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STABILITY OF OSCILLATIONS OF MAGNETO-ELASTIC CONDUCTING PLATES IN THE PRESENCE OF A MAGNETIC FIELD

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Abstract: The paper considers the influence of the diamagnetic vacuum gap on the onset of instability in a dynamic system consisting of two conducting plates in the presence of a magnetic field.

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1. Introduction

Recently, much attention has been paid to the study of theoretical and practical aspects of the theory of magnetism. This is due to the fact that the use of the ideas and methods of this theory has a significant impact on the efficiency of such vital branches of human activity as energy, metallurgy, technology, etc.

Studying the most important problems of the theory of magnetism from the point of view of applied significance, we come to mathematical models that are described by quasi-linear non-stationary equations. The phenomenon of type II superconductivity, in particular, is related to this problem. For this phenomenon, C.P. Bean proposed in 1962 a mathematical model, [1]. This

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model obviously sets for us the following tasks: to study the local and global properties of solutions of the corresponding equations and find estimates for these solutions.

Some of these questionable issues have been considered in the works [2]–[4], [11]–[13].

Of no less importance is the question of stability, in particular, some questions of stability of oscillations of magneto-elastic objects-conducting plates and shells in the presence of a magnetic field. Previously, we considered the problem of stability of the tangential discontinuity formed by two elastic superconducting half-spaces separated by a vacuum gap filled with a constant uniform magnetic field and moving relative to each other [9], [10], and the problem of stability of oscillations of a current-carrying shell containing an incompressible fluid flow [5].

2. Main results

Let us consider the problem of the stability of two plates.

Let two thin ideally conducting plates of thickness h, the bending rigidity of which is equal

$$D = \frac{Eh^3}{12\left(1 - v^2\right)}$$

and the material density ρ, be separated by a vacuum gap filled with a constant uniform magnetic field directed along the axis x. The plate $z = 0$ moves along the axis x with speed v_0, and the plate $z = l$ is stationary.

We will consider perturbations of the plates in the form of flexural waves. As is known, the equations of oscillations of plates in a fixed coordinate system have the form, [8]:

$$\rho h \left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x} \right)^2 \xi_1 = -D \frac{\partial^4 \xi_1}{\partial x^4} - P_{1m},$$

$$\rho h \frac{\partial^2 \xi_2}{\partial t^2} = -D \frac{\partial^4 \xi_2}{\partial x^4} + P_{2m}, \qquad (1)$$

where $\xi\left(x, t\right)$ - are small vertical displacements of points on the plate surface proportional to $\exp\left(i\left(ut - kx\right)\right)$, P_{1m} and P_{2m} is the magnetic pressure on the plate surface.

To determine the perturbation of the magnetic field $\vec{h}$ caused by the deformation of the surface of the plates, as in [8], we use the magneto-static approximation, in accordance with which we put $\vec{h} = grad\psi$, where the function ψ satisfies the Laplace equation $\Delta\psi = 0$ with the boundary conditions [5]

$$\vec{H} \cdot \vec{n}_1 = 0 \quad \text{at} \quad z = 0,$$

$$\vec{H} \cdot \vec{n}_2 = 0 \quad \text{at} \quad z = l.$$

Here $\vec{n}_{1,2}$ is the unit vector of the outward normal to the perturbed surfaces of plates with components [9]

$$n_x = -\frac{\partial \xi}{\gamma x}, \quad n_z = 1.$$

Determining the perturbation $\vec{h}$ from this, we obtain the following expressions for P_{1m} and P_{2m}:

$$P_{1m} = k\frac{H_0^2}{4\pi}\left(\alpha\xi_1 - 2\beta\xi_2\right),$$

$$P_{2m} = -k\frac{H_0^2}{4\pi}\left(\alpha\xi_2 - 2\beta\xi_1\right), \tag{2}$$

where

$$\alpha = \frac{1 + \exp\left(-kl\right)}{1 - \exp\left(-2kl\right)}, \quad \beta = \frac{\exp\left(-kl\right)}{1 - \exp\left(-2kl\right)}$$

are dimensionless parameters that determine the effect of the gap.

Substituting the values $\xi = \xi_0 \exp\left(i\left(kx - \omega t\right)\right)$ and P_{1m}, P_{2m} into the oscillation equations (1), we obtain

$$\left(\left(\omega - kv_0\right)^2 - \Omega^2\right)\xi_1 + 2\beta k\frac{H_0^2}{4\pi\rho h}\xi_2 = 0,$$

$$\left(\omega^2 - \Omega^2\right)\xi_2 + 2\beta K\frac{H_0^2}{4\pi\rho h}\xi_1 = 0.$$

From this it is easy to obtain the dispersion equation of the problem

$$\left(\left(\omega - Kv_0\right)^2 - \Omega^2\right)\left(\omega^2 - \Omega^2\right) = \Omega_0^4. \tag{3}$$

Here

$$\Omega^2 = k^2\left(u_0^2 + \alpha v_A^2\right), \quad v_A^2 = \frac{H_0^2}{4\pi\rho kh},$$

$$\Omega_0^4 = 4\beta^2 K^4 v_A^4,$$

u_0 -being the phase speed of flexural waves in a free plate:

$$u_0 = k\sqrt{\frac{D}{\rho h}}.$$

In the case of a solitary plate $\alpha = 1$, $\beta = 0$ and the dispersion equation takes the form

$$\omega^2 - \Omega^2 = 0$$

for a fixed plate and

$$(\omega - Kv_0)^2 - \Omega^2 = 0$$

for the moving plate.

Thus, the phase speed of flexural waves propagating in a plate in the presence of a magnetic field is

$$\sqrt{u_0^2 + v_A^2}.$$

When approaching a thin plate,

$$kh \ll 1.$$

Therefore, although the value $\frac{H_0^2}{4\pi\rho}$ is usually small in comparison with u_0^2 , even for the currently maximum admissible fields, the presence of the factor $(kh)^{-1}$ makes the influence of the magnetic field noticeable. Returning to the general case, we note once again that the presence of a magnetic field in the gap leads to coupled oscillations of the plates. Dispersion equation (3) in this case has the following solution

$$\omega = \frac{kv_0}{2} \pm \sqrt{\Omega^2 + \left(\frac{kv_0}{2}\right)^2 \pm \sqrt{\Omega_0^4 + k^2 v_0^2 \Omega^2}}.$$

This equation determines the possible types of waves that propagate in a given system.

In the case of fixed plates, this solution has the form:

$$\omega = k\sqrt{u_0^2 + (\alpha \pm 2\beta)\, v_A^2}.$$

Thus, in this system, two types of waves are possible, which propagate with phase speed

$$v_{1\phi} = \sqrt{u_0^2 + (\alpha + 2\beta)\, v_A^2},$$

$$v_{2\phi} = \sqrt{u_2^0 + (\alpha - 2\beta)\, v_A^2}.$$

In the case of moving plates, the frequency becomes complex if

$$\left(\Omega^2 + \frac{k^2 v_0^2}{4}\right)^2 < \Omega_0^4 + k^2 v_0^2 \Omega^2.$$

This inequality is satisfied for the following values of the speed of movement of the plates

$$2v_{1\phi} < v_0 < 2v_{2\phi}.$$

This means that if the speed of the relative motion of the plates is in the interval between the doubled phase velocities, then instability arises.

Next, we will consider the question of the stability of the dynamical system considered above, consisting of plates, and find out how this is affected by the vacuum diamagnetic gap.

Let both plates be stationary and in the domain $z < 0$ there be an incompressible fluid that moves with speed v_0, and the density of which is equal to ρ_0. In the domain $z > l$, the fluid is stationary. Let us consider, as above, perturbations in the form of a plane transverse wave traveling along the axis x, so that

$$\xi\left(x,t\right) \sim \exp\left(i\left(\omega t - kx\right)\right).$$

In this case, the equations for the oscillations of the plates are as follows:

$$\rho h \frac{\partial^2 \xi_1}{\partial t^2} = -D\frac{\partial^4 \xi_1}{\partial x^4} + P_1 - P_{1m},$$

$$\rho h \frac{\partial^2 \xi_2}{\partial t^2} = -D\frac{\partial^4 \xi_2}{\partial x^4} - P_2 + P_{2m}, \tag{4}$$

where P_1, P_2 - hydrodynamic and P_{1m}, P_{2m} - magnetic pressure at the surface of the plates.

Using the equations of hydrodynamics [6],[7]

$$\frac{\partial \vec{v}_1}{\partial t} + v_0 \frac{\partial \vec{v}_1}{\partial x} = -\frac{1}{\rho_0} grad\, P_1,$$

$$div\, \vec{v}_1 = 0 \;\text{ at }\; z < 0,$$

$$\frac{\partial \vec{v}_2}{\partial t} = -\frac{1}{\rho_0} grad\, P_2,$$

$$div\, \vec{v}_2 = 0 \;\text{ at }\; z > l,$$

and considering the boundary conditions

$$\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial t}\right)^2\xi_1=-\frac{1}{\rho_0}\frac{\partial P_1}{\partial t}\ \text{ at }\ z=0,$$

$$\frac{\partial^2\xi_2}{\partial t^2}=-\frac{1}{\rho_0}\frac{\partial P_2}{\partial t}\ \text{ at }\ z=l,$$

for magnetic pressures P_1 and P_2 obtain the following values

$$P_1=\rho_0\frac{(\omega-kv_0)^2}{k}\xi_1,\ \ P_2=-\rho_0\frac{\omega^2}{k}\xi_2. \tag{5}$$

Substituting $\xi\,(x,t)=\xi_0\exp\left(i\,(kt-\omega t)\right)$ and the values $P_1,P_2,\ P_{1m},P_{2m}$ into the equations of oscillations, we obtain the dispersion equation of the problem in the form:

$$\left(\omega^2-\Omega^2\right)\left(\omega^2-\Omega^2+\frac{kv_0}{1+\varepsilon}\left(kv_0-2\omega\right)\right)=4\beta^2k^4v_A^4, \tag{6}$$

where

$$\Omega^2=\frac{D\,k^5}{\rho_0\,(1+\varepsilon)}+\alpha\,k^2v_A^2=k^2\left(u_0^2+\alpha v_A^2\right),$$

u_0 - phase speed of waves propagating in a plate in the presence of fluid [8],

$$u_0=k\sqrt{\frac{Dk}{\rho_0\,(1+\varepsilon)}}.$$

The parameters ε and v_A can be defined by expressions

$$\varepsilon=\frac{\rho kh}{\rho_0},v_A=\frac{H_0}{2\sqrt{\pi\rho_0\,(1+\varepsilon)}},$$

and it is obvious that $\varepsilon\ll 1$.

For solitary plates $\alpha=1,\ \beta=0$; therefore, with a stationary fluid, waves can propagate in the plate for which

$$\omega=\Omega=k\sqrt{u_0^2+V_A^2},$$

and if the fluid moves, then waves with frequency appear

$$\omega=\frac{kv_0}{1+\varepsilon}\pm\sqrt{\Omega^2-\frac{\varepsilon}{(1+\varepsilon)^2}k^2v_0^2}.$$

We see that if the radical expression is negative, that is

$$\Omega^2 < \frac{\varepsilon}{(1+\varepsilon)^2} k^2 v_0^2,$$

then the frequency ω becomes complex. In this case, the instability of our system is possible.

Substituting the value Ω^2 into this inequality, we obtain the condition of possible instability for oscillations in a solitary plate in a fluid flow

$$v_0^2 > \frac{1+\varepsilon}{\varepsilon} \left(u_0^2 + \alpha v_A^2\right),$$

or taking into account that $\varepsilon \ll 1$

$$v_0^2 > \frac{u_0^2 + \alpha v_A^2}{\varepsilon}. \tag{7}$$

In the general case of coupled oscillations at $v_0 = 0$ we get the possible types of oscillations in the form:

$$\omega^2 = k^2 \left(u_0^2 + (\alpha \pm 2\beta)\, v_A^2\right).$$

Thus, in such a system, the presence of a diamagnetic gap leads to the existence of two types of waves with phase speeds

$$v_{1\phi} = \sqrt{u_0^2 + (\alpha - 2\beta)\, v_A^2},$$

$$v_{2\phi} = \sqrt{u_0^2 + (\alpha \pm 2\beta)\, v_A^2}.$$

To study the dispersion equation (6), it is convenient to introduce new dimensionless quantities x, a, b:

$$\frac{\omega}{\Omega} = x, \;\; 2\beta \frac{k^2 v_A^2}{\Omega^2} = a^2, \;\; \frac{k\, v_0}{\Omega} = b, \tag{8}$$

moreover $a^2 \leq 1$.

Then the dispersion equation (6) will take the form:

$$\left(x^2 - 1\right) \left(x^2 - \frac{2bx}{1+\varepsilon} + \frac{b^2}{1+\varepsilon} - 1\right) = a^4. \tag{9}$$

It is easy to show that this equation (9) has complex roots if simultaneously are satisfy the inequalities

$$b^2 > a^2 + \frac{1}{\varepsilon},$$

$$b^2 > 4\left(1 - a^2\right),$$
$$b^2 > 4\left(1 + a^2\right). \tag{10}$$

The domain defined by these inequalities (10) taking into account the fact that $a < 1$, is the domain of instability. Substituting into these inequalities (10) the values of the parameters a^2 and b^2 (see (8)), we obtain the following inequalities

$$v_0^2 > 2\beta v_A^2 + \frac{u_0^2 + \alpha v_A^2}{\varepsilon}, \tag{11}$$

$$2v_{1\phi} < v_0 < 2v_{2\phi}. \tag{12}$$

The first inequality (11) determines the effect of the diamagnetic gap on the plate oscillations and degenerates into inequality (7) for an infinitely large gap. The second inequality (12) defines the domain of instability caused directly by the presence of a diamagnetic gap.

3. Conclusion

It is shown that in the considered dynamic system, consisting of two conducting plates, under the action of a magnetic field, two types of flexural waves are possible: slow and fast with phase speed $v_{1\phi}$ and $v_{2\phi}$. In this case, the system will be unstable if the value of the relative sliding speed v_0 of the plates falls within the interval between $2v_{1\phi}$ and $2v_{2\phi}$, i.e.

$$2v_{1\phi} < v_0 < 2v_{2\phi}.$$

Note that a similar result was obtained in [9] for the problem of the stability of a magneto-elastic system.

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