International Journal of Applied Mathematics
Volume 36 No. 1 2023, 11-33
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v36i1.2

SOLUTIONS OF A TIME FRACTIONAL BLACK-SCHOLES EQUATION UNDER THE CONSTANT ELASTICITY OF VARIANCE PROCESS

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Abstract: In this paper we consider backward problem and inverse source problem for time-fractional Black-Scholes equation, which arises from pricing double barrier option under the constant elasticity of variance process. Main tools are spectral method and Sturm-Liouville theory. The existence, uniqueness and convergence analysis of analytic solutions of the backward problem and inverse source problem of time-fractional Black-Scholes model are established in terms of Whittaker functions or modified Bessel functions with large variables.

AMS Subject Classification: 35R11; 35R30; 34B24
Key Words: fractional Black-Scholes equation; backward problem; inverse source problem; double barrier options; Sturm-Liouville problem

1. Introduction

Fractional partial differential equations have attracted the attention of a wide range of scientists in various fields due to their many applications in the sciences such as mechanics, physics, economics and so on. In this paper we focus on time-fractional Black-Scholes equation, the backward problem

Received: December 25, 2021 © 2023 Academic Publications

$$\begin{cases} (\frac{\partial}{\partial t})^\alpha V(S,t) + L_\gamma V(S,t) = 0, \quad \text{in} \quad (0,T)\times(S_1,S_2) \\ V(S_1,t) = \psi_1(t), V(S_2,t) = \psi_2(t), \\ V(S,T) = \phi(S), \end{cases} \tag{1}$$

and the inverse source problem

$$\begin{cases} (\frac{\partial}{\partial t})^\alpha V(S,t) + L_\gamma V(S,t) = f(S), \quad \text{in} \quad (0,T)\times(S_1,S_2) \\ V(S_1,t) = \psi_1(t), V(S_2,t) = \psi_2(t), \\ V(S,0) = \phi_1(S), V(S,T) = \phi_2(S) \end{cases} \tag{2}$$

which arise from pricing double barrier option under the so-called constant elasticity of variances diffusion model, where

$$(\frac{\partial}{\partial t})^\alpha V(S,t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{V(S,\tau)-V(S,T)}{(\tau-t)^\alpha}d\tau, \quad \alpha\in(0,1)$$

is a fractional derivative,

$$L_\gamma = \frac{1}{2}\sigma^2 S^{2(1+\gamma)}\frac{\partial^2}{\partial S^2} + (r-D)S\frac{\partial}{\partial S} - r, \quad \gamma\in\mathbb{R}/\{0\} \tag{3}$$

stands for a second order differential operator, and V the price of an option, f the cash flow, S the stock underlying, t the minus of the terminal time and the current time, r the riskless interest rate, σ the volatility of the stock price, D the dividend yield, ϕ (or ϕ_i) the payoff function, ψ_i the rebates paid when the corresponding barrier is hit and S_i a positive number, $i = 1, 2$.

Take the common transform

$$V(S,t) = \bar{V}(S,\tau), \quad \tau = T - t, \tag{4}$$

the fractional derivative becomes into

$$\Big(\frac{\partial}{\partial t}\Big)^\alpha V(S,t) = -\frac{\partial}{\partial\tau}\int_0^\tau \frac{\bar{V}(S,\iota)-\bar{V}(S,0)}{\Gamma(1-\alpha)(\tau-\iota)^\alpha}d\iota = -\Big(\frac{\partial}{\partial\tau}\Big)^\alpha \bar{V}(S,\tau).$$

Recalling ${}_0D_\tau^\alpha u(\tau) = \frac{1}{\Gamma(1-\alpha)}\int_0^\tau \frac{u(\iota)}{(\tau-\iota)^\alpha}d\iota$ (Riemann-Liouville fractional derivative), we have

$$\begin{aligned} \Big(\frac{\partial}{\partial\tau}\Big)^\alpha \bar{V}(S,\tau) &= {}_0D_\tau^\alpha \bar{V}(S,\tau) - {}_0D_\tau^\alpha \bar{V}(S,0) \\ &= {}_0D_\tau^\alpha \bar{V}(S,\tau) - \frac{\bar{V}(S,0)}{\Gamma(1-\alpha)\tau^\alpha}. \end{aligned} \tag{5}$$

Besides above, according to the formula on the relation between the Riemann-Liouville fractional derivative and the Caputo fractional derivative given in [25],

$$ {}_aD_x^\alpha u(x) = {}_a^C D_x^\alpha u(x) + \sum_{k=0}^{n-1} \frac{u^k(a)(x-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)}, \quad n-1<\alpha\le n, $$

set $a=0$, $\alpha\in(0,1)$, then we obtain $\left(\frac{\partial}{\partial\tau}\right)^\alpha \bar{V}(S,\tau) = {}_0^C D_\tau^\alpha \bar{V}(S,\tau)$.

Meanwhile, it is easy to verify that $L_\gamma V(S,t) = L_\gamma \bar{V}(S,\tau)$.

Hence the backward problem (1) is translated into a forward problem

$$ \begin{cases} {}_0^C D_x^\alpha \bar{V}(S,\tau) = L_\gamma \bar{V}(S,\tau) \quad \text{in} \quad (0,T)\times(S_1,S_2), \\ \bar{V}(S_1,\tau) = \bar{\psi}_1(\tau), \bar{V}(S_2,\tau) = \bar{\psi}_2(\tau), \\ \bar{V}(S,0) = \bar{\phi}(S), \end{cases} \tag{6} $$

with $\bar{\phi}(S) = \phi(S)$, $\bar{\psi}_i(\tau) = \psi_i(T-\tau)$, $i=1,2$ and the inverse source problem (2) becomes as follows

$$ \begin{cases} {}_0^C D_x^\alpha \bar{V}(S,\tau) = L_\gamma \bar{V}(S,\tau) + f(S) \quad \text{in} \quad (0,T)\times(S_1,S_2), \\ \bar{V}(S_1,\tau) = \bar{\psi}_1(\tau), \bar{V}(S_2,\tau) = \bar{\psi}_2(\tau), \\ \bar{V}(S,0) = \bar{\phi}_2(S), \bar{V}(S,T) = \bar{\phi}_1(S) \end{cases} \tag{7} $$

with $\bar{\phi}_i(S) = \phi_i(S)$, $\bar{\psi}_i(\tau) = \psi_i(T-\tau)$, $i=1,2$.

The classical Black-Scholes equation

$$ \frac{\partial}{\partial t}V(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}V(S,t) + rS\frac{\partial}{\partial S}V(S,t) - rV(S,t) = 0 $$

was introduced in Nobel-prize winning paper [4] by Black and Scholes for evaluating the fair value of the European call option that gives the right to buy a single share of common stock and established a semi-closed formula for the option price, from then on it was extensively used to analyze price of options. Recently, in view of the fact that fractional derivatives and integrals provide a powerful instrument in the description of memory and hereditary properties of different substances, a growing number of researchers have generalized Black-Scholes equation to a fractional type, one can refer to [16, 20, 22, 21, 33]. Motivated by the idea presented in [5] which studied double barrier options for the case $\gamma = 0$ and derived an explicit analytical solution in series form to cut through the difficulty in establishing the analytical solution under the appearances of fractional derivative and two barrier functions, the author generalized the research to a similar model with hyper-Bessel operator [37] and the backward problem (1). Since $\gamma \neq 0$ under the constant elasticity of variance process, the eigenfunction and the eigenvalue of operator L_γ are more difficult to calculated,

$$L_\gamma = \frac{1}{2}\sigma^2 B_{1+\gamma}^2 - \frac{1+\gamma}{2}\sigma^2 S^\gamma B_{1+\gamma} + (r-D)B_1 - r,$$

where $B_1 = S\frac{\partial}{\partial S}$, $B_{1+\gamma} = S^{1+\gamma}\frac{\partial}{\partial S}$ denoting hyper-Bessel operator [8, 18, 19, 24]. Theories of single operator B_θ^α also are studied and applied to solve diffusive transport and Brownian motion [2, 1, 12, 13, 26, 29, 30, 34, 35, 36]. However, there is few result on combination of multiple hyper-Bessel operators, like as $L_\gamma, \gamma \neq 0$. Furthermore, the numerical method are more fit for solving fractional partial differential equations [14, 38]. In this paper, the method relies on the orthonormal system of eigenfunctions of the operator with respect to the space variables. We use Sturm-Liouville theory and some common transformations to construct eigenfunctions and analyze eigenvalues of the two point boundary value problem of L_γ. Then asymptotic behavior of eigenvalues are established and the eigenfunctions are formed in terms of Whittaker functions with one large parameter or modified Bessel functions with large variable in different cases respectively. The calculation is heavy although the method seems standard. At last, we obtain the existence and uniqueness of solution of the backward problem (1). Furthermore, the procedure can be used to solve inverse source problem (2) by taking some modifications. Other similar research interesting can be found [7, 11, 32] and the references therein.

This paper is organized as follows: In §2, the related results of special functions are recalled. In §3, the eigenvalue problem of Sturm-Liouville type equation was considered. In §4, in terms of the results given in §2-3, we establish an analytical solution in series form to forward problem of modified time-fractional Black-Scholes equation. In §5, we give the existence and uniqueness of solution for a backward problem (1) and constructed an example to display applications. In §6, we establish the well posedness of inverse source problem (2). At last, we give conclusions.

2. Preliminaries

In this section, we recall some well-known results of special functions for the convenience in computation.

First, the theory of Bessel function $J_\nu(z)$, $Y_\nu(z)$ and modified Bessel function $I_\nu(z)$, $K_\nu(z)$ can be found in Chapter 10 [28].

Lemma 1. *When $z \to \infty$ with ν fixed, then*

$$J_\nu(z) \sim (\frac{2}{\pi z})^{\frac{1}{2}} \left(\cos\omega \sum_{i=0}^{\infty} \frac{(-1)^i a_{2i}(\nu)}{z^{2i}} - \sin\omega \sum_{i=0}^{\infty} \frac{(-1)^i a_{2i+1}(\nu)}{z^{2i+1}} \right),$$

$$Y_\nu(z) \sim (\frac{2}{\pi z})^{\frac{1}{2}} \left(\sin\omega \sum_{i=0}^{\infty} \frac{(-1)^i a_{2i}(\nu)}{z^{2i}} - \cos\omega \sum_{i=0}^{\infty} \frac{(-1)^i a_{2i+1}(\nu)}{z^{2i+1}} \right)$$

for $|phz| \leq \pi - \delta$, where δ is an arbitrary small positive constant, $\omega = z - \frac{\nu\pi}{2} - \frac{\pi}{4}$ and $a_i(\nu) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots(4\nu^2-(2i-1)^2)}{i!8^i}$, $a_0(\nu) = 1$.

Lemma 2. *When $z \to \infty$ with ν fixed, then*

$$I_\nu(z) \sim (\frac{e^z}{\sqrt{2\pi z}}) \sum_{i=0}^{\infty} (-1)^i \frac{a_i(\nu)}{z^i}, \tag{8}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{i=0}^{\infty} (-1)^i \frac{a_i(\nu)}{z^i} \tag{9}$$

for $|phz| \leq \frac{\pi}{2} - \delta$, where δ is an arbitrary small positive constant.

Lemma 3. *$\mathfrak{L}_\nu(z)$ denotes a modified Bessel function $I_\nu(z)$ or $K_\nu(z)$, then $(\frac{1}{z}\frac{d}{dz})^i (z^\nu \mathfrak{L}_\nu(z)) = z^{\nu-i}\mathfrak{L}_{\nu-i}(z)$.*

Next, Whittaker function $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ can be found in Chapter 13, [28] or [9].

Lemma 4. *When $\kappa \to \infty$ through positive real values with nonnegative number μ fixed, then*

$$M_{\kappa,\mu}(z) = \frac{\sqrt{z}\Gamma(2\mu+1)}{\kappa^\mu} \left(J_{2\mu}(\theta) + \mathrm{env}J_{2\mu}(\theta)\mathcal{O}(\kappa^{-\frac{1}{2}}) \right), \tag{10}$$

$$\begin{aligned} W_{\kappa,\mu}(z) =& \sqrt{z}\Gamma(2\mu+1) \Big(\sin(\kappa\pi - \mu\pi) J_{2\mu}(\theta) \\ & - \cos(\kappa\pi - \mu\pi) Y_{2\mu}(\theta) + \mathrm{env}Y_{2\mu}(\theta)\mathcal{O}(\kappa^{-\frac{1}{2}}) \Big) \end{aligned} \tag{11}$$

uniformly with respect to $z \in (0, A]$ in each case, where A is an arbitrary positive constant and $\mathrm{env}J_\nu(z) = \sqrt{2}J_\nu(z)$, $\theta = 2\sqrt{z\kappa}$.

Lemma 5.

$$M_{-\kappa,\mu}(ze^{\pm\pi i}) = e^{\pm(\mu+\frac{1}{2})\pi i} M_{\kappa,\mu}(z), \tag{12}$$

$$W_{\kappa,-\mu}(z) = W_{\kappa,\mu}(z). \tag{13}$$

Lemma 6.

$$\frac{d^n}{dz^n}(e^{\frac{1}{2}z} z^{\mu-\frac{1}{2}} M_{\kappa,\mu}(z)) = (-1)^n(-2\mu)_n e^{\frac{n}{2}z} z^{\mu-\frac{n+1}{2}} M_{\kappa-\frac{n}{2},\mu-\frac{n}{2}}(z), \tag{14}$$

$$\begin{aligned}&\frac{d^n}{dz^n}(e^{\frac{1}{2}z} z^{\mu-\frac{1}{2}} W_{\kappa,\mu}(z))\\ =&(-1)^n(\frac{1}{2}-\mu-\kappa)_n e^{\frac{n}{2}z} z^{\mu-\frac{n+1}{2}} W_{\kappa-\frac{n}{2},\mu-\frac{n}{2}}(z).\end{aligned} \tag{15}$$

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ with two parameters is defined by ([15, 31]),

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0. \tag{16}$$

Lemma 7. *Let $\alpha < 2$, $\beta \in \mathbb{R}$ and $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$. Then we have the following estimate: $|E_{\alpha,\beta}(z)| \leq \frac{M_0}{1+|z|}, \quad \mu \leq |\arg y| \leq \pi$, where M_0 denotes a positive constant.*

Last, we recall Weierstrass M-test.

Lemma 8. *Let the functions $f_n(t) : \mathbb{S} \to \mathbb{C}$, $M_n = \sup_{t\in\mathbb{S}} |f_n(t)|$, $n = 1, 2, \cdots$. If $\sum_{n=1}^{\infty} M_n$ is convergent, then $F(t) = \sum_{n=1}^{\infty} f_n(t)$ is uniformly convergent on $\mathbb{S}$.*

3. Sturm-Liouville type problem

Under the constant elasticity of variance process, we need to consider the functional form of the fundamental solutions of the following equation

$$L_\gamma U(S) = -\lambda_n U(S), \quad \lambda_n > 0. \tag{17}$$

The fundamental solutions for the cases $\lambda_n < r$ are established by use of Laplace transformation in terms of the Whittaker functions or the modified

Bessel functions [7]. The method is confined since Laplace transform only converges for all complex $r-\lambda_n$ with $Re(r-\lambda_n)>0$. In fact, this is also valid for $\lambda_n>r$ by a direct verification.

Lemma 9. *Suppose $\gamma\neq 0$, then the two functions*

$$\begin{cases} S^{\frac{1}{2}} I_{\frac{1}{2\gamma}}(\sqrt{2(r-\lambda_n)}\frac{1}{\gamma\sigma S^{\gamma}}), \\ S^{\frac{1}{2}} K_{\frac{1}{2\gamma}}(\sqrt{2(r-\lambda_n)}\frac{1}{\gamma\sigma S^{\gamma}}), \end{cases} \qquad r=D, \tag{18}$$

or

$$\begin{cases} e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} S^{\gamma+\frac{1}{2}} M_{\frac{1}{2}(1+\frac{1}{2\gamma}-\frac{r-\lambda_n}{\gamma(r-D)}),\frac{1}{4\gamma}}(\frac{r-D}{\gamma\sigma^2 S^{2\gamma}}), \\ e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} S^{\gamma+\frac{1}{2}} W_{\frac{1}{2}(1+\frac{1}{2\gamma}-\frac{r-\lambda_n}{\gamma(r-D)}),\frac{1}{4\gamma}}(\frac{r-D}{\gamma\sigma^2 S^{2\gamma}}), \end{cases} \qquad r\neq D, \tag{19}$$

form the fundamental solution system of Equ.(17), where $I_\nu(\cdot)$ and $K_\nu(\cdot)$ are modified Bessel functions, where $M_{\kappa,\nu}(\cdot)$ and $W_{\kappa,\nu}(\cdot)$ are Whittaker functions.

Proof. In the following we give the proof for $r\neq D$. Set

$$U(S)=\bar{U}(\xi), \quad \xi=\frac{r-D}{\gamma\sigma^2}S^{-2\gamma}, \tag{20}$$

then (17) is translated into Confluent hypergeometric equation

$$\xi\frac{d^2\bar{U}}{d\xi^2}+(1+\frac{1}{2\gamma}-\xi)\frac{d\bar{U}}{d\xi}-\frac{r-\lambda_n}{2\gamma(r-D)}\bar{U}=0. \tag{21}$$

Based on Confluent hypergeometric equation in Chapter VI of [10], we obtain the fundamental solution systems

$$\Phi(\frac{r-\lambda_n}{2\gamma(r-D)},1+\frac{1}{2\gamma},\frac{r-D}{\gamma\sigma^2 S^{2\gamma}}),\Psi(\frac{r-\lambda_n}{2\gamma(r-D)},1+\frac{1}{2\gamma},\frac{r-D}{\gamma\sigma^2 S^{2\gamma}}). \tag{22}$$

Then, its Whittaker's standard form is expressed by (19).

By a similar method, it is easy to derived fundamental solutions by (18) for the case $r=D$. ☐

We consider the eigenvalues problem

$$\begin{cases} \frac{1}{2}\sigma^2 S^{2(1+\gamma)}\frac{\partial^2 U}{\partial S^2}+(r-D)S\frac{\partial U}{\partial S}+(\lambda_n-r)U=0, \\ U(S_1)=U(S_2)=0. \end{cases} \tag{23}$$

Multiplying $\frac{2}{\sigma^2 S^{2(1+\gamma)}} e^{\{\frac{r-D}{\gamma\sigma^2 S^{2\gamma}}\}}$ on the two sides of Eq.(23), we obtain

$$\begin{cases} \frac{d}{dS}(e^{\frac{D-r}{\gamma\sigma^2 S^{2\gamma}}} \frac{d}{dS}U(S)) + \frac{2(\lambda_n - r)}{\sigma^2 S^{2(1+\gamma)}} e^{\frac{D-r}{\gamma\sigma^2 S^{2\gamma}}} U(S) = 0, \\ U(S_1) = U(S_2) = 0. \end{cases} \tag{24}$$

Set $z = \left(\int_{S_1}^{S_2} e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} dS \right)^{-1} \int_{S_1}^{S} e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} dS$, $S \in [S_1, S_2]$, then $z \in [0,1]$. It is easy to verify that $z(S)$ is a monotone function, then we use $S(z)$ denoting its inverse function. Applying the Liouville transformation, take $\tilde{U}(z) = e^{\frac{D-r}{4\gamma\sigma^2 S^{2\gamma}(z)}} U(S(z))$, then (24) is translated into

$$\begin{cases} \frac{d^2}{dz^2}\tilde{U}(z) - G(z)\tilde{U}(z) = 0, \\ \tilde{U}(0) = \tilde{U}(1) = 0, \end{cases} \tag{25}$$

where

$$G(z) = -(\int_{S_1}^{S_2} e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} dS)^2 (\lambda_n - r - g(z)),$$

$$g(z) = \frac{2(2\gamma+1)(D-r)S^{2\gamma}(z) + 3(D-r)^2}{4\sigma^2 S^{2+4\gamma}(z)} e^{\frac{3(D-r)}{4\gamma\sigma^2 S^{2\gamma}(z)}}.$$

Hence, by a direct computation with Theorem 5.11 in [17], we conclude the following.

Lemma 10. *Let bounded $\Omega \subset L^2(0,1)$, $g \in \Omega$, and $\lambda_n \in \mathbb{C}$ the corresponding eigenvalues of problem (23). Then we have*

$$\lambda_n = r + n^2\pi^2 + \int_0^1 g(z)dz - \int_0^1 g(z)\cos(2n\pi z)dz + \mathcal{O}(\frac{1}{n})$$

for $n \to \infty$ uniformly for $g \in \Omega$.

At last, we denotes $\rho(S) = \frac{2}{\sigma^2 S^{2(1+\gamma)}} e^{\frac{D-r}{\gamma\sigma^2 S^{2\gamma}}}$, then the following well-known results on (24) are holding according to Sturm-Liouville theory.

Lemma 11. *Eigenvalues of homogenous boundary value problem (24) are real and simple, which satisfy*

$$0 \le \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots . \tag{26}$$

Besides, $\lambda_n \to +\infty$ as $n \to +\infty$, $\lambda_0 = 0$ if and only if $r = 0$.

Lemma 12. *Given $\phi_i(x)$ and $\phi_j(x)$ are two eigenfunctions of (24) corresponding to different eigenvalues λ_i and λ_j respectively, then*

$$\int_a^b \rho(x)\phi_i(x)\phi_j(x)dx = 0. \tag{27}$$

Lemma 13. *Suppose $f(x) \in L^2_{\rho(x)}[a,b]$, $\phi_i(x)$ is a eigenfunction of (24) corresponding to the i-th eigenvalue, then there exist*

$$f(x) = \sum_{i=0}^{\infty} c_i\phi_i(x), c_i = \frac{\int_a^b \rho(x)f(x)\phi_i(x)dx}{\int_a^b \rho(x)f^2(x)dx}, \tag{28}$$

where $L^2_{\rho(x)}[a,b]$ denotes a weighted Sobolev space with the norm $L^2_{\rho(x)}[a,b] := \{f(x)|x \in [a,b], \int_a^b \rho(x)f^2(x)dx < \infty\}$.

4. Forward problem of modified time fractional Black-Scholes equation

In this section, we study forward problem of modified time-fractional Black-Scholes equation

$$\begin{cases} {}_0^C\left(t^\theta \frac{\partial}{\partial t}\right)^\alpha V - L_\gamma V = 0, \quad \text{in} \quad (S_1,S_2)\times(0,T) \\ V(S_1,t) = \psi_1(t), V(S_2,t) = \psi_2(t), \\ V(S,0) = \phi(S), \end{cases} \tag{29}$$

where ${}_0^C\left(t^\theta \frac{\partial}{\partial t}\right)^\alpha$ stands for a Caputo-like counterpart of hyper-Bessel operator with a fractional order $\alpha \in (0,1)$, a parameter $\theta < 1$. The special case for $\gamma = 0$ is considered in [37], here we extend that to $\gamma \neq 0$ based on the results established in §2-3.

Theorem 14. *Set constant $T > 0$, assume that $\psi_i(t) \in C^1[0,T_0]$, $i = 1,2$, $\phi(S) \in H^1[S_1,S_2]$ with the compatibility conditions $\psi_1(0) = \phi(S_1)$, $\psi_2(0) = \phi(S_2)$. Then, there exists a unique analytical solution $V(S,t)$ to the problem (29), that is*

$$V(S,t) = \frac{(S-S_1)}{S_2-S_1}\psi_2(t) + \frac{(S_2-S)}{S_2-S_1}\psi_1(t) + \sum_{n=1}^{\infty} W_n(S)R_n(t). \tag{30}$$

Moreover, $V(S,t) \in C([0,T], W^{2,+\infty}[S_1,S_2])$ *and* $\left(\frac{\partial}{\partial t}\right)^{\alpha} V(S,t) \in C([0,T], L^{\infty}[S_1,S_2])$*, where* $W_n(\cdot)$*,* $R_n(\cdot)$ *is defined by (36) and (41) in the following respectively.*

Proof. Set
$$V(S,t) = \tilde{V}(S,t) + H(S,t), \tag{31}$$
where
$$H(S,t) = \frac{(S-S_1)}{S_2-S_1}\psi_2(t) + \frac{(S_2-S)}{S_2-S_1}\psi_1(t), \tag{32}$$
then (29) is transformed into
$$\begin{cases} {}_0^C\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha}\tilde{V}(S,t) - L_{\gamma}\tilde{V}(S,t) = h(S,t), \\ \tilde{V}(S_1,t) = 0, \tilde{V}(S_2,t) = 0, \\ \tilde{V}(S,0) = g(S), \end{cases} \tag{33}$$
where
$$g(S) = \phi(s) + \frac{S-S_1}{S_2-S_1}\psi_2(0) + \frac{S_2-S}{S_2-S_1}\psi_1(0), \tag{34}$$
$$h(S,t) = \frac{(rS_1-DS)}{S_2-S_1}(\psi_2(t)-\psi_1(t)) - {}_0^C\left(t^{\theta}\frac{\partial}{\partial t}\right)^{\alpha} H(S,t). \tag{35}$$

Now set $\tilde{V}(S,t) = W(S)R(t)$ as the formal solution, substitute this expression into the homogeneous problem of (33), then we obtain
$$\begin{cases} L_{\gamma}W + \lambda W = 0, \\ W(S_2) = W(S_1) = 0, \end{cases}$$
where λ is a positive parameter. Applying Lemma 11, there exists a sequence of eigenvalues λ_n, $n \in \mathbb{Z}^+$, and the corresponding eigenfunctions can be constructed as a general solution form by Lemma 9,
$$W_n(S) = \begin{cases} S^{\frac{1}{2}}\Bigg(C_{1n}I_{\frac{1}{2\gamma}}(\sqrt{2(r-\lambda_n)}\frac{1}{\gamma\sigma S^{\gamma}}) \\ \qquad +C_{2n}K_{\frac{1}{2\gamma}}(\sqrt{2(r-\lambda_n)}\frac{1}{\gamma\sigma S^{\gamma}})\Bigg), r = D, \\ e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} S^{\gamma+\frac{1}{2}}\Bigg(\tilde{C}_{1n}M_{\frac{1}{2}(1+\frac{1}{2\gamma}-\frac{r-\lambda_n}{\gamma(r-D)}),\frac{1}{4\gamma}}(\frac{r-D}{\gamma\sigma^2 S^{2\gamma}}) \\ \qquad +\tilde{C}_{2n}W_{\frac{1}{2}(1+\frac{1}{2\gamma}-\frac{r-\lambda_n}{\gamma(r-D)}),\frac{1}{4\gamma}}(\frac{r-D}{\gamma\sigma^2 S^{2\gamma}})\Bigg), r \neq D, \end{cases} \tag{36}$$

where C_{1n} and C_{2n}, at least one of them is not zero, and $\tilde{C}_{1n}$ and $\tilde{C}_{2n}$ are also. Moreover, Lemma 12 confirmed that C_{in} and $\tilde{C}_{in}$, $i = 1, 2$ are uniformly bounded.

In the following, it is sufficient to solve inhomogeneous system (33). Based on Lemma 12-Lemma 13, we seek the solution $\tilde{V}(S,t)$ in the form of

$$\tilde{V}(S,t) = \sum_{n=1}^{\infty} W_n(S) R_n(t). \tag{37}$$

In order to determine $R_n(t)$, we use the basis functions $\{W_n(S)\}_{n=1}^{\infty}$ to expand $h(S,t)$. Set

$$h(S,t) = \sum_{n=1}^{\infty} W_n(S) h_n(t), \tag{38}$$

where

$$h_n(t) = \frac{\int_{S_1}^{S_2} \rho(S) h(S,t) W_n(S) dS}{\int_{S_1}^{S_2} \rho(S) h^2(S,t) dS}. \tag{39}$$

Substituting (37) and (38) into (33), we obtain

$$\begin{cases} {}_0^C\left(t^\theta \frac{\partial}{\partial t}\right)^\alpha R_n(t) + \lambda_n R_n(t) = h_n(t), \\ R_n(0) = \frac{\int_{S_1}^{S_2} \rho(S) g(S) W_n(S) dS}{\int_{S_1}^{S_2} \rho(S) g^2(S) dS}. \end{cases} \tag{40}$$

Set $\mu = 1 - \theta$, $\lambda_n^* = -\frac{\lambda_n}{\mu^\alpha}$, then Lemma 2.5 of [37] implies

$$\begin{aligned} R_n(t) =& R_n(0) E_{\alpha,1}(\lambda_n^* t^{\mu\alpha}) \\ &+ \frac{1}{\mu^\alpha} \int_0^t (t^\mu - \tau^\mu)^{\alpha-1} E_{\alpha,\alpha}(\lambda_n^* (t^\mu - \tau^\mu)^\alpha) h_n(\tau) d(\tau^\mu). \end{aligned} \tag{41}$$

In terms of Proposition 15-Proposition 16 which are given later, by a direct computation with (36) and (37), it follows that

$$\begin{aligned} &\Big|V(S,t)\Big| + \Big|\frac{\partial}{\partial S} V(S,t)\Big| + \Big|\frac{\partial^2}{\partial S^2} V(S,t)\Big| \\ \lesssim& \Big|\tilde{V}(S,t)\Big| + \Big|\frac{\partial}{\partial S} \tilde{V}(S,t)\Big| + \Big|\frac{\partial^2}{\partial S^2} \tilde{V}(S,t)\Big| + \Big|(1 + \frac{\partial}{\partial S}) H(S,t)\Big| \\ \lesssim& |\psi_2(t)| + |\psi_1(t)| + \Big(\|\phi(\cdot)\|_{H^1[S_1,S_2]} + \sum_{i=1}^{2} |\psi_i(0)|\Big) \end{aligned}$$

$$+\sum_{i=1}^{2}\int_0^t (t^\mu-\tau^\mu)^{\alpha-1}\Big(|\psi_i(\tau)|+|{}_0^C\Big(\tau^\theta\frac{\partial}{\partial\tau}\Big)^\alpha\psi_i(\tau)|\Big)d(\tau^\mu). \tag{42}$$

At last, by use of Eq. (29) that

$$\Big|{}_0^C\Big(t^\theta\frac{\partial}{\partial t}\Big)^\alpha V\Big|\lesssim\Big|V(S,t)\Big|+\Big|\frac{\partial}{\partial S}V(S,t)\Big|+\Big|\frac{\partial^2}{\partial S^2}V(S,t)\Big|. \tag{43}$$

According to the estimate (6) in [37], that is

$$\Big|{}_0^C\Big(t^\theta\frac{d}{dt}\Big)^\alpha f(t)\Big|\lesssim t^{-\alpha(1-\theta)}\Big(|f(0)| + \int_0^1(1-\tau)^{-\alpha}\Big(|f(s)|+|s\frac{d}{ds}f(s)|\Big)_{s=t\tau^{\frac{1}{1-\theta}}}d\tau\Big)$$

for $f\in C^1[0,T]$, by a direct computation, (42) and (43) imply

$$\Big|{}_0^C\Big(t^\theta\frac{\partial}{\partial t}\Big)^\alpha V\Big|+\sum_{k=0}^{2}\Big|\frac{\partial^k}{\partial S^k}V(S,t)\Big|\lesssim\|\phi(\cdot)\|_{H^1[S_1,S_2]} +\sum_{i=1}^{2}\Big(|\psi_i(0)|+|\psi_i(t)|+\int_0^t(t^\mu-\tau^\mu)^{\alpha-1}\tau^{-\theta}|\psi_i(\tau)|d\tau + \int_0^t(t^\mu-\tau^\mu)^{\alpha-1}\tau^{\mu(1-\alpha)-1}\times \int_0^1(1-\omega)^{-\alpha}\Big(|\psi_i(s)|+|s\frac{d}{ds}\psi_i(s)|\Big)_{s=\tau\omega^{\frac{1}{\mu}}}d\omega d\tau\Big). \tag{44}$$

Furthermore, Lemma 8 yields series (38) and its second derivative are uniform convergent in $[0,T]\times[S_1,S_2]$ for any positive constant T. The existence of ${}_0^C\Big(t^\theta\frac{\partial}{\partial t}\Big)^\alpha V$ if confirmed by Eq. (29) and (43). Hence the existence of solution to forward problem (29) is established.

In the following we assume V_1 and V_2 are solutions of the problem (29), then set $V=V_1-V_2$ and substitute V into (29), we obtain

$$\begin{cases}{}_0^C\Big(t^\theta\frac{\partial}{\partial t}\Big)^\alpha V-L_\gamma V=0,\\ V(S_1,t)=0, V(S_2,t)=0,\\ V(S,0)=0.\end{cases} \tag{45}$$

Let $\{W_n(S)\}_{n=1}^{\infty}$ denoting eigenfunctions that given by (36), define the function $R_n(t) = \frac{\int_{S_1}^{S_2}\rho(S)V(S,t)W_n(S)dS}{\int_{S_1}^{S_2}\rho(S)V^2(S,t)dS}$. Substituting (37) into (44), then we obtain

$$\begin{cases} {}_0^C\left(t^\theta \frac{d}{dt}\right)^\alpha R_n(t) + \lambda_n R_n(t) = 0, \\ R_n(0) = 0. \end{cases}$$

Using Lemma 2.5 in [37], problem (45) yields $R_n(t) = 0, n = 1, 2, 3, \cdots$, this implies that $\int_{S_1}^{S_2}\rho(S)V(S,t)W_n(S)dS = 0$. In terms of the completeness of $\{W_n(S)\}_{n=1}^{\infty}$, we confirm the solution of (45) that

$$V(S,t) = V_1(S,t) - V_2(S,t) \equiv 0.$$

This yields the uniqueness of solution to the problem (29).

Hence, we complete the proof of Theorem 14. □

Based on Lemma 10-Lemma 13, we give the following estimates.

Proposition 15. *Under the assumption of 14, there exists*

$$|h_n(t)| \lesssim \begin{cases} \frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^2\left(|\psi_i(t)| + |{}_0^C(t^\theta\frac{\partial}{\partial t})^\alpha\psi_i(t)|, r = D; \right. \\ \frac{1}{n^{\frac{3}{2}+\frac{1}{2|\gamma|}}}\sum_{i=1}^2\left(|\psi_i(t)| + |{}_0^C(t^\theta\frac{\partial}{\partial t})^\alpha\psi_i(t)|\right), r \neq D. \end{cases}$$

Proof. Applying the theory of modified Bessel functions (8)-(9), we obtain

$$\begin{aligned} |h_n(t)| &\lesssim \frac{1}{\lambda_n^{\frac{3}{4}}}\|\frac{\partial}{\partial S}(\rho(\cdot)h(\cdot,t))\|_{L^2[S_1,S_2]} \\ &\lesssim \frac{1}{\lambda_n^{\frac{3}{4}}}\sum_{i=1}^2\left(|\psi_i(t)| + |{}_0^C\left(t^\theta\frac{\partial}{\partial t}\right)^\alpha\psi_i(t)|\right) \\ &\lesssim \frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^2\left(|\psi_i(t)| + |{}_0^C\left(t^\theta\frac{\partial}{\partial t}\right)^\alpha\psi_i(t)|\right) \end{aligned}$$

for $r = D$. Taking a same procedure, Lemma 1 and (10)-(11) yield

$$|h_n(t)| \lesssim \frac{1}{n^{\frac{3}{2}+\frac{1}{2|\gamma|}}}\sum_{i=1}^2\left(|\psi_i(t)| + |{}_0^C\left(t^\theta\frac{\partial}{\partial t}\right)^\alpha\psi_i(t)|\right)$$

for $r \neq D$.

These complete the proof of Proposition 15. □

Proposition 16. *Under the assumption of Theorem 14, there exist*

$$|R_n(0)| \lesssim \begin{cases} \frac{1}{n^{\frac{3}{2}}}\Big(\|\phi(\cdot)\|_{H^1[S_1,S_2]} + \sum_{i=1}^{2}|\psi_i(0)|\Big), r = D; \\ \frac{1}{n^{\frac{3}{2}+\frac{1}{2|\gamma|}}}\Big(\|\phi(\cdot)\|_{H^1[S_1,S_2]} + \sum_{i=1}^{2}|\psi_i(0)|\Big), r \neq D \end{cases} \tag{46}$$

and

$$|R_n(t)| \lesssim |R_n(0)| + \frac{1}{n^2}\int_0^t (t^\mu - \tau^\mu)^{\alpha-1}|h_n(\tau)|d(\tau^\mu). \tag{47}$$

Proof. Applying (8)-(9) of modified Bessel functions on the initial datum, we have

$$\begin{aligned} |R_n(0)| &\leq \frac{1}{\lambda_n^{\frac{3}{4}}}\|\frac{d}{dS}(\rho(\cdot)g(\cdot))\|_{L^2[S_1,S_2]} \\ &\leq \frac{C}{\lambda_n^{\frac{3}{4}}}\Big(\|\phi(\cdot)\|_{H^1[S_1,S_2]} + \sum_{i=1}^{2}|\psi_i(0)|\Big) \\ &\lesssim \frac{1}{n^{\frac{3}{2}}}\Big(\|\phi(\cdot)\|_{H^1[S_1,S_2]} + \sum_{i=1}^{2}|\psi_i(0)|\Big), r = D. \end{aligned}$$

Similarly, Lemma 1 and (10)-(11) imply

$$|R_n(0)| \lesssim \frac{1}{n^{\frac{3}{2}+\frac{1}{2|\gamma|}}}\sum_{i=1}^{2}\Big(\|\phi(\cdot)\|_{H^1[S_1,S_2]} + \sum_{i=1}^{2}|\psi_i(0)|\Big), r \neq D.$$

Then, in terms of Lemma 7, we obtain

$$|R_n(t)| \lesssim R_n(0) + \frac{1}{n^2}\int_0^t (t^\mu - \tau^\mu)^{\alpha-1}|h_n(\tau)|d(\tau^\mu).$$

These complete the proof of Proposition 16. □

The forward problem (6) is the special case of (29) for $\theta = 0$, then Theorem 14 becomes as follows.

Theorem 17. *Set constant $T > 0$, assume that $\psi_i(t) \in C^1[0, T_0]$, $i = 1, 2$, $\phi(S) \in H^1[S_1, S_2]$ with the compatibility conditions $\psi_1(0) = \phi(S_1)$, $\psi_2(0) = \phi(S_2)$. Then, there exists a unique analytical solution $V(S, t)$ to the problem (6), that is*

$$V(S,t) = \frac{(S - S_1)}{S_2 - S_1}\psi_2(t) + \frac{(S_2 - S)}{S_2 - S_1}\psi_1(t) + \sum_{n=1}^{\infty} W_n(S)R_n(t), \tag{48}$$

$V \in C([0,T], W^{2,\infty}[S_1,S_2])$, $\left(\frac{\partial}{\partial t}\right)^\alpha V \in C([0,T], L^\infty[S_1,S_2])$, *where* $W_n(\cdot)$, $R_n(\cdot)$ *is defined by (36) and (41) with* $\theta = 0$, *respectively.*

5. Backward problem of time fractional Black-Scholes equation

In this section, we consider the backward problem (1) which arise from pricing double barrier option under the so-called constant elasticity of variances diffusion model.

Theorem 18. *Assume that* $T > 0$, $\psi_i(t) \in C^1[0,T]$, $i = 1, 2$ *and* $\phi(S) \in H^1[S_1,S_2]$ *with the compatibility conditions that* $\psi_1(T) = \phi(S_1)$, $\psi_2(T) = \phi(S_2)$. *Then the backward problem (1) has a unique analytical solution*

$$\begin{aligned} V(S,t) =& \frac{(S-S_1)}{S_2-S_1}\psi_2(T-t) + \frac{(S_2-S)}{S_2-S_1}\psi_1(T-t) \\ &+ \sum_{n=1}^{\infty} W_n(S) R_n(T-t), \end{aligned} \tag{49}$$

$V(S,t) \in C([0,T], W^{2,\infty}[S_1,S_2])$, $\left(\frac{\partial}{\partial t}\right)^\alpha V(S,t) \in C([0,T], L^\infty[S_1,S_2])$, *where* $W_n(\cdot)$, $R_n(\cdot)$ *is defined by (36) and (41) with* $\theta = 0$, *respectively.*

Proof. Under the inverse transformation (4), the forward problem (6) is translated into backward problem (1), then by a direct computation with (46), we have

$$\begin{aligned} &\sum_{k=0}^{2} \left| \frac{\partial^k}{\partial S^k} V(S,t) \right| + \left| \left(\frac{\partial}{\partial t}\right)^\alpha V(S,t) \right| \lesssim \|\phi(S)\|_{H^1[S_1,S_2]} \\ &+ \sum_{i=1}^{2} \Big(|\psi_i(T)| + |\psi_i(T-t)| + \int_t^T (\tau - t)^{\alpha-1} |\psi_i(T-\tau)| d\tau \\ &+ \int_t^T (\tau-t)^{\alpha-1}(T-\tau)^{-\alpha} \int_0^1 (1-\omega)^{-\alpha} \\ &\quad \times \Big(|\psi_i(T-s)| + |(T-s)\frac{d}{ds}\psi_i(T-s)| \Big)_{s=T-(T-\tau)\omega} d\omega d\tau \Big). \end{aligned}$$

Then, Theorem 17 yields Theorem 18. □

Whether a formula can be efficiently used to compute is one of the main criteria of assessing its practical usefulness. From a numerical point of view, the suitability of a particular numerical method in the context of solving partial differential equations usually goes with the governing equation, rather than being subject to boundary conditions. Therefore, we construct an example with special conditions to illustrate the implementation of formula (49).

Example 19. Consider the following backward problem

$$\begin{cases} {}_0^C\left(t^\theta \frac{\partial}{\partial t}\right)^\alpha V + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^{2(1+\gamma)}} V + (r-D)S\frac{\partial}{\partial S}V - rV\right) = 0, \\ V(S_1,t) = V(S_2,t) = 0, \\ V(S,T) = \frac{\sqrt{2}\sigma}{2} S^{1+\gamma} e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} \sin\frac{\pi(S-S_1)}{S_2-S_1}. \end{cases}$$

Whereupon (49) in Theorem 18, the unique solution is given by

$$V(S,t) = \frac{2A}{\pi} E_{\alpha,1}(-\lambda_1(T-t)^\alpha) W_1(S), \tag{50}$$

where undetermined function $W_1(S)$ is defined by (36) and

$$A = \int_{S_1}^{S_2} \frac{\sqrt{2}}{\sigma S^{1+\gamma}} e^{\frac{D-r}{2\gamma\sigma^2 S^{2\gamma}}} \sin\frac{\pi(S-S_1)}{S_2-S_1} W_1(S) dS.$$

Since $V(S,t) \in C([0,T], W^{2,\infty}[S_1,S_2])$, then

$$V(S,T) = \lim_{t\to T} V(S,t) = \frac{2A}{\pi} W_1(S).$$

Applying the terminal datum on $t=T$, we have

$$W_1(S) = \frac{\sqrt{2}\pi}{4A} \sigma S^{1+\gamma} e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} \sin\frac{\pi(S-S_1)}{S_2-S_1}. \tag{51}$$

Multiply $\frac{\sqrt{2}}{\sigma S^{1+\gamma}} e^{\frac{D-r}{2\gamma\sigma^2 S^{2\gamma}}} \sin\frac{\pi(S-S_1)}{S_2-S_1}$ on two sides of (51) and integrate from S_1 to S_2, then we obtain $A = \frac{\pi}{2}$ and

$$W_1(S) = \frac{\sqrt{2}}{2} \sigma S^{1+\gamma} e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} \sin\frac{\pi(S-S_1)}{S_2-S_1}. \tag{52}$$

Hence, substitute (52) into (50), we arrive at

$$V(S,t) = \frac{\sqrt{2}}{2} \sigma S^{1+\gamma} e^{\frac{r-D}{2\gamma\sigma^2 S^{2\gamma}}} \sin\frac{\pi(S-S_1)}{S_2-S_1} E_{\alpha,1}(-\lambda_1(T-t)^\alpha).$$

6. Inverse source problem of time fractional Black-Scholes equation

In pricing double barrier options involving cash flow, we establish the unique pair of solutions of inverse source problem (2) of time-fractional Black-Scholes model by considering its equivalent problem (7) in $(S_1, S_2) \times (0, T)$.

Theorem 20. *Assume that $\psi_i(t) \in C^1[0,T]$, $\phi_i(S) \in H^4[S_1, S_2]$, $i = 1, 2$ with the compatibility conditions that*

$$\psi_1(0) = \phi(S_1) = \psi_1(T), \quad \psi_2(0) = \phi(S_2) = \psi_2(T),$$

and a positive numbers T, $S_1 < S_2$. Then, there exists a unique pair of analytical solutions $(V(S,t), f(S))$ of the problem (2), where $V(S,t) \in C([0,T], W^{2,\infty}[S_1, S_2])$, $\left(\frac{\partial}{\partial t}\right)^{\alpha} V(S,t) \in C([0,T], L^{\infty}[S_1, S_2])$, $f(S) \in C([0,T], L^{\infty}[S_1, S_2])$.

Proof. Under the transform (31), problem (7) becomes into

$$\begin{cases} {}_0^C D_t^{\alpha} \tilde{V}(S,t) - L_{\gamma} \tilde{V}(S,t) = G(S,t), \\ \tilde{V}(S_1, t) = 0, \tilde{V}(S_2, t) = 0, \\ \tilde{V}(S, 0) = g_2(S), \tilde{V}(S,T) = g_1(S), \end{cases} \tag{53}$$

where

$$g_2(S) = \phi_1(S) + \frac{S - S_1}{S_2 - S_1} \psi_2(0) + \frac{S_2 - S}{S_2 - S_1} \psi_1(0), \tag{54}$$

$$g_1(S) = \phi_2(S) + \frac{S - S_1}{S_2 - S_1} \psi_2(T) + \frac{S_2 - S}{S_2 - S_1} \psi_1(T), \tag{55}$$

$$G(S,t) = f(S) + h(S,t) \tag{56}$$

with $h(S,t)$ is defined in (35).

We seek a formal solution as the sum of infinite series to the problem (54) by use of the complete orthogonal system $\{W_n(S)\}_{n=1}^{\infty}$ given in (36) as follows:

$$\tilde{V}(S,t) = \sum_{n=1}^{\infty} W_n(S) R_n(t), G(S,t) = \sum_{n=1}^{\infty} W_n(S)(f_n + h_n(t)), \tag{57}$$

where $h_n(t)$ is given in (38), $R_n(t)$ and f_n are the unknowns to be determined, which satisfy

$$\begin{cases} {}_0^C D_t^{\alpha} R_n(t) + \lambda_n R_n(t) = f_n + h_n(t), \\ R_n(0) = g_{2n}, R_n(T) = g_{1n}, \end{cases} \tag{58}$$

where $R_n(t)$ and f_n are defined as

$$f_n = \frac{2}{S_2 - S_1} \int_{S_1}^{S_2} f(S,t) W_n(S) dS,$$

$$g_{in} = \frac{2}{S_2 - S_1} \int_{S_1}^{S_2} g_i(S) W_n(S) dS.$$

By a direct computation, it yields that

$$\begin{aligned}\sum_{n=1}^{\infty} |g_{2n}| &\leq \frac{1}{\lambda_n^2} \|g_2^{(4)}(\cdot)\|_{L^2[S_1,S_2]} \\ &\leq \frac{C}{\lambda_n^2}\Big(\|\phi_2(\cdot)\|_{H^4[S_1,S_2]} + \sum_{i=1}^{2} |\psi_i(0)|\Big) \\ &\lesssim \frac{1}{n^4}\Big(\|\phi_2(\cdot)\|_{H^4[S_1,S_2]} + \sum_{i=1}^{2} |\psi_i(0)|\Big). \end{aligned} \tag{59}$$

Similarly, there exists

$$\sum_{n=1}^{\infty} |g_{1n}| \lesssim \frac{1}{n^4}\Big(\|\phi_1(\cdot)\|_{H^4[S_1,S_2]} + \sum_{i=1}^{2} |\psi_i(T)|\Big). \tag{60}$$

According to Lemma 7 for the Mittag-Leffler function, we have

$$1 - E_{\alpha,1}(-\lambda_n T^{\alpha}) \neq 0, \tag{61}$$

for large n. Besides, in terms of Lemma 2.5 of [37], we obtain

$$\begin{aligned} R_n(t) =&(g_{2n} - \frac{f_n}{\lambda_n}) E_{\alpha,1}(-\lambda_n t^{\alpha}) + \frac{f_n}{\lambda_n} \\ &+ \int_0^t (t^{\rho} - \tau^{\rho})^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) h_n(\tau) d\tau. \end{aligned} \tag{62}$$

Then, applying the datum on $t = T$, we derive

$$\begin{aligned} g_{1n} =&(g_{2n} - \frac{f_n}{\lambda_n}) E_{\alpha,1}(-\lambda_n T^{\alpha}) + \frac{f_n}{\lambda_n} \\ &+ \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\tau)^{\alpha}) h_n(\tau) d\tau. \end{aligned}$$

This yields

$$f_n = \lambda_n(1 - E_{\alpha,1}(-\lambda_n T^\alpha))^{-1}\Big(g_{1n} - g_{2n}E_{\alpha,1}(-\lambda_n T^\alpha) - \int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)h_n(\tau)d\tau\Big). \tag{63}$$

By analysis of (63), we obtain

$$|f_n| \lesssim \frac{1}{n^2}\Big(\|\phi_i(\cdot)\|_{H^4[S_1,S_2]} + \sum_{i=1}^{2}(|\psi_i(0)| + |\psi_i(T)|) + \int_0^t (t-\tau)^{-1}|h_n(\tau)|d\tau\Big). \tag{64}$$

This means $f(S) = \sum_{n=1}^{\infty} W_n(S)f_n$ is convergent, then Lemma 8 Weierstrass M-test implies the series is uniformly convergent.

Substituting (63) into (62), we arrive at

$$\begin{aligned} R_n(t) = &\frac{1 - E_{\alpha,1}(-\lambda_n t^\alpha)}{1 - E_{\alpha,1}(-\lambda_n T^\alpha)}g_{1n} + \frac{E_{\alpha,1}(-\lambda_n t^\alpha) - E_{\alpha,1}(-\lambda_n T^\alpha)}{1 - E_{\alpha,1}(-\lambda_n T^\alpha))}g_{2n} \\ &+ \frac{1}{(1 - E_{\alpha,1}(-\lambda_n T^\alpha))}\Big((1 - E_{\alpha,1}(-\lambda_n T^\alpha)) \\ &\times \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t^-\tau)^\alpha)f_n(\tau)d\tau - (1 - E_{\alpha,1}(-\lambda_n t^\alpha)) \\ &\times \int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)f_n(\tau)d\tau\Big). \end{aligned} \tag{65}$$

Substitute (63) and (65) into (57), we obtain $\tilde{V}(S,t)$ and $f(S)$. The uniformly convergence of series of representation of the pair function $(\tilde{V}(S,t), f(S))$ and $\frac{\partial}{\partial S}\tilde{V}(S,t)$, $\frac{\partial^2}{\partial S^2}\tilde{V}(S,t)$, ${}_0^C D_t^\alpha \tilde{V}(S,t)$ are confirmed by the given conditions in Theorem 6.1 which can be done by taking a similar procedure in the proof of Theorem 14.

Hence, by use of the inverse transform (31) and (4), we confirm Theorem 20 holding. □

Acknowledgements. The authors wish to thank the support of Qinglan Project of Jiangsu Province. This work was supported by NNSF of China (No.11326152) and NSF of Jiangsu Province of China (No. BK20130736).

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