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ON THE EQUIVALENCE BETWEEN ERGODICITY AND WEAK MIXING FOR OPERATORS SEMIGROUPS

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Abstract: We prove the equivalence between ergodicity and weak mixing of an invariant probability measure m for strongly continuous contraction semigroups of linear operators on $L^2(m)$ satisfying the sector condition.

The same result is proved for subordinated semigroups in the Bochner sense by the one-sided stable sudordinators.

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1. Introduction

Ergodic theory is still an important branch in mathematics because of its multiple applications in other mathematical branches and also in other fields of science such as physics, biology, chemistry, etc. see monograph [4] and the references therein. Recently, important new applications of ergodicity appear in economics, social sciences, quantum dynamics etc. We refer to [14, 12, 13, 19] and the related references. On the other hand, it is known that weak mixing is a sufficient condition for ergodicity and is sometimes more convenient to handle. Therefore, it seems to be worthwhile to investigate systems for which

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there is equivalence between the two notions. The present paper constitutes a contribution in this subject.

Let $(X, \mathcal{A}, m)$ be a probability space, let $\mathbb{P}$ be a strongly continuous contraction semigroup on $L^2(m)$ such that m is $\mathbb{P}$-invariant, and let A_m be the infinitesimal generator of $\mathbb{P}$.

The probability measure m is said to be $\mathbb{P}$-ergodic if 0 is a simple eigenvalue of the generator A_m and it is said to be $\mathbb{P}$-weakly mixing if 0 is the unique eigenvalue of the generator A_m and it is simple. It is clear that weak mixing implies ergodicity but the converse fails in general (cf. [4] for example). However the equivalence between egodicity and weak mixing is proved for stationary Gaussian processes [4], symmetric stable processes [14], symmetric semi-stable processes [10], stationary symmetric infinitely divisible processes [2], and stationary infinitely divisible processes [15].

In this note, we suppose that $\mathbb{P}$ satisfies the so-called Sector Condition (Definition 5) and we prove using standard arguments that m is $\mathbb{P}$-ergodic if and only if m is $\mathbb{P}$-weakly mixing (Theorem 7). If $\mathbb{P}$ is the transition function of a Markov process, then our result may be applied to symmetric processes, some Lévy processes and some non necessarily infinitely divisible processes.

In the second part of this note, we consider the subordinate $\mathbb{P}_\alpha$ of (a general semigroup) $\mathbb{P}$ by means of any one-sided stable sudordinator of order $\alpha \in]0, 1[$ and we prove by similar arguments, the equivalence between egodicity and weak mixing for m with respect to $\mathbb{P}_\alpha$ (Theorem 11).

2. Ergodicity and weak mixing

The present note is concerned with strongly continuous contraction semigroups on $L^2(m)$ and their Bochner subordination. Since these notions are well known even for operators on abstract Banach spaces, we adapt them on $L^2(m)$. We will refer essentially to chapter 4 of [8] and the related references (cf. also [1, 16, 17]).

Let $(E, \mathcal{A}, m)$ be a probability space and let $L^2(m) = L^2(E, \mathcal{A}, m)$ be the space of complex-valued square integrable functions. We denote by $< .,. >_m$ the inner product in the Hilbert space $L^2(m)$ and by $\|.\|_m$ the associated norm. As usually, the equality between two functions $\varphi_1, \varphi_2 \in L^2(m)$ means that $< \varphi_1, \psi >_m = < \varphi_2, \psi >_m$ for all $\psi \in L^2(m)$. The set $\mathbb{R}$ is always endowed with the Borel σ- algebra $\mathcal{B}$ and the Lebesgue measure λ.

Definition 1. A strongly continuous contraction semigroup on $L^2(m)$ is

a family $\mathbb{P} := (P_t)_{t\geq 0}$ of linear operators $P_t : L^2(m) \to L^2(m), t \geq 0$ such that $P_0 = I$ the identical operator,
(1) $\mathbb{P}$ is a semigroup, i.e. $P_{s+t} = P_s P_t; \quad s, t \geq 0$,
(2) $\mathbb{P}$ is strongly continuous, i.e. $\lim_{t\to 0} \|P_t\varphi - \varphi\|_m = 0; \varphi \in L^2(m)$,
(3) $\mathbb{P}$ is a contraction, i.e. $\|P_t\varphi\|_m \leq \|\varphi\|_m; \varphi \in L^2(m)$.

The *generator* A_m of $\mathbb{P}$ is defined by

$$A_m(\varphi) := \lim_{t\to 0} \frac{1}{t}(P_t\varphi - \varphi),$$

on its domain $D(A_m)$ which is the set of all functions $\varphi \in L^2(m)$ for which this limit exists in $L^2(m)$. It is known that:

1. $D(A_m)$ is dense in $L^2(m)$ and A_m is closed.

2. If $\varphi \in D(A_m)$ then $P_t\varphi \in D(A_m)$ and $A_m(P_t\varphi) = P_t(A_m\varphi)$, for each $t > 0$.

For the following notions about invariance and ergodicity, we will refer to [5], Part I.

Let $\mathbb{P}$ be a strongly continuous contraction semigroup on $L^2(m)$ and let A_m be the associated generator. The probability measure m is said to be $\mathbb{P}$-invariant if

$$< P_t\varphi\,, 1 >_m = < \varphi\,, 1 >_m; \quad t \geq 0, \varphi \in L^2(m). \tag{1}$$

Definition 2. The measure m is said to be $\mathbb{P}$-ergodic if m is $\mathbb{P}$-invariant and
(E1) *If $\varphi \in L^2(m)$ and $P_t\varphi = \varphi$, for all $t > 0$, then φ is constant.*

It is known that (E1) is equivalent to
(E2) 0 *is a simple eigenvalue of the generator A_m.*

Definition 3. The measure m is said to be $\mathbb{P}$-weakly mixing if m is $\mathbb{P}$-invariant and
(M1) *If $\varphi \in L^2(m)$ and $\ell \in \mathbb{R}$ with $P_t\varphi = e^{i\ell t}\varphi$ for all $t > 0$, then φ is constant.*

It is known that (M1) is equivalent to
(M2) 0 *is the unique eigenvalue of the generator A_m and it is simple.*

Proposition 4. *Suppose that $P_t(L^2(m)) \subset D(A_m), t > 0$ and there exists some constant $C > 0$ such that*

$$\|A_m P_t \varphi\|_m \;\leq\; \frac{C}{t}\,\|\varphi\|_m; \quad t > 0, \varphi \in L^2(m). \tag{2}$$

Then m is $\mathbb{P}$-ergodic if and only if m is $\mathbb{P}$-weakly mixing.

Proof. Suppose that m is $\mathbb{P}$-ergodic. Let $\psi \in L^2(m)$ and $\ell \in \mathbb{R}$ such that $\|\psi\|_m \neq 0$ and

$$P_t\psi \;=\; e^{i\ell t}\psi; \quad t > 0. \tag{3}$$

Hence

$$P_{s+t}\psi = e^{i\ell(s+t)}\psi; \quad s, t > 0. \tag{4}$$

Since $P_t\psi \in D(A_m)$ then, by differentiation of (4) and by taking the norm, we deduce that

$$\|A_m P_t\psi\|_m \;=\; |\ell|\,\|\psi\|_m; \quad t > 0. \tag{5}$$

Now, combining (5) and (2) applied to ψ, we obtain

$$|\ell| \;\leq\; \frac{C}{t}; \quad t > 0. \tag{6}$$

By letting $t \to \infty$ in (6) we get $\ell = 0$. Thus (3) becomes

$$P_t\psi \;=\; \psi; \quad t > 0. \tag{7}$$

By the $\mathbb{P}$-ergodicity of m we deduce from (7) that ψ is constant and we conclude that m is $\mathbb{P}$-weakly mixing. □

For the following notion, we refer to [18].

Definition 5. $\mathbb{P}$ is said to satisfy the Sector Condition if there exists a constant $M > 0$ such that for all $\varphi, \psi \in D(A_m)$

$$|< \varphi, -A_m\psi >_m| \;\leq\; M \;< \varphi, -A_m\varphi >_m^{(1/2)} \cdot < \psi, -A_m\psi >_m^{(1/2)}. \tag{8}$$

For the proof of the following important result, we follow the proof of Lemma 2.1 in [18].

Proposition 6. *If $\mathbb{P}$ satisfies the Sector Condition then we have $P_t(L^2(m)) \subset D(A_m)$ and there exists some constant $C > 0$ such that*

$$\|A_m P_t \varphi\|_m \le \frac{C}{t} \|\varphi\|_m; \quad t > 0, \varphi \in L^2(m). \tag{9}$$

Proof. Following [9] Chapter 9, §1.6, the Sector Condition (8) implies that $\mathbb{P}$ admits a holomorphic extension in the sector $\{t \in \mathbb{C} : |\arctan(t)| < \arctan(M)\}$. Hence the Cauchy integral formula yields the estimate

$$\|(d/dt) P_t \varphi\|_m \le \frac{C}{t} \|\varphi\|_m; \quad t > 0, \varphi \in L^2(m) \tag{10}$$

for some constant $C > 0$. Since $(d/dt)P_t\varphi = A_m P_t \varphi$ for $\varphi \in D(A_m)$, A_m is closed, and $D(A_m)$ is dense in $L^2(m)$ then (10) implies that $P_t(L^2(m)) \subset D(A_m)$ for $t > 0$ and (9) holds. □

Combining Propositions 4 and 6, we deduce the first result of this note.

Theorem 7. *Suppose that $\mathbb{P}$ satisfies the Sector Condition. Then m is $\mathbb{P}$-ergodic if and only if m is $\mathbb{P}$-weakly mixing.*

Examples 8.

1. If $\mathbb{P}$ is m-symmetric, that is

   $$< P_t\varphi, \psi >_m = < \varphi, P_t\psi >_m; \quad t > 0, \varphi, \psi \in L^2(m),$$

   then $\mathbb{P}$ satisfies the sector condition for $M = 1$. If a Markov process X has $\mathbb{P}$ as transition function, then X is symmetric.

2. Suppose that $P_t\varphi = \mu_t * \varphi$ where $\mu := (\mu_t)_{t \ge 0}$ is a convolution semigroup on $\mathbb{R}^d$ with negative definite function $\Upsilon : \mathbb{R}^d \to \mathbb{C}$ defined by the Fourier transform $\hat{\mu}_t = \exp(-t\Upsilon(x)), t > 0$ (cf. [1] for more details). Following [8] Example 4.7.32, $\mathbb{P}$ satisfies the Sector Condition if and only if

   $$|\Upsilon(y)| \le L(1 + \Re\Upsilon(y)), \quad y \in \mathbb{R}^d,$$

   for some constant $L > 0$.

   Notice that such semigroups are always transitions functions of general Lévy processes (cf. [16] Chapters 1 and 2 for example).

Remark 9. In view of the result established in [15], the Sector "Condition" is only a sufficient condition in order to obtain the equivalence between ergodicity and weak mixing.

3. Equivalence for fractional power semigroups

For the following classical concepts, we refer to [1, 8, 16, 17, 20].

For $\alpha \in]0,1[$ and $t \geq 0$, let $\eta_{t,\alpha}$ be the probability measure on $[0,\infty[$ defined by its Laplace transform

$$\mathcal{L}\,(\eta_{t,\alpha})(r) \;=\; \exp(-tr^{\alpha}); \quad r > 0. \tag{11}$$

The family $\eta_\alpha := (\eta_{t,\alpha})_{t\geq 0}$ is a vaguely continuous convolution semigroup of probability measures on $[0,\infty[$, called one-sided stable subordinator of order α.

Let $\mathbb{P}$ be a strongly continuous contraction semigroup on $L^2(m)$ and $\alpha \in]0,1[$. The Bochner integral

$$P_{t,\alpha}\varphi := \int_0^\infty P_s\varphi\,\eta_{t,\alpha}(ds); \quad t \geq 0, \varphi \in L^2(m), \tag{12}$$

defines a strongly continuous contraction semigroup $\mathbb{P}_\alpha := (P_{t,\alpha})_{t\geq 0}$ on $L^2(m)$. $\mathbb{P}_\alpha$ is said to be subordinated to $\mathbb{P}$ by means of η_α in the sense of Bochner.

Suppose that m is $\mathbb{P}$-invariant. By integration of (1) with respect to the probability measure $\eta_{t,\alpha}$, we get

$$< P_{t,\alpha}\varphi\,,\,1 >_m \,=\, < f\,,\,1 >_m; \quad t \geq 0, \varphi \in L^2(m). \tag{13}$$

Hence m is $\mathbb{P}_\alpha$-invariant. The converse does not hold in general (cf. [6]).
Let $A_{\alpha,m}$ be the generator of $\mathbb{P}_\alpha$, then "roughly speaking", $A_{\alpha,m}$ is the fractional power of order α of the generator A_m of $\mathbb{P}$, that is $D(A_m) \subset D(A_{\alpha,m})$ and

$$A_{\alpha,m}\varphi \;=\; -(-A_m)^{\alpha}\varphi; \quad \varphi \in D(A_m).$$

The proof of the next important result is inspired from the proof of Theorem 1 pp. 263-264 in [20] (cf. also [7]).

Proposition 10. *Let $\mathbb{P}$ be a strongly continuous contraction semigroup on $L^2(m)$ such that m is $\mathbb{P}$-invariant. Then for each $\alpha \in]0,1[$ we have $P_{t,\alpha}(L^2(m)) \subset D(A_{\alpha,m}), t > 0$ and there exists some constant $C > 0$ such that*

$$\|A_{\alpha,m}P_{t,\alpha}\varphi\|_m \;\leq\; \frac{C}{t}\,\|\varphi\|_m; \quad t > 0, \varphi \in L^2(m). \tag{14}$$

Proof. Let S be the Banach algebra of complex Borel measures on $[0,\infty[$, with convolution as multiplication and normed by the total variation $\|.\|_S$.

According to [3] Example 3, $t \to \eta_{t,\alpha}$ is continuously differentiable from $]0,\infty[$ to S and

$$\|\eta'_{t,\alpha}\|_S < \infty; \quad t > 0, \tag{15}$$

where $\eta'_{t,\alpha} := (d/dt)\eta_{t,\alpha}$.

On the other hand, since $\mathbb{P}$ is contractive then it is uniformly bounded and therefore the differentiation with respect to t under the integral sign in (12) is justified and we have

$$P'_{t,\alpha}\varphi = \int_0^\infty P_s\varphi\, \eta'_{t,\alpha}(ds); \quad t \geq 0, \varphi \in L^2(m), \tag{16}$$

where $P'_{t,\alpha} := (d/dt)P_{t,\alpha}$. Since

$$(d/dt)P_{t,\alpha}\varphi \; = \; A_{\alpha,m}P_{t,\alpha}\varphi; \quad \varphi \in D(A_{\alpha,m}),$$

$A_{\alpha,m}$ is closed, and $D(A_{\alpha,m})$ is dense in $L^2(m)$ then (16) implies that $P_{t,\alpha}(L^2(m)) \subset D(A_{\alpha,m}), t > 0$.

Now, following [20], page 264, we have

$$P'_{t,\alpha}\varphi = \frac{1}{t}\int_0^\infty P_{rt^{1/\alpha}}\varphi\, \eta'_{1,\alpha}(dr); \quad t \geq 0, \varphi \in L^2(m), \tag{17}$$

by a convenient change of variables in (16). Therefore, by taking the norm in (17) and by using (15) and the contraction property of $\mathbb{P}$ we get

$$\|A_{\alpha,m}P_{t,\alpha}\varphi\|_m \; \leq \; \frac{1}{t}\, \|\eta'_{1,\alpha}\|_S\, \|\varphi\|_m; \quad t > 0, \varphi \in L^2(m) \tag{18}$$

and we conclude by (18) that (14) holds. □

Combining Proposition 10 and Proposition 4 (applied to $\mathbb{P}_\alpha$ instead of $\mathbb{P}$) we deduce the second result of this note.

Theorem 11. *Let $\mathbb{P}$ be a strongly continuous contraction semigroup on $L^2(m)$ such that m is $\mathbb{P}$-invariant. Then, for each $\alpha \in]0,1[$, the probability measure m is $\mathbb{P}_\alpha$-ergodic if and only if m is $\mathbb{P}_\alpha$-weakly mixing.*

Remark 12. The subordination in the sense of Bochner may be obtained by any subordinator, i.e. a vaguely continuous convolution semigroup of probability measures on $[0,\infty[$. It seems to be worthwhile to characterize subordinators for which Theorem 11 holds.

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