1,3,4) \otimes \tilde{x}_3 \leq (7,17,22) \\ &\Re(\tilde{\lambda}) \in [0,1] \text{ and } \tilde{x}_1, \tilde{x}_2 \geq \tilde{0}. \end{aligned}$$

After solving the above problem, we get the Fuzzy Pareto optimal solution:

Fuzzy variable	Pareto optimal solution	Ranking value
$\tilde{x}_1$	(-2,1.5,4.28)	1.32
$\tilde{x}_2$	(-3.54,0.25,3.46)	0.105
$\tilde{x}_3$	(2.79,3.6,8.74)	4.68
$\tilde{\lambda}$	(-1.41,0.9,1.9)	0.57

Table 7: The Fuzzy Pareto optimal solution for Example 3

Where the fuzzy objective function values are:

	Proposed Method		Cheng Method [8]	
Obj.	Obj. value	RV	Obj. value	RV
$\tilde{z}_1$	(6.3,18.4,33.5)	19.2	(15.23, 22.18, 29.13)	22.18
$\tilde{z}_2$	(17.6,30.7,46.8)	31.5	(21.91, 31.42, 43.30)	32.01
$\tilde{z}_3$	(7.6,19.6,36.8)	20.9	(16.07, 23.03, 36.54)	24.66

Table 8: The Fuzzy Pareto optimal solution for Example 3 compared to Cheng Method [8]

Remark 15. Note that the ranking values of the Pareto optimal solution obtained using the Chengs' method [8] are larger than the ones obtained by the proposed method. However, the solution obtained by their method does not satisfy the constraints using the binary operations used in this paper. Therefore, the solution is not a basic feasible solution (see Table 8).

5. Conclusion

The fully fuzzy multi-objective linear programming problem with n-polygonal fuzzy numbers is studied in this paper where the n-PFNs are more generalized form of fuzzy numbers. A method is proposed to construct a fuzzy Pareto optimal solution based on the min-max approach. Numerical applications are presented to test the method and compared with other methods. The numerical results indicate that the proposed method gives more generalized solutions to the FFMOLP problems. It also provides a simple, logical, and clear approach to dealing with the FFMOLP problems with n-PFN. Furthermore, using the n-PFN to represent the fuzziness in the problem enables us to solve a wide range of problems.

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