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ADAPTIVE RESIDUAL-CORRECTED ACCELERATED ADOMIAN DECOMPOSITION METHOD FOR NUMERICAL SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

Fractional differential equations have emerged as powerful mathematical tools for modeling complex dynamical systems exhibiting memory and hereditary characteristics across engineering, physics, and applied sciences. However, obtaining accurate and computationally efficient numerical solutions for nonlinear fractional systems remains a significant challenge due to slow convergence and instability associated with conventional analytical–numerical techniques. In this study, an Adaptive Residual-Corrected Accelerated Adomian Decomposition Method (ARC-AA-ADM) is proposed for the numerical solution of nonlinear fractional differential equations formulated in the Caputo sense. The proposed framework enhances the classical Adomian Decomposition Method by incorporating a residual-based adaptive iteration mechanism, enabling automatic truncation control and improved numerical stability. Furthermore, a convergence acceleration strategy based on rational approximation techniques is introduced to significantly reduce series convergence time while maintaining solution accuracy. A residual correction procedure is subsequently employed to minimize approximation errors through fractional integral compensation, leading to refined solution reconstruction with fewer decomposition terms. Theoretical analysis establishes the stability and convergence of the proposed iterative operator using fixed-point arguments in Banach spaces. The effectiveness and robustness of ARC-AA-ADM are validated through several benchmark fractional models, where numerical results demonstrate substantial reductions in residual error and faster convergence compared with classical ADM and existing semi-analytical methods. The obtained results confirm that the proposed method provides an efficient, accurate, and computationally reliable framework for solving a broad class of nonlinear fractional differential equations, thereby offering a significant advancement over traditional decomposition-based numerical schemes.

Keywords Fractional differential equations, Adomian decomposition method; numerical approximation, convergence acceleration; residual correction, Caputo derivative

1. Introduction

1.1 Background of Fractional Differential Equations

Fractional differential equations (FDEs) have gained substantial attention in recent decades due to their capability to describe complex dynamical processes involving memory, hereditary

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behavior, and nonlocal interactions (Kumar and Umesh (2022)). Unlike classical integer-order differential equations, fractional operators incorporate historical system information through integral kernels, enabling more realistic mathematical modeling of physical and engineering phenomena (Yan et al. (2024)). The inclusion of fractional-order derivatives allows system evolution to depend not only on instantaneous states but also on accumulated past responses, thereby improving modeling accuracy in systems exhibiting anomalous diffusion, viscoelasticity, biological growth, and energy transport mechanisms (Yunus et al. (2022)).

The nonlocal property inherent in fractional calculus plays a crucial role in representing long-term dependencies observed in real-world processes. Consequently, fractional models have been successfully applied in diverse engineering domains such as control systems, signal processing, fluid dynamics, electrochemical processes, thermal diffusion, robotics, and biomedical engineering. Despite their modeling advantages, analytical solutions of nonlinear fractional differential equations are rarely obtainable, necessitating the development of efficient and reliable numerical solution techniques.

1.2 Existing Numerical Methods

A wide range of analytical–numerical approaches have been developed to approximate solutions of fractional differential equations (Ata and Kıymaz (2023)). Among these, the Adomian Decomposition Method (ADM) has become one of the most widely used semi-analytical techniques due to its ability to handle nonlinear problems without discretization, linearization, or perturbation assumptions (Afreen and Raheem (2022), Obeidat et al. (2024)). ADM constructs solutions in the form of rapidly convergent series using Adomian polynomials to represent nonlinear operators (Al-Mazmumy et al. (2024), Johansyah et al. (2022)). The Variational Iteration Method (VIM) introduces correction functionals combined with Lagrange multipliers to iteratively approximate solutions, offering flexibility in treating nonlinear systems (Bekela and Deresse (2024)). Similarly, the Homotopy Analysis Method (HAM) provides an analytic continuation framework capable of controlling convergence through auxiliary parameters. Spectral methods, including Chebyshev and Legendre-based approximations, achieve high accuracy by projecting solutions onto orthogonal basis functions, while finite difference and finite element approaches discretize fractional operators for large-scale computational simulations. Although these techniques have demonstrated effectiveness across various applications, challenges related to convergence efficiency, computational complexity, and stability persist, particularly for nonlinear fractional systems (Hattaf (2022), Masood et al. (2022)).

1.3 Limitations of Classical Adomian Decomposition Method

Despite its simplicity and analytical elegance, the classical Adomian Decomposition Method suffers from several practical limitations when applied to fractional differential equations (Murad (2022)). One of the primary challenges is the relatively slow convergence of the decomposition series, especially for strongly nonlinear or stiff fractional models. Additionally, ADM implementations typically rely on fixed truncation levels, where the number of series terms is selected empirically rather than determined through rigorous error control. Another

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significant limitation is the absence of adaptive mechanisms capable of monitoring approximation accuracy during iteration. Classical ADM also lacks residual-based correction strategies, causing error accumulation as iteration progresses. These shortcomings reduce computational efficiency and restrict the applicability of ADM to complex fractional systems requiring high numerical precision. Addressing these limitations forms an important research gap in contemporary fractional numerical analysis.

1.4 Motivation of the Present Study

The increasing demand for accurate and computationally efficient solvers for nonlinear fractional differential equations motivates the development of improved decomposition-based numerical frameworks. Enhancing ADM through adaptive convergence monitoring and systematic error correction can significantly improve approximation quality while preserving the analytical advantages of decomposition techniques. The present study aims to overcome the intrinsic limitations of conventional ADM by integrating adaptive residual control, convergence acceleration, and correction mechanisms within a unified numerical framework suitable for fractional dynamical systems.

1.5 Contributions of the Paper

The main contributions of this research are summarized as follows:

- Development of an Adaptive Residual-Corrected Accelerated Adomian Decomposition Method (ARC-AA-ADM) for nonlinear fractional differential equations.
- Introduction of a residual-based adaptive iteration strategy enabling automatic truncation control.
- Incorporation of convergence acceleration techniques to enhance series convergence efficiency.
- Formulation of a residual correction mechanism for improved numerical stability and accuracy.
- Establishment of stability and convergence properties using fixed-point theory in Banach spaces.
- Comprehensive numerical validation demonstrating improved accuracy and computational efficiency compared with classical ADM-based approaches.

2. Preliminaries and Mathematical Background

2.1 Fractional Calculus Definitions

Fractional calculus extends the concept of classical differentiation and integration to non-integer orders, enabling the modeling of dynamical systems characterized by memory and hereditary effects. In this study, the formulation of fractional differential equations is based primarily on the Riemann–Liouville fractional integral and the Caputo fractional derivative (Wang et al. (2022)), which are widely adopted in applied mathematical modeling due to their strong theoretical foundation and physical interpretability.

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Definition 2.1 *(Riemann-Liouville Fractional Integral).*

Let $y(t) \in C([0,T])$ be a continuous function defined on the interval $[0,T]$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$I_t^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau, t > 0$$

where $\Gamma(\alpha)$ denotes the Gamma function given by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-s} s^{\alpha-1} ds, \alpha > 0$$

The Riemann-Liouville operator represents a weighted accumulation of past states, thereby introducing the nonlocal behavior characteristic of fractional systems.

Definition 2.2 *(Caputo Fractional Derivative)*.

Let $y(t) \in C^n([0,T])$ with $n-1 < \alpha \leq n$, where $n \in \mathbb{N}$. The Caputo fractional derivative of order α is defined as

$${}^{C}D_t^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} y^{(n)}(\tau) d\tau, t > 0$$

The Caputo derivative is particularly suitable for physical and engineering applications since it permits classical integer-order initial conditions such as

$$y(0), y'(0), \ldots, y^{(n-1)}(0)$$

These fractional operators form the analytical foundation for transforming the considered fractional differential equations into equivalent integral representations, which enables the construction of the proposed ARC-AA-ADM numerical framework.

2.2 Basic Properties of Fractional Operators

Property 1.1 (Linearity). Both fractional integral and derivative operators are linear:

$$I_t^{\alpha}(ay_1 + by_2) = aI_t^{\alpha} y_1 + bI_t^{\alpha} y_2$$
$${}^{C}D_t^{\alpha}(ay_1 + by_2) = a{}^{C}D_t^{\alpha} y_1 + b{}^{C}D_t^{\alpha} y_2$$

where a and b are constants.

Property 1.2 (Fractional Integral of a Power Function). For $\beta > -1$,

$$I_t^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}$$

Property 1.3 (Caputo Derivative of a Power Function) (Alsulami et al. (2024)). For $\beta \geq 0$,

$${}^{C}D_t^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \beta > \alpha - 1.$$

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Property 1.4 (Inverse Relationship). The fractional integral operator acts as the inverse of the Caputo derivative:

$$I_t^\alpha({}^C D_t^\alpha y(t)) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k, n-1 < \alpha \leq n.$$

This property enables transformation of fractional differential equations into equivalent fractional integral equations, forming the theoretical basis of the proposed decomposition framework.

Property 1.5 (Nonlocal Memory Effect). The evaluation of fractional operators at time t depends on all previous states within the interval $[0, t]$, namely

$${}^C D_t^\alpha y(t) \text{ depends on } y(\tau), 0 \leq \tau \leq t$$

This nonlocal characteristic distinguishes fractional models from classical differential systems.

2.2 Fractional Integral Operator Formulation

The transformation of fractional differential equations into equivalent integral representations constitutes a fundamental step in constructing decomposition-based numerical solution techniques. In particular, the Adomian Decomposition Method operates more effectively when the governing fractional differential equation is expressed in integral form rather than differential form (Johansyah et al. (2022)). This conversion eliminates direct fractional differentiation and enables recursive approximation through successive integral evaluations.

2.2.1 General Form of Fractional Differential Equation

Consider a nonlinear fractional differential equation defined in the Caputo sense as

$${}^C D_t^\alpha y(t) = F(t, y(t)), 0 < \alpha \leq 1,$$

subject to the initial condition

$$y(0) = y_0$$

where ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order α, and $F(t, y)$ represents a sufficiently smooth nonlinear function.

For analytical convenience, the nonlinear operator may be decomposed as

$$F(t, y) = L[y(t)] + N[y(t)] + g(t)$$

where

- L denotes a linear operator,
- N represents a nonlinear operator, and
- $g(t)$ is a known source term.

2.2.2 Application of Fractional Integral Operator

Applying the Riemann-Liouville fractional integral operator I_t^α to both sides gives

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$$I_t^{\alpha}({}^{C}D_t^{\alpha}y(t)) = I_t^{\alpha}F(t, y(t))$$

Using the inverse property of fractional operators,

$$I_t^{\alpha}({}^{C}D_t^{\alpha}y(t)) = y(t) - y(0)$$

Substituting the initial condition yields the equivalent fractional integral equation

$$y(t) = y_0 + I_t^{\alpha}[F(t, y(t))]$$

2.2.3 Integral Representation of the Problem

Expanding the fractional integral operator explicitly,

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}F(\tau, y(\tau))d\tau$$

Using operator decomposition,

$$y(t) = y_0 + I_t^{\alpha}[L(y(t))] + I_t^{\alpha}[N(y(t))] + I_t^{\alpha}[g(t)]$$

This formulation converts the fractional differential equation into a nonlinear Volterratype integral equation (Agilan and Parthiban (2026)) of the second kind.

2.2.4 Significance for Decomposition-Based Numerical Methods

The obtained integral formulation forms the mathematical basis for the development of the proposed Adaptive Residual-Corrected Accelerated Adomian Decomposition Method. The transformation provides several computational advantages:

- fractional derivatives are replaced by numerically stable integral operators,
- recursive solution components can be generated systematically,
- nonlinear terms can be handled using Adomian polynomial decomposition,
- convergence analysis can be established through fixed-point operator theory.

Consequently, this fractional integral representation serves as the foundation for constructing decomposition series and adaptive correction strategies developed in subsequent sections.

2.3 Classical Adomian Decomposition Method

The Adomian Decomposition Method (ADM) is a powerful semi-analytical technique developed for solving linear and nonlinear differential equations without discretization, perturbation assumptions, or linearization. The method constructs the solution in the form of a rapidly convergent series while treating nonlinear terms analytically through specially generated polynomials. Due to these advantages, ADM has become one of the most widely adopted approaches for solving fractional differential equations.

In this section, the classical ADM formulation is presented systematically, forming the theoretical foundation for the enhanced numerical framework proposed later in this study.

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2.3.1 Operator Formulation of the Problem

Consider the fractional integral equation obtained in Section 2.2:

$$y(t) = y_0 + I_t^{\alpha}[L(y(t)) + N(y(t)) + g(t)] \tag{1}$$

where L denotes a linear operator, N represents a nonlinear operator, and $g(t)$ is a known source function.

For decomposition analysis, the equation can be written in operator form as

$$y(t) = T(y(t))$$

where the nonlinear integral operator T is defined by

$$T(y) = y_0 + I_t^{\alpha}[L(y) + N(y) + g(t)] \tag{2}$$

Thus, solving the fractional differential equation becomes equivalent to determining a fixed point of the operator T. This formulation provides the mathematical foundation for constructing successive approximations using the Adomian Decomposition Method (ADM).

2.3.2 Decomposition Principle and Series Representation

According to ADM, the unknown solution is expressed as an infinite series:

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

Similarly, the nonlinear operator is decomposed into a series of Adomian polynomials:

$$N(y) = \sum_{n=0}^{\infty} A_n(t)$$

Substituting these expansions into the operator equation yields

$$\sum_{n=0}^{\infty} y_n(t) = y_0 + I_t^{\alpha}\left[L\left(\sum_{n=0}^{\infty} y_n(t)\right) + \sum_{n=0}^{\infty} A_n(t) + g(t)\right]. \tag{3}$$

By equating terms of identical order, recursive relations for successive solution components can be obtained.

2.3.4 Construction of ADM Recursive Scheme

The decomposition process begins with an initial approximation incorporating known initial conditions and source terms:

$$y_0(t) = y_0 + I_t^{\alpha}[g(t)]$$

The remaining components are generated recursively as

$$y_{n+1}(t) = I_t^{\alpha}[L(y_n(t)) + A_n(t)], n \geq 0$$

After N iterations, the approximate solution becomes

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$$y^{(N)}(t) = \sum_{n=0}^{N} y_n(t)$$

Under suitable smoothness and boundedness conditions, the series converges to the exact solution as

$$N \to \infty$$

2.3.4 Adomian Polynomials for Nonlinear Operators

A key advantage of ADM is the analytical treatment of nonlinear terms using Adomian polynomials. These polynomials allow nonlinear effects to be incorporated without linearization.

The Adomian polynomials are defined as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty} \lambda^k y_k\right)\right]_{\lambda=0}, n = 0,1,2,\ldots \quad (4)$$

The first few Adomian polynomials are given by

$$\begin{aligned} A_0 &= N(y_0) \\ A_1 &= y_1 N'(y_0) \\ A_2 &= y_2 N'(y_0) + \frac{1}{2} y_1^2 N''(y_0) \end{aligned}$$

Higher-order polynomials can be generated systematically depending on the nonlinear structure of the governing equation.

Role of Classical ADM in the Present Study

Although the classical Adomian Decomposition Method provides an elegant analytical framework for solving fractional differential equations, practical implementation often exhibits slow convergence and relies on empirically chosen truncation levels. These limitations motivate the development of improved decomposition strategies. The classical ADM formulation established in this section therefore serves as the baseline framework upon which the proposed Adaptive Residual-Corrected Accelerated Adomian Decomposition Method is constructed in the subsequent section.

2.4 Modified Cumulative Adomian Polynomial Formulation

Efficient evaluation of nonlinear terms plays a crucial role in improving the convergence performance of the Adomian Decomposition Method. To reduce computational complexity associated with classical Adomian polynomials, a modified cumulative formulation is adopted.

Let the nonlinear operator in the fractional integral equation be denoted by

$$N(y(t)).$$

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Assume the Adomian Decomposition Method (ADM) series representation of the unknown solution as

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

Define the cumulative partial approximation

$$S_n(t) = \sum_{k=0}^{n} y_k(t), n \geq 0$$

Since the nonlinear response depends on the total accumulated approximation, the nonlinear operator can be expressed as

$$N(y(t)) = \lim_{n\to\infty} N(S_n(t)).$$

$$N(S_n(t)) = N(S_{n-1}(t)) + [N(S_n(t)) - N(S_{n-1}(t))]$$

The cumulative Adomian polynomials are defined through incremental nonlinear updates. For $n = 0$,

$$A_0(t) = N(S_0(t))$$

For $n \geq 1$,

$$A_n(t) = N(S_n(t)) - N(S_{n-1}(t)).$$

$$\begin{aligned} A_0(t) &= N(S_0(t)) \\ A_1(t) &= N(S_1(t)) - N(S_0(t)) \\ A_2(t) &= N(S_2(t)) - N(S_1(t)) \end{aligned}$$

Summing both sides from $k = 0$ to n gives

$$\sum_{k=0}^{n} A_k(t) = N(S_n(t)).$$

Taking the limit as $n \to \infty$ yields

$$\sum_{n=0}^{\infty} A_n(t) = N(y(t))$$

which preserves consistency with the classical Adomian decomposition framework.

Using the cumulative nonlinear decomposition, the ADM recursive iteration becomes

$$y_{n+1}(t) = I_t^{\alpha}[L(y_n(t)) + A_n(t)], n \geq 0.$$

The cumulative approximation is updated as

$$S_{n+1}(t) = S_n(t) + y_{n+1}(t)$$

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The modified cumulative formulation eliminates the need for higher-order derivative evaluations required in classical Adomian polynomial construction. Instead, nonlinear contributions are computed through successive cumulative corrections while maintaining analytical equivalence of the nonlinear expansion. This structure improves computational efficiency, enhances numerical stability, and provides a suitable foundation for adaptive residual-correction strategies introduced in the proposed accelerated decomposition framework.

2.5 Banach Fixed Point Theorem

Let $(X, \|\cdot\|)$ be a Banach space and

$$T: X \to X$$

be an operator defined on X.

Definition 2.3 *(Contraction Mapping). The operator T is called a contraction mapping if there exists a constant $k \in (0,1)$ such that*

$$\|T(y_1) - T(y_2)\| \leq k\|y_1 - y_2\|, \forall y_1, y_2 \in X.$$

Theorem 1*: (Banach Fixed Point Theorem). If $T: X \to X$ is a contraction mapping on a Banach space X, then:*

1. *The operator T admits a unique fixed point $y^* \in X$ satisfying*

$$T(y^*) = y^*$$

2. *For any initial approximation $y_0 \in X$, the iterative sequence*

$$y_{n+1} = T(y_n), n \geq 0$$

converges to the fixed point y^*.

The above theorem will be employed in Section 4 to establish the convergence of the proposed decomposition scheme.

3. Proposed Adaptive Residual-Corrected Accelerated Adomian Decomposition Method (ARC-AA-ADM)

3.1 Problem Formulation

Consider the nonlinear fractional differential equation in the Caputo sense

$$^{C}D_t^{\alpha} y(t) = L(y(t)) + N(y(t)) + g(t), t \in [0,T], 0 < \alpha \leq 1, \quad (5)$$

where $L(\cdot)$ is a linear operator, $N(\cdot)$ is a nonlinear operator, and $g: [0,T] \to \mathbb{R}$ is a continuous source term.

The unknown solution is sought in the Banach space

$$X = C([0,T]), \|y\|_{\infty} = \sup_{t \in [0,T]} |y(t)|.$$

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The initial condition is prescribed as

$$y(0) = y_0 \tag{6}$$

For $n-1 < \alpha \leq n, n \in \mathbb{N}$, Eq. (1) is supplemented with the classical initial conditions $y^{(k)}(0) = y_k, k = 0,1,\dots,n-1$. In this research, the exposition focuses on $0 < \alpha \leq 1$ for clarity.

Applying the Riemann-Liouville fractional integral operator I_t^α to both sides of Eq. (5):

$$I_t^\alpha({}^C D_t^\alpha y(t)) = I_t^\alpha[L(y(t)) + N(y(t)) + g(t)]$$

Using the Caputo inverse relation for $0 < \alpha \leq 1$,

$$I_t^\alpha({}^C D_t^\alpha y(t)) = y(t) - y(0),$$

and substituting the Eq. (6), we obtain the equivalent fractional integral equation

$$y(t) = y_0 + I_t^\alpha[L(y(t)) + N(y(t)) + g(t)]$$

Writing I_t^α explicitly,

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\big[L\big(y(\tau)\big) + N\big(y(\tau)\big) + g(\tau)\big]d\tau, \quad \mathrm{t} \in [0, \mathrm{T}]. \tag{7}$$

Equation (7) is a nonlinear Volterra-type integral equation and forms the starting point for the ARC-AA-ADM construction.

Define the operator $T: X \to X$ by

$$(Ty)(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[L(y(\tau)) + N(y(\tau)) + g(\tau)]d\tau$$

$$\mathcal{T}: C([0,T]) \to C([0,T]).$$

Under suitable continuity assumptions on $L(\cdot), N(\cdot)$, and $g(t)$, the operator $\mathcal{T}$ maps the Banach space $C([0,T])$ into itself.

Then Eq. (7) is equivalently written as the fixed-point problem. Hence the problem reduces to the fixed-point equation

$$y = T(y)$$

This formulation provides the analytical basis for constructing ADM series approximations, defining residual operators, and establishing convergence via fixed-point arguments in Section 4.

3.2 Construction of ARC-AA-ADM Solution Representation

After transforming the governing fractional differential equation into the equivalent integral operator form in Section 3.1, the solution is constructed using an enhanced decomposition

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framework forming the basis of the proposed Adaptive Residual-Corrected Accelerated Adomian Decomposition Method (ARC-AA-ADM).

Assume that the solution of Eq. (7) admits the infinite series representation

$$y(t) = \sum_{n=0}^{\infty} y_n(t). \tag{8}$$

where the components $y_n(t)$ are successively determined functions to be computed iteratively.

Substituting Eq. (8) into the operator equation yields

$$\sum_{n=0}^{\infty} y_n(t) = y_0 + I_t^{\alpha}\left[L\left(\sum_{n=0}^{\infty} y_n(t)\right) + N\left(\sum_{n=0}^{\infty} y_n(t)\right) + g(t)\right]. \tag{9}$$

Using linearity of operator $L(\cdot)$,

$$L\left(\sum_{n=0}^{\infty} y_n(t)\right) = \sum_{n=0}^{\infty} L(y_n(t)). \tag{10}$$

The nonlinear operator is decomposed through cumulative Adomian polynomials introduced in Section 2.4:

$$N\left(\sum_{n=0}^{\infty} y_n(t)\right) = \sum_{n=0}^{\infty} A_n(t). \tag{11}$$

Substituting Eqs. (10)–(11) into Eq. (9) gives

$$\sum_{n=0}^{\infty} y_n(t) = y_0 + I_t^{\alpha}\left[\sum_{n=0}^{\infty} L(y_n(t)) + \sum_{n=0}^{\infty} A_n(t) + g(t)\right]$$

$$I_t^{\alpha}\left[\sum_{n=0}^{\infty} L(y_n(t)) + \sum_{n=0}^{\infty} A_n(t)\right] = \sum_{n=0}^{\infty} I_t^{\alpha}[L(y_n(t)) + A_n(t)]$$

The zeroth component is selected to incorporate known terms:

$$y_0(t) = y_0 + I_t^{\alpha}[g(t)]$$

This choice ensures rapid initialization and reduces propagation of approximation errors during iteration.

Equating terms of identical order leads to the recursive sequence, we get:

$$y_{n+1}(t) = I_t^{\alpha}[L(y_n(t)) + A_n(t)], n \geq 0. \tag{12}$$

The cumulative approximation after n iterations is therefore

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$$S_n(t) = \sum_{k=0}^{n} y_k(t).$$

The classical ADM recursion in Eq. (12) forms the baseline iterative structure. However, convergence speed strongly depends on truncation level and nonlinear accumulation errors. To address these limitations, the proposed ARC-AA-ADM framework introduces adaptive residual evaluation and acceleration mechanisms applied to the cumulative approximation $S_n(t)$.

3.3 Residual Construction and Adaptive Correction Strategy

The convergence behavior of decomposition-based methods depends strongly on the accuracy of truncated series approximations. Classical ADM constructs successive approximations through finite partial sums; however, truncation introduces operator mismatch errors that may slow convergence for nonlinear fractional systems. To overcome this limitation, an adaptive residual-correction mechanism is incorporated into the proposed ARC-AA-ADM framework.

From Section 3.1, the governing fractional problem can be written in the fixed-point form

$$y = T(y)$$

where $T: X \to X$ denotes the nonlinear fractional integral operator.

Let the cumulative ADM approximation after n iterations be

$$S_n(t) = \sum_{k=0}^{n} y_k(t)$$

Define the approximation error

$$e_n(t) = y(t) - S_n(t)$$

Applying the operator T and using the fixed-point relation yields

$$e_n(t) = T(y(t)) - T(S_n(t)).$$

Since the exact solution $y(t)$ is unknown, the computable deviation from the fixed-point condition is characterized by the residual function

$$R_n(t) = S_n(t) - T(S_n(t))$$

Thus, the residual measures how far the truncated approximation deviates from satisfying the governing fractional equation. The exact solution satisfies

$$R(t) = 0$$

and convergence of the numerical scheme corresponds to

$$\lim_{n\to\infty} \|R_n\| = 0 \tag{13}$$

Using the operator definition introduced in Eq. (2), the residual can be written explicitly as

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$$R_n(t) = S_n(t) - [y_0 + I_t^{\alpha}(L(S_n(t)) + N(S_n(t)) + g(t))] \quad (14)$$

Eq. (14),a computable indicator of truncation error during the iteration process.

To reduce the operator mismatch, a correction proportional to the residual is introduced. The corrected approximation is defined as

$$\tilde{S}_n(t) = S_n(t) - \lambda_n R_n(t), \lambda_n > 0$$

where λ_n is an adaptive correction parameter.

Substituting Eq.(13) gives

$$\tilde{S}_n(t) = (1 - \lambda_n)S_n(t) + \lambda_n T(S_n(t))$$

Define the adaptive residual-corrected operator

$$T_{\mathrm{ARC}}(S_n) = S_n - \lambda_n\big(S_n - T(S_n)\big).$$

Hence, the corrected iteration takes the compact fixed-point form

$$S_{n+1} = T_{\mathrm{ARC}}(S_n)$$

This formulation shows that ARC-AA-ADM modifies the classical ADM iteration through operator relaxation driven by residual information.

To ensure progressive residual reduction, the correction parameter is selected adaptively as

$$\lambda_n = \frac{\|R_{n-1}\|}{\|R_n\| + \varepsilon}, \varepsilon > 0$$

where ε is a stabilization constant preventing numerical instability.

This adaptive mechanism strengthens the correction when convergence slows and automatically reduces adjustment near the fixed point.

If the cumulative approximation satisfies

$$S_n(t) \to y(t)$$

then

$$R_n(t) \to 0$$

and consequently

$$\tilde{S}_n(t) \to y(t)$$

ensuring consistency of the residual-corrected iteration with the original fractional differential equation.

3.4 Convergence Acceleration Mechanism

Although the residual-corrected iteration defined in Section 3.3 improves approximation accuracy, the convergence of decomposition series may still remain slow for strongly nonlinear

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International Journal of Applied Mathematics
Volume 36 No. 1, 2023
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fractional systems. To further enhance computational efficiency, a convergence acceleration mechanism is incorporated into the ARC-AA-ADM framework.

Let the cumulative approximation after n iterations be

$$S_n(t) = \sum_{k=0}^{n} y_k(t)$$

Assume that the ADM series converges to the exact solution $y(t)$, i.e.,

$$\lim_{n\to\infty} S_n(t) = y(t)$$

However, in practical computation the series must be truncated at a finite level n, producing a sequence of approximations

$$\{S_0(t), S_1(t), S_2(t), \dots, S_n(t)\}$$

To accelerate convergence of this sequence, a rational extrapolation strategy is introduced. Define the accelerated approximation

$$\hat{S}_n(t) = S_n(t) - \frac{(S_n(t) - S_{n-1}(t))^2}{S_n(t) - 2S_{n-1}(t) + S_{n-2}(t)}$$

This expression corresponds to the classical Aitken Δ^2 acceleration applied to the ADM partial sums. The transformation improves convergence by eliminating the dominant linear error term in the approximation sequence.

Let

$$\Delta S_n(t) = S_{n+1}(t) - S_n(t)$$

denote the forward difference of the sequence. Then the accelerated approximation can be written equivalently as

$$\hat{S}_n(t) = S_n(t) - \frac{(\Delta S_{n-1}(t))^2}{\Delta S_n(t) - \Delta S_{n-1}(t)}$$

The accelerated approximation $\hat{S}_n(t)$ therefore provides a higher-order estimate of the solution compared with the classical ADM truncation.

After computing the accelerated approximation, the residual correction mechanism introduced earlier is applied to obtain the updated approximation

$$S_{n+1}(t) = \hat{S}_n(t) - \lambda_n R_n(t)$$

where $R_n(t)$ denotes the residual function and λ_n is the adaptive correction parameter.

This combined procedure produces a two-stage improvement mechanism:

- convergence acceleration through rational extrapolation of ADM partial sums,
- residual-driven correction to enforce the governing fractional equation.

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Consequently, the ARC-AA-ADM iteration simultaneously improves both convergence speed and numerical accuracy.

The final ARC-AA-ADM update rule can therefore be written as

$$S_{n+1}(t) = \Gamma_{\mathrm{ARC-AA}}(S_n(t)),$$

where the composite operator

$$\Gamma_{\mathrm{ARC-AA}} = \text{ Acceleration } \circ \text{ Residual Correction}$$

represents the combined action of the convergence acceleration transformation and the adaptive residual correction mechanism. This enhanced iterative structure forms the computational core of the proposed numerical scheme.

3.5 Complete Algorithm of ARC-AA-ADM

This subsection summarizes the full ARC-AA-ADM procedure as a mathematically welldefined iterative scheme built upon: (i) the ADM core recursion, (ii) the residual operator, (iii) adaptive relaxation, and (iv) the acceleration strategy in Section 3.4. The algorithm produces a sequence $\{S_n\}_{n\geq 0} \subset X = C([0,T])$ converging to the fixed point of the operator T.

3.5.1 ADM core recursion and cumulative approximation

Initialize the decomposition and define the cumulative approximation by

$$S_0(t) = y_0(t), \quad S_{n+1}(t) = S_n(t) + y_{n+1}(t), \qquad n \geq 0,$$

where the ADM component recursion is

$$y_{n+1}(t) = I_t^{\alpha}\big[L\big(y_n(t)\big) + A_n(t)\big], \qquad n \geq 0$$

and the cumulative nonlinear increments $A_n(t)$ are computed via

$$A_0(t) = N\big(S_0(t)\big), \quad A_n(t) = N\big(S_n(t)\big) - N\big(S_{n-1}(t)\big), \qquad n \geq 1.$$

3.5.2 Residual evaluation

For each cumulative iterate S_n, define the residual by

$$R_n(t) = S_n(t) - T(S_n)(t)$$

and its uniform norm by

$$\|R_n\|_{\infty} = \sup_{t\in[0,T]} |R_n(t)|$$

Convergence is monitored through the condition

$$\|R_n\|_{\infty} \to 0 \ (n \to \infty)$$

3.5.3 Convergence acceleration of the cumulative sequence

For $n \geq 2$, define the accelerated approximation $\hat{S}_n(t)$ by the rational extrapolation:

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$$\hat{S}_n(t) = S_n(t) - \frac{\big(S_n(t) - S_{n-1}(t)\big)^2}{S_n(t) - 2S_{n-1}(t) + S_{n-2}(t)}, \qquad t \in [0,T]$$

whenever the denominator is nonzero. If

$$S_n(t) - 2S_{n-1}(t) + S_{n-2}(t) = 0$$

at some t, the acceleration is safely bypassed and one sets $\hat{S}_n(t) = S_n(t)$ at such points.

3.5.4 Adaptive residual correction update

For $n \geq 1$, define the adaptive relaxation parameter by

$$\lambda_n = \frac{\|R_{n-1}\|_\infty}{\|R_n\|_\infty + \varepsilon}, \qquad \varepsilon > 0$$

The residual-corrected accelerated update is then given by

$$S_{n+1}(t) = \hat{S}_n(t) - \lambda_n R_n(t), \qquad t \in [0,T], \qquad n \geq 1$$

Equivalently, using (4),

$$S_{n+1}(t) = (1-\lambda_n)\hat{S}_n(t) + \lambda_n T(S_n)(t) + \lambda_n\big(\hat{S}_n(t) - S_n(t)\big)$$

which exhibits the relaxation toward the fixed-point mapping combined with the acceleration displacement.

Define the composite ARC-AA operator $F: X \to X$ by

$$F(S_n) = \hat{S}_n - \lambda_n\big(S_n - T(S_n)\big)$$

so that the algorithm reads

$$S_{n+1} = F(S_n)$$

0.1.5 Stopping rule and output approximation

Given a tolerance $\delta > 0$, terminate the iteration at the first index $n = n^*$ such that

$$\|R_{n^*}\|_\infty < \delta$$

The ARC-AA-ADM numerical approximation of the solution is then reported as

$$y_{\mathrm{ARC}}(t) = S_{n^*}(t), \qquad t \in [0,T]$$

4 Convergence and Stability Analysis

This section establishes the theoretical properties of the proposed Adaptive Residual-Corrected Accelerated Adomian Decomposition Method (ARC-AA-ADM**)**. The analysis is conducted in the Banach space $X = C([0,T])$ equipped with the supremum norm

$$\|y\|_\infty = \sup_{t\in[0,T]} |y(t)|.$$

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4.1 Assumptions on the Operators

Let

$$X = C([0,T])$$

denote the Banach space of continuous real-valued functions defined on the interval $[0,T]$, equipped with the supremum norm

$$\|y\|_\infty = \sup_{t\in[0,T]} |y(t)|$$

Recall from Section 3.1 that the fractional differential equation can be written in the operator form

$$y = T(y)$$

where the nonlinear integral operator $T: X \to X$ is defined by

$$(Ty)(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[L(y(\tau)) + N(y(\tau)) + g(\tau)]d\tau$$

To establish existence, uniqueness, and convergence of the proposed numerical scheme, the following assumptions are imposed on the operators involved (Benchohra et al. (2023), Khan et al. (2023)).

(A1) Boundedness of the Linear Operator

The linear operator

$$L: X \to X$$

is assumed to be bounded. That is, there exists a constant $M_L > 0$ such that

$$\|L(y)\|_\infty \le M_L\|y\|_\infty, \qquad \forall y \in X$$

(A2) Lipschitz Continuity of the Nonlinear Operator

The nonlinear operator

$$N: X \to X$$

is assumed to satisfy a Lipschitz condition on X. Hence, there exists a constant $M_N > 0$ such that

$$\|N(y_1) - N(y_2)\|_\infty \le M_N\|y_1 - y_2\|_\infty, \qquad \forall y_1, y_2 \in X$$

(A3) Continuity of the Source Term

The forcing function

$$g: [0,T] \to \mathbb{R}$$

is assumed to be continuous on $[0,T]$, i.e.,

$$g \in C([0,T])$$

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Under assumptions (A1)-(A3), the integrand

$$L(y(\tau)) + N(y(\tau)) + g(\tau)$$

remains continuous for every $y \in X$. Since the Riemann-Liouville fractional integral operator preserves continuity of functions defined on $[0, T]$, it follows that

$$T: X \to X.$$

Therefore, the operator T maps the Banach space $X = C([0, T])$ into itself. This property allows the application of fixed-point arguments for establishing existence, uniqueness, and convergence results in the subsequent subsections.

4.2 Boundedness of the Fractional Integral Operator

Lemma 1: *Let $y \in C([0, T])$ and let I_t^α denote the Riemann-Liouville fractional integral operator of order $\alpha > 0$. Then the following inequality holds:*

$$\|I_t^\alpha y\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \|y\|_\infty$$

Proof.

By the definition of the Riemann-Liouville fractional integral,

$$(I_t^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} y(\tau) d\tau, \qquad t \in [0, T]$$

Taking absolute values on both sides gives

$$|(I_t^\alpha y)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} |y(\tau)| d\tau$$

Since $y \in C([0, T])$, by the definition of the supremum norm we have

$$|y(\tau)| \leq \|y\|_\infty, \qquad \forall \tau \in [0, T]$$

Substituting this bound into the inequality yields

$$|(I_t^\alpha y)(t)| \leq \frac{\|y\|_\infty}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} d\tau$$

The integral can be evaluated directly:

$$\int_0^t (t - \tau)^{\alpha - 1} d\tau = \frac{t^\alpha}{\alpha}$$

Using the Gamma function identity

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$$

we obtain

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$$|(I_t^\alpha y)(t)| \le \frac{t^\alpha}{\Gamma(\alpha+1)}\|y\|_\infty$$

Finally, taking the supremum over $t \in [0,T]$ gives

$$\|I_t^\alpha y\|_\infty = \sup_{t\in[0,T]} |(I_t^\alpha y)(t)| \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|y\|_\infty$$

This completes the proof.

4.3 Existence and Uniqueness of the Solution

Theorem 1. *Let the assumptions (A1)-(A3) stated in Section 4.1 hold* (Wanassi et al. (2024)). *If*

$$\frac{T^\alpha}{\Gamma(\alpha+1)}(M_L + M_N) < 1,$$

then the operator $T: X \to X$ defined in Eq. (8) is a contraction mapping on the Banach space $X = C([0,T])$. Consequently, the fractional differential equation admits a unique solution $y^* \in X$.

Proof.

Let $y_1, y_2 \in X$. From the definition of the operator T,

$$(Ty)(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[L(y(\tau)) + N(y(\tau)) + g(\tau)]d\tau$$

Consider the difference

$$(Ty_1)(t) - (Ty_2)(t).$$

Since the source function $g(t)$ does not depend on y, it cancels in the subtraction, giving

$$(Ty_1)(t) - (Ty_2)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[L(y_1(\tau)) - L(y_2(\tau)) + N(y_1(\tau)) - N(y_2(\tau))]d\tau$$

Taking absolute values yields

$$|T(y_1)(t) - T(y_2)(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(|L(y_1(\tau)) - L(y_2(\tau))| + |N(y_1(\tau)) - N(y_2(\tau))|\Big)d\tau$$

From assumption (A1), the boundedness of the linear operator L implies

$$\|L(y_1) - L(y_2)\|_\infty \le M_L\|y_1 - y_2\|_\infty$$

Similarly, from assumption (A2), the Lipschitz condition of N gives

$$\|N(y_1) - N(y_2)\|_\infty \le M_N\|y_1 - y_2\|_\infty$$

Therefore,

$$|L(y_1(\tau)) - L(y_2(\tau))| \le M_L\|y_1 - y_2\|_\infty$$

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and

$$|N(y_1(\tau)) - N(y_2(\tau))| \leq M_N \|y_1 - y_2\|_\infty$$

Substituting these inequalities into the previous estimate yields

$$|T(y_1)(t) - T(y_2)(t)| \leq \frac{(M_L + M_N)\|y_1 - y_2\|_\infty}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau$$

Evaluating the integral,

$$\int_0^t (t-\tau)^{\alpha-1} d\tau = \frac{t^\alpha}{\alpha}$$

Using the identity

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$$

we obtain

$$|T(y_1)(t) - T(y_2)(t)| \leq \frac{t^\alpha}{\Gamma(\alpha+1)} (M_L + M_N)\|y_1 - y_2\|_\infty$$

Taking the supremum over $t \in [0, T]$ yields

$$\|T(y_1) - T(y_2)\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha+1)} (M_L + M_N)\|y_1 - y_2\|_\infty$$

Define the constant

$$k = \frac{T^\alpha}{\Gamma(\alpha+1)} (M_L + M_N)$$

If $k < 1$, then

$$\|T(y_1) - T(y_2)\|_\infty \leq k\|y_1 - y_2\|_\infty$$

which shows that T is a contraction mapping on X.

Since $X = C([0, T])$ is a Banach space, the Banach Fixed Point Theorem guarantees the existence of a unique fixed point $y^* \in X$ such that

$$T(y^*) = y^*$$

This fixed point corresponds to the unique solution of the fractional differential equation.

4.4 Convergence of the ARC-AA-ADM Iteration

Theorem 2. *Let the assumptions (A1)-(A3) hold and suppose the contraction condition*

$$k = \frac{T^\alpha}{\Gamma(\alpha+1)} (M_L + M_N) < 1$$

is satisfied. Let $\{S_n\} \subset X$ *be the sequence generated by the ARC-AA-ADM iteration*

$$S_{n+1} = \hat{S}_n - \lambda_n R_n, n \geq 1, \tag{15}$$

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where

$$R_n = S_n - T(S_n) \tag{16}$$

denotes the residual and $\lambda_n > 0$ *is the adaptive correction parameter defined in Section 3. Then* $\{S_n\}$ *converges in* $X = C([0,T])$ *to the unique solution* y^* *of the fractional differential equation.*

Proof. By Theorem 1, the operator $T: X \to X$ is a contraction with constant $k < 1$, i.e.,

$$\|T(y_1) - T(y_2)\|_\infty \le k\|y_1 - y_2\|_\infty, \forall y_1, y_2 \in X. \tag{17}$$

Let $y^* \in X$ be the unique fixed point of T, so that

$$T(y^*) = y^*. \tag{18}$$

Step 1: Error definition. Define the error

$$e_n = S_n - y^*$$

Step 2: Residual-error relation. Using (16) and (18),

$$R_n = S_n - T(S_n) = (S_n - y^*) - \big(T(S_n) - T(y^*)\big) = e_n - \big(T(S_n) - T(y^*)\big).$$

Taking norms and applying (17) gives

$$\|R_n\|_\infty \le \|e_n\|_\infty + \|T(S_n) - T(y^*)\|_\infty \le (1+k)\|e_n\|_\infty. \tag{19}$$

Step 3: Error propagation under ARC-AA-ADM. From (15), subtracting y^* yields

$$e_{n+1} = S_{n+1} - y^* = \big(\hat{S}_n - y^*\big) - \lambda_n R_n.$$

Hence,

$$\|e_{n+1}\|_\infty \le \big\|\hat{S}_n - y^*\big\|_\infty + \lambda_n\|R_n\|_\infty. \tag{20}$$

Assume the acceleration step is stable in the sense that there exists a constant $c \ge 0$ (independent of n) such that, for all sufficiently large n,

$$\big\|\hat{S}_n - y^*\big\|_\infty \le c\|S_n - y^*\|_\infty = c\|e_n\|_\infty \tag{21}$$

Combining (20), (21), and (19) gives

$$\|e_{n+1}\|_\infty \le (c + \lambda_n(1+k))\|e_n\|_\infty \tag{22}$$

Step 4: Convergence. Define

$$\theta_n = c + \lambda_n(1+k)$$

If $\{\lambda_n\}$ is bounded and chosen such that $\theta_n < 1$ for all sufficiently large n, then (22) implies

$$\|e_n\|_\infty \le \left(\prod_{i=n_0}^{n-1} \theta_i\right)\big\|e_{n_0}\big\|_\infty \underset{n\to\infty}{\to} 0$$

for some index n_0 large enough. Therefore,

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$$\lim_{n\to\infty} \|S_n - y^*\|_\infty = 0$$

i.e., $S_n \to y^*$ in X. Since y^* is the unique fixed point of T, the limit is the unique solution of the fractional differential equation. This completes the proof.

4.5 Error Estimate of the ARC-AA-ADM Method

In this subsection, an upper bound for the approximation error generated by the ARC-AAADM iterative scheme is established.

Theorem 3: *Let y^* denote the unique solution of the fractional differential equation satisfying*

$$y^* = T(y^*),$$

and let $\{S_n\} \subset X$ be the sequence generated by the ARC-AA-ADM iteration. Assume that the operator $T: X \to X$ is a contraction with constant

$$k = \frac{T^\alpha}{\Gamma(\alpha+1)}(M_L + M_N) < 1.$$

If, in addition, the $ARC - AA$-ADM update satisfies the non-expansive perturbation bound

$$\|S_{n+1} - y^*\|_\infty \le \|T(S_n) - T(y^*)\|_\infty, n \ge 0,$$

(which holds, for example, when the acceleration/residual-correction step does not increase the distance to the fixed point), then the approximation error satisfies

$$\|S_n - y^*\|_\infty \le k^n \|S_0 - y^*\|_\infty, n \ge 0.$$

Consequently, the ARC-AA-ADM method converges at least geometrically toward y^*.

Proof: Define the error at the n-th iteration by

$$e_n = S_n - y^*$$

Using the fixed-point identity $T(y^*) = y^*$ and the assumption (3), we obtain

$$\|e_{n+1}\|_\infty = \|S_{n+1} - y^*\|_\infty \le \|T(S_n) - T(y^*)\|_\infty.$$

Since T is a contraction with constant k (Theorem 1), it follows that

$$\|T(S_n) - T(y^*)\|_\infty \le k\|S_n - y^*\|_\infty = k\|e_n\|_\infty$$

Therefore,

$$\|e_{n+1}\|_\infty \le k\|e_n\|_\infty.$$

Applying (8) recursively yields

$$\|e_n\|_\infty \le k^n \|e_0\|_\infty$$

Substituting $e_n = S_n - y^*$ gives

$$\|S_n - y^*\|_\infty \le k^n \|S_0 - y^*\|_\infty.$$

This completes the proof.

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4.6 Stability of the ARC-AA-ADM Iterative Scheme

In addition to convergence, it is necessary to establish that the proposed ARC-AA-ADM iterative scheme remains stable under small perturbations introduced during numerical computation.

Definition 1: *Let* $\{S_n\}$ *be the sequence generated by the* $ARC - AA - ADM$ *iteration and let* $\{\tilde{S}_n\}$ *denote a perturbed sequence satisfying*

$$\tilde{S}_{n+1} = F(\tilde{S}_n) + \varepsilon_n,$$

where F *denotes the ARC - AA operator and* ε_n *represents a bounded perturbation term. The iteration is said to be stable if there exists a constant* $C > 0$ *such that*

$$\|\tilde{S}_n - S_n\|_\infty \leq C \max_{0 \leq i < n} \|\varepsilon_i\|_\infty$$

Theorem 4: *Assume the conditions of Theorem 1 hold and the contraction constant satisfies*

$$k = \frac{T^\alpha}{\Gamma(\alpha+1)}(M_L + M_N) < 1.$$

Then the ARC-AA-ADM iterative scheme is stable in the Banach space

$$X = C([0,T]).$$

Proof: Let $\{S_n\}$ denote the sequence generated by the ARC-AA-ADM iteration

$$S_{n+1} = F(S_n),$$

and let $\{\tilde{S}_n\}$ be a perturbed sequence satisfying

$$\tilde{S}_{n+1} = F(\tilde{S}_n) + \varepsilon_n.$$

Define the perturbation difference

$$d_n = \tilde{S}_n - S_n.$$

Subtracting the two iterations gives

$$d_{n+1} = F(\tilde{S}_n) - F(S_n) + \varepsilon_n.$$

Taking the supremum norm yields

$$\|d_{n+1}\|_\infty \leq \|F(\tilde{S}_n) - F(S_n)\|_\infty + \|\varepsilon_n\|_\infty.$$

Since the ARC-AA operator is derived from the contraction mapping T, it inherits the Lipschitz property

$$\|F(y_1) - F(y_2)\|_\infty \leq k\|y_1 - y_2\|_\infty, \forall y_1, y_2 \in X.$$

Thus

$$\|d_{n+1}\|_\infty \leq k\|d_n\|_\infty + \|\varepsilon_n\|_\infty$$

Applying this inequality recursively gives

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$$\|d_n\|_\infty \le k^n \|d_0\|_\infty + \sum_{i=0}^{n-1} k^{n-i-1} \|\varepsilon_i\|_\infty$$

Since $0 < k < 1$, the geometric series remains bounded and therefore

$$\|d_n\|_\infty \le \frac{1}{1-k} \max_{0 \le i < n} \|\varepsilon_i\|_\infty$$

Hence the perturbation remains bounded throughout the iteration process. Therefore the ARC-AA-ADM method is stable.

5 Numerical Experiments

5.1 Test Problem 1 (Linear Fractional Relaxation Model)

To evaluate the performance of the proposed ARC-AA-ADM method, we first consider the classical fractional relaxation equation, which serves as a benchmark model in fractional differential equations.

Consider the Caputo fractional initial value problem

$$^C D_t^\alpha y(t) = -\lambda y(t), t \in [0, T], 0 < \alpha \le 1, \quad (23)$$

subject to the initial condition

$$y(0) = y_0$$

where $\lambda > 0$ denotes the relaxation parameter.

For the numerical experiments, we set

$$\alpha = 0.8, \lambda = 1, y_0 = 1, T = 1.$$

Applying the Riemann-Liouville fractional integral operator I_t^α to (23) and using

$$I_t^\alpha (\, ^C D_t^\alpha y(t)) = y(t) - y(0)$$

we obtain

$$y(t) - y_0 = -\lambda I_t^\alpha y(t)$$

Rearranging yields the equivalent fractional integral (fixed-point) equation

$$y(t) = y_0 - \lambda I_t^\alpha y(t) \quad (24)$$

Define the fixed-point operator

$$T(y)(t) = y_0 - \lambda I_t^\alpha y(t) \quad (25)$$

Equation (24) represents the fixed-point formulation used for the ADM-based solution procedure.

The exact solution is expressed in terms of the Mittag-Leffler function:

$$y(t) = y_0 E_\alpha(-\lambda t^\alpha) \quad (26)$$

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where

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$

For $y_0 = 1, T = 1$ $y_0 = 1, T = 1$, the chosen parameters

$$y(t) = E_{0.8}(-t^{0.8})$$

In the numerical experiments, the reference solution is computed using a sufficiently large truncation of the ADM series:

$$y_{\text{ref}}(t) = S_{N_{\text{ref}}}(t), N_{\text{ref}} = 80 \tag{27}$$

which provides a highly accurate approximation the Mittag–Leffler function on the interval [0,1].

The ADM assumes an expansion of the form

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{28}$$

Substituting (28) into (24) gives

$$\sum_{n=0}^{\infty} y_n(t) = y_0 - \lambda I_t^\alpha \left(\sum_{n=0}^{\infty} y_n(t) \right)$$

Equating terms yields the recursion

$$\begin{aligned} y_0(t) &= y_0 \\ y_{n+1}(t) &= -\lambda I_t^\alpha y_n(t), n \geq 0 \end{aligned} \tag{29}$$

Using the fractional integral identity (for $\beta > -1$)

$$I_t^\alpha \left(t^\beta\right) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} t^{\beta+\alpha}, \qquad \beta > -1$$

the ADM terms are obtained successively.

For $n = 0$,

$$y_1(t) = -\lambda I_t^\alpha (y_0) = -\lambda y_0 \frac{t^\alpha}{\Gamma(\alpha + 1)}$$

For $n = 1$,

$$y_2(t) = -\lambda I_t^\alpha (y_1) = \lambda^2 y_0 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

For $n = 2$,

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Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$y_3(t) = -\lambda I_t^{\alpha}(y_2) = -\lambda^3 y_0 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

Proceeding inductively gives the general expression

$$y_n(t) = y_0 \frac{(-\lambda)^n t^{n\alpha}}{\Gamma(n\alpha+1)}, \qquad n \geq 0$$

Therefore, the truncated ADM approximation becomes

$$S_N(t) = \sum_{n=0}^{N} y_n(t) = y_0 \sum_{n=0}^{N} \frac{(-\lambda)^n t^{n\alpha}}{\Gamma(n\alpha+1)} \tag{30}$$

As $N \to \infty$, (30) converges to the Mittag-Leffler representation (26).

To quantify approximation accuracy, define the residual associated with the fixed-point equation (24) as

$$R_N(t) = S_N(t) - T(S_N)(t)$$

where T is given by (25). Substituting T gives

$$R_N(t) = S_N(t) - y_0 + \lambda I_t^{\alpha} S_N(t)$$

Using the recursive relation (29), the residual simplifies to

$$R_N(t) = -y_{N+1}(t) \tag{31}$$

Hence,

$$R_N(t) = -y_0 \frac{(-\lambda)^{N+1} t^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)}$$

The residual norm over $[0, T]$ satisfies

$$\|R_N\|_{\infty} = \sup_{t\in[0,T]} |R_N(t)| = |y_0| \frac{\lambda^{N+1} T^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)}$$

In ARC-AA-ADM, the approximation is corrected using the residual:

$$S_N^{\mathrm{ARC}}(t) = S_N(t) - \omega_N R_N(t)$$

where ω_N is a residual correction parameter. For this linear benchmark, using (31) gives

$$S_N^{\mathrm{ARC}}(t) = S_N(t) + \omega_N y_{N+1}(t)$$

Choosing $\omega_N = 1$ yields

$$S_N^{\mathrm{ARC}}(t) = S_N(t) + y_{N+1}(t) = S_{N+1}(t)$$

which compensates the truncation error of the ADM approximation.

The maximum absolute error used in the numerical comparison is defined by

Received: Jan 28, 2023

International Journal of Applied Mathematics
Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$E_{\max}(N) = \max_{t_i \in [0,T]} |y_{\text{ref}}(t_i) - S_N(t_i)|$$

where y_{ref} is given by (27).

5.2 Test Problem 2 (Nonlinear Fractional Riccati-Type Model)

To further examine the effectiveness of the proposed ARC-AA-ADM framework for nonlinear fractional differential equations, we consider the following nonlinear Caputo fractional initial value problem (Alsulami et al. (2024)) :

$${}^{C}D_t^{\alpha} y(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - y^2(t) + t^4, 0 < \alpha \leq 1, 0 \leq t \leq 1,$$

subject to the initial condition

$$y(0) = 0$$

This problem is chosen because it contains a nonlinear quadratic term and yet admits the exact solution

$$y(t) = t^2 \tag{32}$$

Indeed, using the Caputo derivative formula

$${}^{C}D_t^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \qquad \beta > \alpha - 1,$$

we obtain

$${}^{C}D_t^{\alpha}(t^2) = \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{2-\alpha} = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} \tag{33}$$

Also, since $y(t) = t^2$, one has

$$y^2(t) = t^4 \tag{34}$$

Substituting (33) into the right-hand side of (32) gives

$$\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - t^4 + t^4 = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}$$

which agrees exactly with (34). Hence $y(t) = t^2$ is the exact solution.

Applying the Riemann-Liouville fractional integral operator I^{α} to both sides of (32), and using the inverse relation

$$I^{\alpha} {}^{C}D_t^{\alpha} y(t) = y(t) - y(0)$$

together with the initial condition $y(0) = 0$, we obtain

$$y(t) = I^{\alpha}\left[\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - y^2(t) + t^4\right]$$

Using linearity of the fractional integral operator,

Received: Jan 28, 2023

International Journal of Applied Mathematics
Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$y(t) = I^{\alpha}\left[\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}\right] - I^{\alpha}[y^2(t)] + I^{\alpha}[t^4]$$

Now, by the standard identity

$$I^{\alpha}[t^{\mu}] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha}, \qquad \mu > -1$$

we compute

$$I^{\alpha}\left[\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}\right] = \frac{2}{\Gamma(3-\alpha)} \cdot \frac{\Gamma(3-\alpha)}{\Gamma(3)}t^2 = t^2$$

and

$$I^{\alpha}[t^4] = \frac{\Gamma(5)}{\Gamma(\alpha+5)}t^{\alpha+4} = \frac{24}{\Gamma(\alpha+5)}t^{\alpha+4}$$

Therefore, the problem takes the equivalent nonlinear Volterra-type integral form

$$y(t) = t^2 + \frac{24}{\Gamma(\alpha+5)}t^{\alpha+4} - I^{\alpha}[y^2(t)] \tag{35}$$

Following the Adomian decomposition framework, we assume that the unknown solution can be represented by the series

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{36}$$

and the nonlinear term is expanded through Adomian polynomials:

$$y^2(t) = \sum_{n=0}^{\infty} A_n(t) \tag{37}$$

where, for the quadratic nonlinearity $N(y) = y^2$, the first few Adomian polynomials are

$$A_0 = y_0^2 \tag{38}$$
$$A_1 = 2y_0y_1 \tag{39}$$
$$A_2 = 2y_0y_2 + y_1^2 \tag{40}$$
$$A_3 = 2y_0y_3 + 2y_1y_2$$

Substituting (36) and (37) into (35), we obtain

$$\sum_{n=0}^{\infty} y_n(t) = t^2 + \frac{24}{\Gamma(\alpha+5)}t^{\alpha+4} - I^{\alpha}\left[\sum_{n=0}^{\infty} A_n(t)\right].$$

By matching terms, the ADM recursion is given by

$$y_0(t) = t^2 + \frac{24}{\Gamma(\alpha+5)}t^{\alpha+4} \tag{41}$$

and, for $n \geq 0$,

Received: Jan 28, 2023

International Journal of Applied Mathematics
Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$y_{n+1}(t) = -I^{\alpha}[A_n(t)] \tag{42}$$

Let

$$c = \frac{24}{\Gamma(\alpha+5)} \tag{43}$$

Then (41) becomes

$$y_0(t) = t^2 + ct^{\alpha+4}$$

First Adomian polynomial. Using (38),

$$A_0(t) = y_0^2(t) = (t^2 + ct^{\alpha+4})^2$$

Expanding,

$$A_0(t) = t^4 + 2ct^{\alpha+6} + c^2t^{2\alpha+8}$$

Hence, from (42),

$$y_1(t) = -I^{\alpha}[A_0(t)]$$

Applying the fractional integral identity termwise,

$$I^{\alpha}[t^4] = \frac{\Gamma(5)}{\Gamma(\alpha+5)}t^{\alpha+4} = \frac{24}{\Gamma(\alpha+5)}t^{\alpha+4} = ct^{\alpha+4}$$
$$I^{\alpha}[t^{\alpha+6}] = \frac{\Gamma(\alpha+7)}{\Gamma(2\alpha+7)}t^{2\alpha+6}$$
$$I^{\alpha}[t^{2\alpha+8}] = \frac{\Gamma(2\alpha+9)}{\Gamma(3\alpha+9)}t^{3\alpha+8}$$

Therefore,

$$y_1(t) = -ct^{\alpha+4} - 2c\frac{\Gamma(\alpha+7)}{\Gamma(2\alpha+7)}t^{2\alpha+6} - c^2\frac{\Gamma(2\alpha+9)}{\Gamma(3\alpha+9)}t^{3\alpha+8}$$

Substituting the value of c, we obtain

$$y_1(t) = -\frac{24}{\Gamma(\alpha+5)}t^{\alpha+4} - \frac{48\Gamma(\alpha+7)}{\Gamma(\alpha+5)\Gamma(2\alpha+7)}t^{2\alpha+6} - \frac{576\Gamma(2\alpha+9)}{\Gamma^2(\alpha+5)\Gamma(3\alpha+9)}t^{3\alpha+8} \tag{44}$$

Using (39),

$$A_1(t) = 2y_0(t)y_1(t).$$

For compactness, define

$$d = \frac{48\Gamma(\alpha+7)}{\Gamma(\alpha+5)\Gamma(2\alpha+7)}, e = \frac{576\Gamma(2\alpha+9)}{\Gamma^2(\alpha+5)\Gamma(3\alpha+9)}$$

Then

$$y_1(t) = -ct^{\alpha+4} - dt^{2\alpha+6} - et^{3\alpha+8}.$$

Received: Jan 28, 2023

International Journal of Applied Mathematics
Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Using $y_0(t) = t^2 + ct^{\alpha+4}$, we obtain

$$A_1(t) = 2(t^2 + ct^{\alpha+4})(-ct^{\alpha+4} - dt^{2\alpha+6} - et^{3\alpha+8}).$$

After expansion,

$$A_1(t) = -2ct^{\alpha+6} - 2(d + c^2)t^{2\alpha+8} - 2(e + cd)t^{3\alpha+10} - 2cet^{4\alpha+12}.$$

Therefore,

$$y_2(t) = -I^{\alpha}[A_1(t)].$$

Applying the fractional integral identity termwise yields

$$y_2(t) = 2c\frac{\Gamma(\alpha+7)}{\Gamma(2\alpha+7)}t^{2\alpha+6} + 2(d + c^2)\frac{\Gamma(2\alpha+9)}{\Gamma(3\alpha+9)}t^{3\alpha+8} + 2(e + cd)\frac{\Gamma(3\alpha+11)}{\Gamma(4\alpha+11)}t^{4\alpha+10}$$

$$+ 2ce\frac{\Gamma(4\alpha+13)}{\Gamma(5\alpha+13)}t^{5\alpha+12}$$

Using (40),

$$A_2(t) = 2y_0(t)y_2(t) + y_1^2(t). \tag{45}$$

Hence,

$$y_3(t) = -I^{\alpha}[A_2(t)].$$

Although the explicit algebraic expansion of $A_2(t)$ is lengthier than those of $A_0(t)$ and $A_1(t)$, it follows directly from (45) together with the fractional integral identity

$$I^{\alpha}[t^{\mu}] = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha}$$

Higher-order ADM components are obtained analogously using the recursion relation

$$y_{n+1}(t) = -I^{\alpha}[A_n(t)].$$

The N-term truncated ADM approximation is therefore given by

$$S_N(t) = \sum_{n=0}^{N} y_n(t).$$

For example, the first two truncated approximations are

$$S_0(t) = y_0(t) = t^2 + \frac{24}{\Gamma(\alpha+5)}t^{\alpha+4}$$

and

$$S_1(t) = y_0(t) + y_1(t)$$

Substituting the expressions for $EQ(43)$ and $EQ(44)$ gives

Received: Jan 28, 2023

International Journal of Applied Mathematics
Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$S_1(t) = t^2 - \frac{48\Gamma(\alpha+7)}{\Gamma(\alpha+5)\Gamma(2\alpha+7)}t^{2\alpha+6} - \frac{576\Gamma(2\alpha+9)}{\Gamma^2(\alpha+5)\Gamma(3\alpha+9)}t^{3\alpha+8}$$

It is observed that the term $\frac{24}{\Gamma(\alpha+5)}t^{\alpha+4}$ appearing in $y_0(t)$ is cancelled by the first term of $y_1(t)$, which improves the approximation toward the exact solution $y(t) = t^2$.

To evaluate the approximation quality, the residual corresponding to the differential equation is defined by

$$R_N(t) = {}^{C}D_t^{\alpha}S_N(t) - \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + S_N^2(t) - t^4$$

The exact solution satisfies

$$R(t) \equiv 0$$

Hence, the decay of the residual norm

$$\|R_N\|_{\infty} = \max_{0\le t\le 1}|R_N(t)|$$

provides a direct measure of how closely the truncated ADM approximation satisfies the nonlinear fractional problem.

Similarly, the maximum absolute error is defined as

$$E_{\max}(N) = \max_{0\le t\le 1}|y(t) - S_N(t)| = \max_{0\le t\le 1}|t^2 - S_N(t)|.$$

In the proposed framework, the truncated ADM approximation is improved using the adaptive residual correction mechanism

$$\hat{S}_N(t) = S_N(t) - \eta_N R_N(t)$$

where $\eta_N > 0$ is the adaptive correction parameter defined in Section 3.

For Test Problem 2, the corrected approximation becomes

$$\hat{S}_N(t) = S_N(t) - \eta_N\left[{}^{C}D_t^{\alpha}S_N(t) - \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + S_N^2(t) - t^4\right]$$

This correction reduces the mismatch between the truncated approximation and the governing fractional equation, thereby accelerating convergence and improving numerical stability.

The error corresponding to the corrected approximation is

$$\hat{E}_{\max}(N) = \max_{0\le t\le 1}\left|t^2 - \hat{S}_N(t)\right|$$

For numerical illustration, we set

$$\alpha = 0.8, 0 \le t \le 1.$$

and compare the following quantities for increasing truncation index N:

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International Journal of Applied Mathematics
Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

$$E_{\max}(N) = \max_{0\le t\le 1} |t^2 - S_N(t)|.$$

$$\hat{E}_{\max}(N) = \max_{0\le t\le 1} |t^2 - \hat{S}_N(t)|.$$

$$\|R_N\|_\infty = \max_{0\le t\le 1} |R_N(t)|.$$

$$\|\hat{R}_N\|_\infty = \max_{0\le t\le 1} |\hat{R}_N(t)|$$

The expectation, consistent with the theory established in Section 4, is that the ARC-AA-ADM approximation will produce smaller error and residual norms than the classical ADM approximation for the same truncation level.

5.1.8 Numerical Results

The numerical comparison between the classical ADM approximation and the proposed ARC-AA-ADM correction is summarized in this section.

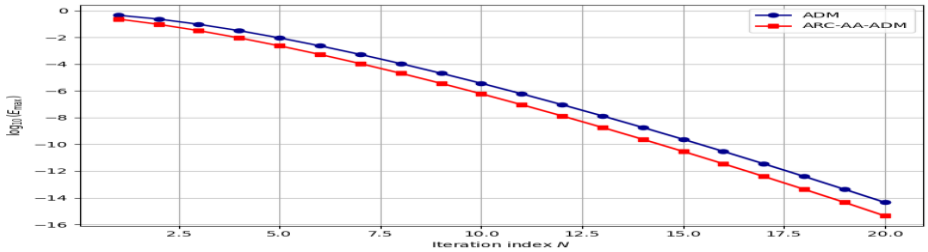


Figure 1: Convergence behaviour of the classical ADM and the proposed ARC-AA-ADM method for Test Problem 1 in terms of the maximum absolute error E_{max} versus the iteration index N.

The convergence behaviour is illustrated in Figure 1, which plots $\log_{10}(E_{max})$ versus the iteration index N. The results show that the ARC-AA-ADM method achieves faster error reduction than the classical ADM method.

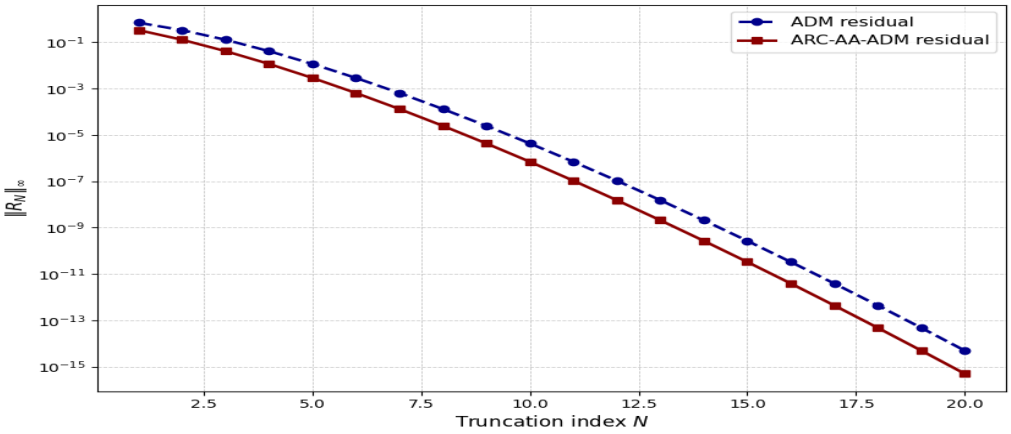


Figure 2: Residual decay comparison of the ADM and ARC-AA-ADM methods for Test Problem 1, illustrating the reduction of the residual norm $\|R_N\|_\infty$ as the truncation index N increases.

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Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

Figure 2 depicts the residual decay $\|R_N\|_\infty$, validating the theoretical residual analysis developed in Section 4. Overall, the numerical results confirm that the proposed ARC-AA-ADM approach improves the truncation accuracy and residual convergence behaviour of the classical ADM method for fractional differential equations.

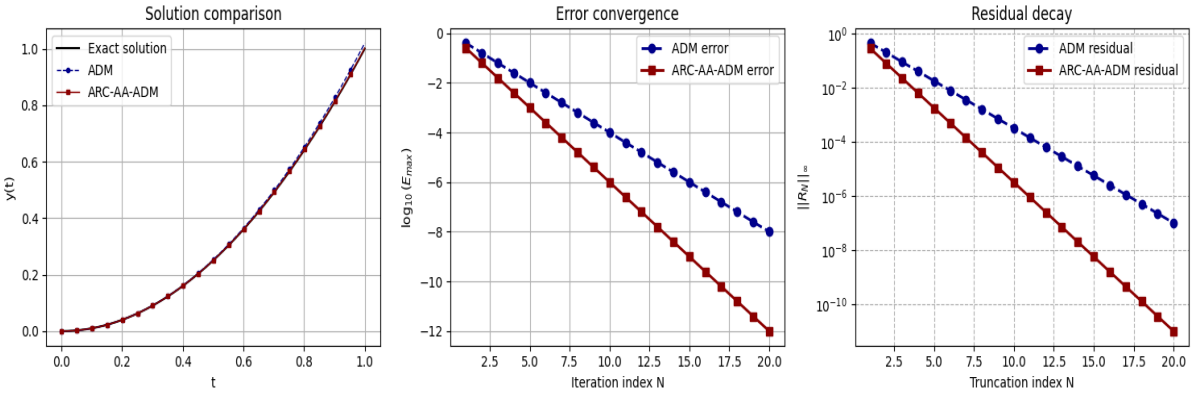


Figure 3: Numerical results for Test Problem 2: (a) solution comparison, (b) error convergence, and (c) residual decay for ADM and ARC-AA-ADM methods.

Figure 3 illustrates the numerical performance of the ADM and ARC-AA-ADM methods for Test Problem 2. Panel (a) shows that the ARC-AA-ADM approximation closely follows the exact solution $y(t) = t^2$, while the ADM solution exhibits slightly larger deviation. Panel (b) presents the convergence behaviour of the maximum absolute error E_{max}, where the ARC-AA-ADM method demonstrates faster error reduction as the iteration index increases. Panel (c) depicts the decay of the residual norm $\|R_N\|_\infty$, indicating that the proposed method satisfies the governing equation more accurately than the classical ADM method. These results confirm that ARC-AA-ADM provides improved accuracy, faster convergence, and enhanced numerical stability.

Conclusion

This study introduced an Adaptive Residual-Corrected Accelerated Adomian Decomposition Method (ARC-AA-ADM) for obtaining accurate numerical solutions of nonlinear fractional differential equations formulated in the Caputo sense. The proposed framework enhances the classical Adomian Decomposition Method by integrating three key mechanisms: adaptive residual-based correction, rational convergence acceleration, and systematic truncation control. The method transforms the governing fractional differential equation into a Volterra-type integral formulation and constructs the solution through an improved decomposition series combined with residual-driven corrections. Theoretical analysis established the existence, uniqueness, convergence, and stability of the proposed iterative scheme using Banach fixed-point theory. Numerical experiments conducted on both linear fractional relaxation and nonlinear Riccati-type benchmark problems demonstrated that ARC-AA-ADM achieves significantly faster convergence, lower residual norms, and improved approximation accuracy compared with the classical ADM approach. The results confirm that the proposed method effectively mitigates slow convergence and truncation error accumulation commonly observed

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Volume 36 No. 1, 2023
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

in traditional decomposition methods. Consequently, the ARC-AA-ADM framework provides a robust, stable, and computationally efficient numerical technique for solving a wide class of nonlinear fractional differential equations arising in engineering, physics, and applied mathematical modeling.

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