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APPROXIMATE SOLUTIONS AND VECTOR OPTIMIZATION OVER CONE WITH Kρ-LOCALLY CONNECTED FUNCTION AND ITS GENERALIZATIONS

Vani Sharma[1], Mamta Chaudhary[2 §]
[1,2] Department of Mathematics
Satyawati College (University of Delhi)
Ashok Vihar, Delhi 100052, INDIA

Abstract: In this paper (strictly) locally ρ-K-connected, locally (naturally) quasi ρ-K-connected and (strictly) locally pseudo ρ-K-connected functions are defined for a vector optimization problem over cones. Involving these functions necessary and sufficient optimality conditions are obtained for an approximate weak quasi efficient solution of this problem. Approximate Wolfe type and Mond-Weir type duals are formulated and duality results are established.

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1. Introduction

Various generalizations of convex functions have appeared in literature. Among them we recall the class of arcwise connected functions introduced by Ortega and Rheinboldt [10] defined on arc wise connected sets. Kaul et al. [6] defined locally connected sets by reducing the width of the arc part. These sets include arc wise connected sets [1] and locally star shaped sets [2]. The authors then defined locally connected functions, locally Q-connected on a locally connected set. These functions include semi-locally convex functions and semi-locally

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§Correspondence author

quasiconvex functions defined earlier by Kaul and Kaur [8] as special cases. Kaul and Lyall [7] defined the directional derivative (with respect to an arc) of a real valued function and called it the right differential, at a point of a locally connected set. They also defined locally P-connected functions in terms of their right differentials and obtained a number of sufficient optimality criteria for a non-linear programming problem involving these functions. Lyall et al. [9] further extended these results to the multiple objective programming. Recently, Suneja et al. [18] extended these results to the vector optimization problem over cones with generalized cone locally connected functions. Necessary and sufficient optimality conditions and Wolfe and Mond-Weir type duality results are obtained for a weak quasi efficient solution involving these functions. Vial [19] defined ρ-convex functions. Jeyakumar [4, 5] defined ρ-pseudo convex and ρ-quasi convex functions. Later on Preda and Niculescu [11, 12] defined ρ-locally arc-wise connected, ρ-locally Q-connected and ρ-locally P-connected functions and gave necessary and sufficient optimality conditions and Wolfe and Mond-Weir type duality results for minimax and non-linear multiple objective programming problem. After that authors like Stancu-Minasian [13, 14, 15], Stancu-Minasian and Andreea Madalina Stancu [17] obtained these results for non-linear programming problems involving these functions.

2. Definitions and preliminaries

Let $S \subseteq R^n$ be a nonempty set and $K \subseteq R^m$ be closed convex pointed cone with non-empty interior. The positive dual cone K^* of K is given by

$$K^* = \{y^* \in R^m : x^T y^* \geq 0, \text{ for all } x \in K\}.$$

Also, Flores-Bezan et al. [3] have shown that $k \in \operatorname{int} K \Leftrightarrow \lambda^T k > 0$, for all $\lambda \in K^* \setminus \{0\}$.

Definition 2.1 ([6]). A set $S \subseteq R^n$ is said to be locally connected if for each $x, x^* \in S$ there exists a maximum positive number $a(x^*, x) \leq 1$ and a vector valued function $H_{x^*,x} : [0,1] \to S$ such that

$$H_{x^*,x}(\lambda) \in S, \quad 0 < \lambda < a(x^*, x), \tag{1}$$

$H_{x^*,x}$ is continuous in the interval $]0, a(x^*, x)[$ and

$$H_{x^*,x}(0) = x^*, \quad H_{x^*,x}(1) = x\,. \tag{2}$$

Let $S \subseteq R^n$ be a locally connected set with respect to the are $H_{x^*,x} : [0,1] \to S$ satisfying (1) and (2). $f : S \to R^m$be a vector valued function.

Definition 2.2 ([7]). The function f is said to have a right derivative (or right differential) at $x^* \in S$ with respect to $H_{x^*,x}$ if

$$\lim_{\lambda \to 0^+} \frac{f(H_{x^*,x}(\lambda)) - f(x^*)}{\lambda}$$

exists. This limit is denoted by $(df)^+(H_{x^*,x}(0^+))$. If $\lim_{\lambda \to 0^+} \frac{H_{x^*,x}(\lambda) - x^*}{\lambda}$ exists then it is denoted by $dH^+_{x^*,x}(0)$, and is called directional derivative of $H_{x^*,x}$ at $\lambda = 0$.

Generalizing the concept of ρ-locally arc-wise connected function [16], we define new notions of locally ρ-K-connected function and its generalizations. For this purpose we consider $\rho \in R^m$, $d : S \times S \to R_+$, $f : S \to R^m$ and a closed convex pointed cone K in R^m having non-empty interior. We say that f is

(i) locally ρ-K-connected (ρKLCN) at $x^* \in S$ with respect to $H_{x^*,x}$ if for every $x \in S$

$$f(x) - f(x^*) - (df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in K.$$

If $\rho = 0$ then the above definition reduces to locally K-connected function.

(ii) Strictly locally ρ-K-connected (ρKSLCN) at $x^* \in S$ with respect to $H_{x^*,x}$ if for every $x \in S$, $x \neq x^*$

$$f(x) - f(x^*) - (df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in \operatorname{int} K.$$

(iii) locally quasi ρ-K-connected (ρKLQCN) at $x^* \in S$ with respect to $H_{x^*,x}$ if for every $x \in S$,

$$f(x) - f(x^*) \notin \operatorname{int} K \Rightarrow -(df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in K.$$

(iv) locally naturally quasi ρ-K-connected (ρKLNQCN) at $x^* \in S$ with respect to $H_{x^*,x}$ if for every $x \in S$,

$$-[f(x) - f(x^*)] \in K \Rightarrow -(df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in K.$$

(v) locally pseudo ρ-K-connected (ρKLPCN) at $x^* \in S$ with respect to $H_{x^*,x}$ if for every $x \in S$,

$$f(x^*) - f(x) \in \operatorname{int} K \Rightarrow -(df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in \operatorname{int} K.$$

(vi) strictly locally pseudo ρ-K-connected (ρKSPLCN) at $x^* \in S$ with respect to $H_{x^*,x}$ if for every $x \in S$

$$-[f(x) - f(x^*)] \in K \Rightarrow -(df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in \operatorname{int} K.$$

If f is ρKLCN at each $x^* \in S$, then f is said to be ρKLCN on S. Same is applied for its generalizations.

Theorem 2.3. *If f is locally ρ-K-connected at $x^* \in S$ with respect to the arc $H_{x^*,x}$, then f is locally naturally quasi ρ-K-connected at x^* with respect to the same arc $H_{x^*,x}$.*

Proof. Let f be locally ρ-K-connected at $x^* \in S$ with respect to the arc $H_{x^*,x}$, then for every $x \in S$

$$f(x) - f(x^*) - (df)^+(H_{x^*,x}) - \rho d(x^*, x) \in K. \tag{3}$$

Let

$$-[f(x) - f(x^*)] \in K\,. \tag{4}$$

Adding (3) and (4), we get

$$-(df)^+(H_{x^*,x}) - \rho d(x^*, x) \in K.$$

Hence f is locally naturally quasi ρ-K-connected at x^* with respect to the same arc $H_{x^*,x}$. □

The converse of the above theorem may not hold as can be seen from the following example.

Example 2.4. Let $S = \{(x_1, x_2) : x_1^2 + x_2^2 \geq 1,\ x_1 \neq 2x_2,\ x_1 > 0,\ x_2 > 0\}$ and $H_{x^*,x}(\lambda) = (\{(1-\lambda)x_1^{*^2} + \lambda x_1^2\}^{1/2}, \{(1-\lambda)x_2^{*^2} + \lambda x_2^2\}^{1/2})$.

Then S is a locally connected set with respect to the arc given by $H_{x^*,x}(\lambda)$.

Let $x^* = (1, 1)$ and $K = \{(x_1, x_2) : -x_1 \leq x_2,\ x_2 \geq 0\}$. Let us define $f : S \to R^2$ as

$$f(x_1, x_2) = \begin{cases} (-x_1^2 x_2^2, x_1^2 - x_2^2), & \text{if } x_1 > 1, x_2 > 1, \\ (-1, 0), & \text{otherwise.} \end{cases}$$

Let $\rho = (0, -1)$ and

$$d(x^*, x) = \begin{cases} \frac{1}{x_1+x_2}, & \text{if } x_1 > 1, x_2 > 1, \\ (x_1 + x_1^*)^2 + (x_2 + x_2^*)^2, & \text{if } x_1 \leq 1, x_2 \leq 1 \text{ and} \\ & \quad (x_1 + x_1^*)^2 + (x_2 + x_2^*)^2 \leq 1, \\ \frac{1}{(x_1+x_1^*)^2+(x_2+x_2^*)^2}, & \text{if } x_1 \leq 1, x_2 \leq 1 \text{ and} \\ & \quad (x_1 + x_1^*)^2 + (x_2 + x_2^*)^2 > 1. \end{cases}$$

Now

$$(df)^+(H_{x^*,x}(0^+)) = \begin{cases} (2 - x_1^2 - x_2^2, x_1^2 - x_2^2), & \text{if both the components} \\ & \quad \text{of } H_{x^*,x}(\lambda) > 1, \\ (0, 0), & \text{otherwise.} \end{cases}$$

f is locally naturally quasi ρ-K-connected at x^*, because $f(x) - f(x^*) \in -K$.

This implies $1 - x_1^2 x_2^2 \leq x_2^2 - x_1^2$ and $x_2^2 - x_1^2 \geq 0$. So

$$-(df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in K.$$

But f is not locally ρ-K-connected at x^*, because for $x = (2, 2)$

$$f(x) - f(x^*) - (df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) = (-9, \frac{1}{4}) \notin K.$$

Theorem 2.5. *If f is locally quasi ρ-K-connected at $x^* \in S$ with respect to the arc $H_{x^*,x}$ then f is locally naturally quasi ρ-K-connected with respect to the same arc $H_{x^*,x}$.*

The converse of the above theorem may not hold as can be seen by the following example.

Example 2.6. Let S be a locally connected set with respect to the arc $H_{x^*,x}(\lambda)$ as defined in Example 2.4.

Let $x^* = (1, 1)$ and $K = \{(x_1, x_2) : x_1 \leq 0,\ x_2 \leq x_1\}$.

Let $f : S \to R^2$ be defined as

$$f(x_1, x_2) = \begin{cases} \left(\frac{1}{x_1^2}, \frac{1}{x_2^2}\right), & \text{if } x_1 > 1, x_2 > 1, \\ (1, 1), & \text{otherwise.} \end{cases}$$

Let $\rho = (1, 1)$ and $d(x^*, x)$ is defined as in Example 2.4.

Now

$$(df)^+(H_{x^*,x}(0^+)) = \begin{cases} (1-x_1^2, 1-x_2^2), & \text{if both the components} \\ & \quad \text{of } H_{x^*,x}(\lambda) > 1, \\ (0,0), & \text{otherwise.} \end{cases}$$

f is locally naturally quasi ρ-K-connected at x^* because

$$f(x) - f(x^*) \in -K.$$

This implies

$$1 - x_1^2 \geq 0 \text{ and } x_1^2 \geq x_2^2.$$

So

$$-(df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in K.$$

But f is not locally quasi ρ-K-connected at x^*, because for $x = (2,2)$

$$f(x) - f(x^*) = \left(\frac{-3}{4}, \frac{-3}{4}\right) \notin \operatorname{int} K.$$

But

$$-(df)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) = \left(\frac{11}{4}, \frac{11}{4}\right) \notin K.$$

Remark 2.1. The following diagram illustrates the relation between ρKLCN, ρKLQCN and ρKLNQCN:

$$\rho\text{KLCN} \ \underset{\not\leftarrow}{\rightarrow} \ \rho\text{KLNQCN} \ \underset{\not\rightarrow}{\leftarrow} \ \rho\text{KLQCN}$$

3. Optimality conditions

Consider the following vector optimization problem:

$$\begin{aligned} \text{(VP)} \quad & K\text{-minimize } f(x) \\ & \text{subject to } -g(x) \in Q, \end{aligned}$$

where $f : S \to R^m$, $g : S \to R^p$, $S \subseteq R^n$ is a locally connected set such that for each $x, x^* \in S$, there exists a vector valued function $H_{x^*,x}(\lambda)$ satisfying the conditions (1) and (2) and the right differentials of f and g exist at x^* with respect to the same arc $H_{x^*,x}$.

Let $K \subseteq R^m$ and $Q \subseteq R^p$ be closed convex cones with nonempty interiors and $X^0 = \{x \in S : -g(x) \in Q\}$ be the set of all feasible solutions of (VP).

Definition 3.1. Let $\varepsilon \in R$, $\varepsilon \geq 0$ and $e \in K \setminus \{0\}$. $x^* \in X^0$ is said to be εe-quasi efficient solution of (VP) if

$$f(x) - f(x^*) \notin -K \setminus \{0\} - \varepsilon\|x - x^*\|e, \quad \text{for all } x \in X^0,$$

and a weak εe-quasi efficient solution of (VP) if

$$f(x) - f(x^*) \notin -\operatorname{int} K - \varepsilon\|x - x^*\|e, \quad \text{for all } x \in X^0.$$

When $\varepsilon = 0$, εe-quasi efficient solution (weak εe-quasi efficient solution) of (VP) will coincide with efficient solution (weak efficient solution) of (VP).

We now give the generalized Slater's type cone constraint qualification which will be used in obtaining the necessary optimality conditions.

Definition 3.2. The function g is said to satisfy generalized Slater's type cone constraint qualification at x^* if g is locally σ-Q-connected at x^* and there exists $\hat{x} \in S$ such that

$$-g(\hat{x}) + \sigma d(x^*, \hat{x}) \in \operatorname{int} Q.$$

Theorem 3.3 ([17]). *Suppose that $x^* \in X^0$ be a weak εe-quasi efficient solution of (VP). If $(df)^+(H_{x^*,x}(0^+)) + \varepsilon\|dH^+_{x^*,x}(0)\|e$ and $(dg)^+(H_{x^*,x}(0^+))$ are K-subconvexlike and Q-subconvexlike functions of x respectively with respect to the same arc $H_{x^*,x}$. Then there exist $\alpha \in K^*$, $\beta \in Q^*$ not both zero such that*

$$\alpha^T(df)^+(H_{x^*,x}(0^+)) + \beta^T(dg)^+(H_{x^*,x}(0^+)) + \varepsilon\|dH^T_{x^*,x}(0)\|\alpha^T e \geq 0, \quad \textit{for all } x \in S. \tag{5}$$

$$\beta^T g(x^*) = 0\,. \tag{6}$$

Theorem 3.4. *Suppose that the hypothesis of Theorem 3.3 holds. Then there exist $\alpha \in K^*$, $\beta \in Q^*$ (not both zero) such that conditions* (5) *and* (6) *hold. If g satisfies the generalized Slater's type cone constraint qualification at x^* then $\alpha \neq 0$.*

Proof. If possible, let $\alpha = 0$. Then form (5), we get

$$\beta^T (dg)^+(H_{x^*,x}(0^+)) \geq 0, \quad \text{for all } x \in S. \tag{7}$$

Since g satisfies the generalized Slater's type cone constraint qualification at x^*, therefore

$$g(x) - g(x^*) - (dg)^+(H_{x^*,x}(0^+)) - \sigma d(x^*, x) \in Q, \quad \text{for all } x \in S$$

and there exist $\hat{x} \in S$ such that

$$-g(\hat{x}) + \sigma d(x^*, \hat{x}) \in \operatorname{int} Q.$$

As $\beta \in Q^*$, so we get

$$\beta^T [g(x) - g(x^*) - (dg)^+(H_{x^*,x}(0^+)) - \sigma d(x^*, x)] \geq 0, \text{ for all } x \in S \tag{8}$$

and

$$-\beta^T g(\hat{x}) + \beta^T \sigma d(x^*, \hat{x}) > 0. \tag{9}$$

Adding (7) and (8) and using (6), we get

$$-\beta^T g(x) - \beta^T \sigma d(x^*, x) \geq 0, \text{ for all } x \in S \tag{10}$$

which contradicts (9). Hence $\alpha \neq 0$. □

Now, we will establish some sufficient optimality conditions for (VP).

Theorem 3.5. *Suppose $x^* \in X^0$, $F : S \to R^m$, defined as $F(x) = f(x) + \varepsilon\|x - x^*\|e$, is locally ρ-K-connected and g is locally σ-Q-connected at x^* and there exist $0 \neq \alpha \in K^*$ and $\beta \in Q^*$ satisfying the conditions* (5) *and* (6)*, then x^* is a weak εe-quasi efficient solution of (VP) provided*

$$\alpha^T \rho + \beta^T \sigma \geq 0. \tag{11}$$

Proof. Suppose that x^* is not a weak εe-quasi efficient solution of (VP), then there exists $x \in X^0$ such that

$$f(x^*) - f(x) \in \operatorname{int} K + \varepsilon\|x - x^*\|e.$$

Since $0 \neq \alpha \in K^*$, it follows that

$$\alpha^T (f(x^*) - f(x) - \varepsilon\|x - x^*\|e) > 0$$

which gives

$$\alpha^T(F(x^*) - F(x)) > 0. \tag{12}$$

Since F is locally ρ-K-connected and g is locally σ-Q-connected at x^*, therefore

$$F(x) - F(x^*) - (dF)^+(H_{x^*,x}(0^+)) - \rho d(x^*, x) \in K$$

and

$$g(x) - g(x^*) - (dg)^+(H_{x^*,x}(0^+)) - \sigma d(x^*, x) \in Q.$$

Consider

$$\begin{aligned}
&\alpha^T(F(x) - F(x^*)) \geq \alpha^T((dF)^+(H_{x^*,x}(0^+)) + \alpha^T \rho d(x^*, x) \\
&\geq \alpha^T((df)^+(H_{x^*,x}(0^+)) + \varepsilon \|dH^+_{x^*,x}(0)\|e) + \alpha^T \rho d(x^*, x) \\
&\geq -\beta^T(dg)^+(H_{x^*,x}(0^+)) - \beta^T \sigma d(x^*, x) \\
&\geq -\beta^T(g(x) - g(x^*)) \\
&= -\beta^T g(x) \geq 0, \text{ which contradicts (12).}
\end{aligned}$$

□

Theorem 3.6. *Suppose $x^* \in X^0$, $F : S \to R^m$, defined as $F(x) = f(x) + \varepsilon\|x - x^*\|e$, is locally pseudo ρ-K-connected and g is locally quasi σ-Q-connected at x^* and there exist, $0 \neq \alpha \in K^*$ and $\beta \in Q^*$ satisfying the conditions* (5) *and* (6) *then x^* is a weak εe-quasi efficient solution of (VP) provided* (11) *holds.*

Proof. Let $x^* \in X^0$. Then $\beta^T g(x) \leq 0$.
Also $\beta^T g(x^*) = 0$, it follows that

$$\beta^T(g(x) - g(x^*)) \leq 0.$$

If $\beta \neq 0$, we have $g(x) - g(x^*) \in \operatorname{int} Q$.
Since g is locally quasi σ-Q-connected at x^*, therefore, we get

$$-(dg)^+(H_{x^x,x}(0^+)) - \sigma d(x^*, x) \in Q$$

which gives

$$\beta^T(dg)^+(H_{x^*,x}(0^+)) + \beta^T \sigma d(x^*, x) \leq 0.$$

The above inequality holds if $\beta = 0$. On using (5), we get

$$\alpha^T(df)^+(H_{x^*,x}(0^+)) + \varepsilon\|dH^+_{x^*,x}(0)\|\alpha^T e \geq \beta^T \sigma d(x^*, x)$$

$$\geq -\alpha^T \rho d(x^*, x)\,.$$

Thus

$$\begin{aligned} & -\alpha^T (dF)^+ (H_{x^*,x}(0^+)) - \alpha^T \rho d(x^*, x) \leq 0 \\ \Rightarrow \quad & -(dF)^+ (H_{x^*,x}(0^+)) - \rho d(x^*, x) \notin \operatorname{int} K \end{aligned}$$

Since F is locally pseudo ρ-K-connected at x^*, we get

$$\begin{aligned} & -(F(x) - F(x^*)) \notin \operatorname{int} K \\ \Rightarrow \quad & (f(x) - f(x^*)) \notin -\operatorname{int} K - \varepsilon \|x - x^*\| e \end{aligned}$$

Hence x^* is a weak εe-quasi efficient solution of (VP). □

Theorem 3.7. *Suppose $x^* \in X^0$, $F : S \to R^m$, defined as $F(x) = f(x) + \varepsilon \|x - x^*\| e$, is locally pseuds ρ-K-connected and g is locally naturally quasi σ-Q-connected at x^* and there exist $0 \neq \alpha \in K^*$ such that condition* (5) *holds and condition* (6) *holds for all $\beta \in Q^*$. Then x^* is a weak εe-quasi efficient solution of (VP) provided* (11) *holds.*

Remark 3.1. In the above result, if the assume F to be strictly locally pseudo ρ-K-connected at x^*, then x^* is a weak εe-quasi efficient solution of (VP).

4. Duality

We now formulate Approximate Wolfe-type dual which generalizes the Wolfe-type dual given by Stancu-Minasian [16].

(AWD) K-maximize $\phi(u, \alpha, \beta) = f(u) + \beta^T g(u) k$

subject to

$$\alpha^T (df)^+ (H_{u,x}(0^+)) + \beta^T (dg)^+ (H_{u,x}(0^+)) + \varepsilon \|dH^+_{u,x}(0)\| \alpha^T e \geq 0 \quad \text{for all } x \in X^0$$

where $k \in \operatorname{int} K$ is a fixed element, $0 \neq \alpha \in K^*$, $\alpha^T k = 1$, $\beta \in Q^*$, $u \in S$.

Let

$$W = \{(u, \alpha, \beta) \in S \times R^m \times R^p \mid \alpha^T (df)^+ (H_{u,x}(0^+)) + \beta^T (dg)^+ (H_{u,x}(0^+))$$

$$+\,\varepsilon\|dH^{+}_{u,x}(0)\|\alpha^{T}e\geq 0,\ \text{for all}\ \ x\in X^{0},\ 0\neq\alpha\in K^{*},$$
$$\alpha^{T}k=1,\ \beta\in Q^{*},\ u\in S\}$$

denote the set of all feasible points of (AWD).

We now prove the various duality results for (VP) and (AWD) by assuming the functions involved to be locally ρ-connected with respect to the cone.

Theorem 4.1 (Approximate Weak Duality). *Suppose that $F:S\to R^{m}$ is defined as $F(x)=f(x)+\varepsilon\|x-u\|e$. Let x and (u,α,β) be feasible points for (VP) and (AWD) respectively. If F is locally ρ-K-connected and g is locally σ-Q-connected at u, then*

$$f(x)-\phi(u,\alpha,\beta)\notin-\operatorname{int}K-\varepsilon\|x-u\|e,$$

provided (11) *holds.*

Proof. If possible, suppose

$$f(x)-\phi(u,\alpha,\beta)\in-\operatorname{int}K-\varepsilon\|x-u\|e.$$

Since $0\neq\alpha\in K^{*}$, $\beta\in Q^{*}$ and $x\in X^{0}$, therefore

$$\alpha^{T}(f(u)+\beta g(u)k-f(x)-\varepsilon\|x-u\|e)>0\geq\beta^{T}g(x).$$

This implies that

$$\alpha^{T}f(u)+\beta^{T}g(u)>\alpha^{T}f(x)+\beta^{T}g(x)+\varepsilon\|x-u\|\alpha^{T}e. \tag{13}$$

Since F is locally ρ-K-connected, therefore

$$F(x)-F(u)-(dF)^{+}(H_{u,x}(0^{+}))-\rho d(x^{*},x)\in K.$$

That implies

$$f(x)+\varepsilon\|x-u\|e-f(u)$$
$$-\lim_{\lambda\to 0^{+}}\frac{[f(H_{u,x}(\lambda))+\varepsilon\|H_{u,x}(\lambda)-u\|e-f(u)]}{\lambda}-\rho d(x^{*},x)\in K.$$

Thus

$$f(x)-f(u)-(df)^{+}(H_{u,x}(0^{+}))+\varepsilon\|x-u\|e-\varepsilon\|dH^{+}_{u,x}(0)\|e-\rho d(x^{*},x)\in K.$$

Since $\alpha \in K^*$, therefore

$$\alpha^T f(x) - \alpha^T f(u) - \alpha^T (df)^+(H_{u,x}(0^+)) \\ + \varepsilon\|x-u\|\alpha^T e - \varepsilon\|dH^+_{u,x}(0)\|\alpha^T e - \alpha^T \rho d(x^*,x) \geq 0. \tag{14}$$

Now since g is locally σ-Q-connected and $\beta \in Q^*$, we get

$$\beta^T g(x) - \beta^T g(u) - \beta^T (dg)^+(H_{u,x}(0^+)) - \beta^T \sigma d(x^*,x) \geq 0. \tag{15}$$

Adding (14) and (15), we get

$$\alpha^T f(x) - \alpha^T f(u) + \beta^T g(x) - \beta^T g(u) + \varepsilon\|x-u\|\alpha^T e \\ \geq \alpha^T (df)^+(H_{u,x}(0^+)) + \beta^T (dg)^+(H_{u,x}(0^+)) \\ + \varepsilon\|dH^+_{u,x}(0)\|\alpha^T e + (\alpha^T \rho + \beta^T \sigma) d(x^*,x).$$

By the dual feasibility of (u,α,β) and (11), we get

$$\alpha^T f(x) - \alpha^T f(u) + \beta^T g(x) - \beta^T g(u) + \varepsilon\|x-u\|\alpha^T e \geq 0.$$

This gives

$$\alpha^T f(u) + \beta^T g(u) \leq \alpha^T f(x) + \beta^T g(x) + \varepsilon\|x-u\|e$$

which contradicts (13).

Hence $f(x) - \phi(u,\alpha,\beta) \notin -\operatorname{int} K - \varepsilon\|x-u\|e$. □

Theorem 4.2 (Approximate Strong Duality). *Let $x^* \in X$ be a weak εe-quasi efficient solution of (VP). Let $(df)^+(H_{x^*,x}(0^+)) + \varepsilon\|dH^+_{x^*,x}(0)\|e$ and $(dg)^+(H_{x^*,x}(0^+))$ are K-subconvexlike and Q-subconvexlike functions of x respectively with respect to the same arc $H_{x^*,x}$. Suppose that g satisfies the generalized Slater's-type cone constraint qualification at x^*, then there exist $0 \neq \alpha^* \in K^*$, $\beta^* \in Q^*$ such that (x^*,α^*,β^*) is feasible for (AWD). Moreover, if the conditions of Approximate Weak Duality Theorem 4.1 hold then (x^*,α^*,β^*) is a weak εe-quasi efficient solution of (AWD).*

Proof. Suppose all the conditions of Theorem 3.4 are satisfied therefore there exist $0 \neq \alpha \in K^*$ and $\beta \in Q^*$ such that conditions (5) and (6) hold.

Setting

$$\alpha^* = \frac{\alpha}{\alpha k}, \ \beta^* = \frac{\beta}{\alpha k}.$$

We get ${\alpha^*}^T k = 1$ where $0 \neq \alpha^* \in K^*$. By condition (5), (x^*, α^*, β^*) is feasible solution for (AWD). If possible let (x^*, α^*, β^*) be not a weak εe-quasi efficient solution of (AWD), then there exists (u, α, β) feasible solution for (AWD) such that $\phi(u, \alpha, \beta) - \phi(x^*, \alpha^*, \beta^*) \in \operatorname{int} K + \varepsilon \|u - x^*\| e$.

Since by condition (6), $\alpha^*, g(x^*) = 0$, we get

$$\phi(u, \alpha, \beta) - f(x^*) \in \operatorname{int} K + \varepsilon \|u - x^*\| e,$$

which is a contradiction to Approximate Weak Duality Theorem 4.1. Hence (x^*, α^*, β^*) must be a weak εe-quasi efficient solution of (AWD). □

Theorem 4.3 (Strict Converse Duality Theorem). *Suppose $F : S \to R^m$, where $F(x) = f(x) + \varepsilon \|x - u\| e$. Let x and (u, α, β) be feasible points for (VP) and (AWD) respectively such that*

$$\alpha^T (f(x) + \varepsilon \|x - u\| e) = \alpha^T f(u) + \beta^T g(u). \tag{16}$$

If F is strictly locally ρ-K-connected and g is locally $\sigma - Q$ connected at u then $x = u$ provided (11) *holds.*

Proof. Let if possible $x \neq u$. Since F is strictly locally ρ-K-connected and g is locally σ-Q-connected at u, therefore

$$F(x) - F(u) - (dF)^+(H_{u,x}(0^+)) - \rho d(x^*, x) \in \operatorname{int} K$$

and

$$g(x) - g(u) - (dg)^+(H_{u,x}(0^+)) - \sigma d(x^*, x) \in Q\,.$$

Thus

$$f(x) - f(u) - (df)^+(H_{u,x}(0^+)) + \varepsilon \|x - u\| e - \varepsilon \|dH^+_{u,x}(0)\| e - \rho d(x^*, x) \in \operatorname{int} K\,.$$

Since $0 \neq \alpha \in K^*$and $\beta \in Q^*$, therefore

$$\begin{aligned} &\alpha^T f(x) - \alpha^T f(u) - \alpha^T (df)^+(H_{u,x}(0^+)) + \varepsilon \|x - u\| \alpha^T e \\ &\quad - \varepsilon \|dH^+_{u,x}(0)\| \alpha^T e - \alpha^T \rho d(x^*, x) > 0 \end{aligned} \tag{17}$$

and

$$\beta^T g(x) - \beta^T g(u) - \beta^T (dg)^+(H_{u,x}(0^+)) - \beta^T \sigma d(x^*, x) \geq 0. \tag{18}$$

Adding (17) and (18), we get

$$\alpha^T f(x) - \alpha^T f(u) + \beta^T g(x) - \beta^T g(u) + \varepsilon \|x - u\| \alpha^T e$$

$$> \alpha^T (df)^+ (H_{u,x}(0^+)) + \beta^T (dg)^+ (H_{u,x}(0^+))$$
$$+ \varepsilon \|dH^+_{u,x}(0)\| \alpha^T e + (\alpha^T \rho + \beta^T \sigma) d(x^*, x).$$

Using the dual feasibility of (u, α, β) and (11), we have

$$\alpha^T f(x) - \alpha^T f(u) + \beta^T g(x) - \beta^T g(u) + \varepsilon \|x - u\| \alpha^T e > 0$$

This implies

$$\alpha^T (f(x) + \varepsilon \|x - u\| e) > \alpha^T f(u) + \beta^T g(u),$$

which is a contradiction to (16). Hence $x = u$.

We now associate the following Approximate Mond-Weir type dual with (VP),

$$\text{(AVD)} \quad K\text{-maximize } f(u)$$
$$\text{subject to } \alpha^T (df)^+ (H_{u,x}(0^+)) + \beta^T (dg)^+ (H_{u,x}(0^+))$$
$$+ \varepsilon \|d(H^+_{u,x}(0)\| \alpha^T e \geq 0, \quad \text{for all } x \in X^0$$
$$\beta^T g(u) \geq 0$$
$$u \in S,\ 0 \neq \alpha \in K^*,\ \beta \in Q^*$$

where $e \in K \setminus \{0\}$. □

Theorem 4.4 (Approximate Weak Duality). *Suppose that $F : S \to R^m$ is defined by $F(x) = f(x) + \varepsilon \|x - u\| e$. Let $x \in X^0$ and (u, α, β) be dual feasible. If F is locally pseudo ρ-K-connected and g is locally quasi σ-Q-connected at u, then*

$$f(x) - f(u) \notin - \operatorname{int} K - \varepsilon \|x - u\| e$$

provided (11) *holds.*

Theorem 4.5 (Approximate Strong Duality). *Let x^* be a weak εe-quasi efficient solution of (VP). Let $(df)^+(H_{x^*,x}(0^+)) + \varepsilon \|dH^+_{x^*,x}(0)\| e$ and $(dg)^+(H_{x^*,x}(0^+))$ be K-subconvexlike and Q-subconvexlike functions of x respectively with respect to the same arc $H_{x^*,x}$. Let g satisfies the generalized Slater's type cone constraint qualification at x^*. Then there exist $0 \neq \alpha^* \in K^*$, $\beta^* \in Q^*$ such that (x^*, α^*, β^*) is feasible for (AVD). Moreover, if for each feasible (x, α, β) hypotheses of Theorem 4.4 holds then (x^*, α^*, β^*) is a weak εe-quasi efficient solution of (AVD).*

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