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# ON SECOND HYPER-ZAGREB INDEX OF CORONA PRODUCTS RELATED TO R-GRAPHS

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**Abstract:** The cognitive and evidential features of the graph discipline are significantly influenced by the implementation of graph operations. Molecular descriptor acts as a fundamental network invariant relevant to a particular molecular structure in the framework of chemical graph theory. The semi-total point graph features the edges of subdivision graph as well as the edges of the original graph. In this paper, we explore combinatorial inequalities associated with the edges, vertices and its corresponding neighborhood notions along with the inclusion of other molecular descriptors in the computations for the determination of exact expressions of second hyper-Zagreb index for certain corona products involving the semi-total point graph.

**AMS Subject Classification:** 05C09, 05C76, 05C92
**Key Words:** second Hyper-Zagreb Index, semi-total point graphs, corona product, graph operations

## 1. Introduction

A molecule's topology is mostly a non-numerical component of mathematics that reflects various features of a molecular network. Numerous quantitative molecular attributes are frequently represented by precise numerals. A chemical compound must convert itself to a molecular graph in order to evaluate

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topological invariants, with the atoms of the molecule corresponding to vertices and the atomic linkages displayed as edges. The molecular descriptors are relevant in diverse domains, but degree-based invariants are specifically essential in the realm of chemical graph theory.

For a molecular graph $\Upsilon = (V, E)$, $V(\Upsilon)$ indicates the vertex set and $E(\Upsilon)$ represents the edge set. $p$ and $q$ denote cardinality of vertex and edge set for graph, $\Upsilon$ respectively. Let $d(w)$ denote the degree of vertex $w$ in $\Upsilon$ and $e = vw$ is the edge joining the vertex $v$ with vertex $w$. The line graph of $\Upsilon$, $L(\Upsilon)$ is the graph wherein the edges of $\Upsilon$ correspond to vertices of $L(\Upsilon)$ and two edges of $L(\Upsilon)$ are adjacent iff they are incident in $\Upsilon$. $\delta_\Upsilon$ and $\Delta_\Upsilon$ indicates minimum and maximum degree for the graph, $\Upsilon$ respectively.

Wiener index is the primitive and extensively explored topological invariant and forms a classic example among the distance based topological descriptors[6]. Since then, a vast array of molecular descriptors have come into prominence and significant progress has been made in the discovery of topological network indices. Randić index has numerous implementations in chemical and therapeutic domains [10, 9]. Consequently, generalizations on randić index embarked the study on another descriptor called the general randić index[11], $R_\alpha$ $(\alpha \in \mathbb{R})$,

$$R_\alpha(\Upsilon) = \sum_{vw \in E(\Upsilon)} (d_\Upsilon(v).d_\Upsilon(w))^\alpha.$$

Introduced by Gutman, the First and the Second Zagreb Index are the widely known topological descriptors defined on the basis of the node-degree to specify $\pi$-electron energy of the molecules, [7, 8]. Many works on another invariant, namely the forgotten topological index, have also been carried out, [20]. Many graph operational expressions have been determined with respect to the index [26],

$$F(\Upsilon) = \sum_{vw \in E(\Upsilon)} \left(d_\Upsilon(v)^2 + d_\Upsilon(w)^2\right) \text{ or } \sum_{w \in V(\Upsilon)} d_\Upsilon(w)^3.$$

Bo Zhou and Nenad Trinajstić identified the sum-connectivity index, $\chi$. Subsequently, numerous characteristics and combinatorial inequalities for corresponding invariant were determined in [17],

$$\chi(\Upsilon) = \sum_{vw \in E(\Upsilon)} (d_\Upsilon(v) + d_\Upsilon(w))^{-1/2}.$$

Consequently, on generalizing the first Zagreb index with the sum-connectivity index [18] embarked the study on another descriptor called the general sum-connectivity index, $\chi_\alpha$ $(\alpha \in \mathbb{R})$. Certain theoretical underpinnings for the invariant have also been determined in relation to certain corona product operations for composite graphs, [16],

$$\chi_\alpha(\Upsilon) = \sum_{vw\in E(\Upsilon)} (d_\Upsilon(v) + d_\Upsilon(w))^\alpha .$$

In [12], the study on the Hyper Zagreb index was initiated and certain results relevant to the mathematical techniques of graph operations were obtained,

$$HM(\Upsilon) = \sum_{vw\in E(\Upsilon)} (d_\Upsilon(v) + d_\Upsilon(w))^2.$$

Many re-formulations and re-definitions have been carried out with reference to the Zagreb indices in [13, 14, 19] and the redefined Zagreb index variants have been proposed. The third redefined Zagreb index, $ReZG_3(\Upsilon)$ can be determined as:

$$ReZG_3(\Upsilon) = \sum_{vw\in E(\Upsilon)} (d_\Upsilon(v)d_\Upsilon(w))(d_\Upsilon(v) + d_\Upsilon(w)).$$

The study on another variant of topological descriptor, the second hyper Zagreb index was initiated in [21] and computations of the invariant related to certain chemical compounds have also been established, [22, 23]. It is described as:

$$HM_2(\Upsilon) = \sum_{vw\in E(\Upsilon)} \big(d_\Upsilon(v).d_\Upsilon(w)\big)^2.$$

To express the forgotten topological index, graph invariant earlier denoted by $\xi_4$, [24], later renamed as the $Y$-Index (Yemen Index) in [5] is defined as:

$$Y(\Upsilon) = \sum_{vw\in E(\Upsilon)} \big(d_\Upsilon(v)^3 + d_\Upsilon(w)^3\big) \text{ or } \sum_{w\in V(\Upsilon)} d_\Upsilon(w)^4.$$

The investigation of various graph operations has always given a broad opportunity for research in linked sectors of the field. The corona product $\Upsilon_1 \circ \Upsilon_2$ of graphs $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$ is acquired while taking a replica of $\Upsilon_1$ with $p_{\Upsilon_1}$ replicas of $\Upsilon_2$ and by linking every node of the $p_{th}$ replica of $\Upsilon_2$ to the $p_{th}$ node of $\Upsilon_1$; $1 \leq p \leq p_{\Upsilon_1}$. $S(\Upsilon)$ or subdivision graph is generated by replacing every link of the graph with a degree two node, keeping the original nodes unchanged. $R(\Upsilon)$ or semi-total point graph is the one including the edges of $S(\Upsilon)$ along with the edges of $\Upsilon$.

Outcomes of certain graph operational series relevant to the semi-total point graph have been initiated by Jie Lan and Bo Zhou, [15]. Discrete inequalities have been determined for the first entire Zagreb index and general sum-connectivity index with respect to the semi-total point graph associated corona products, [16, 25].

## 2. Methodology

In this paper, certain combinatorial exact expressions for second hyper-Zagreb index linked with some corona graph products predicated mainly on the semi-total point graph, are proposed. We have included the corona operations correlated with vertex, edge, vertex neighborhood and edge neighborhood products for our computations.

### 2.1. R-Vertex Corona Product

For $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$, the $R$-Vertex Corona Product indicated as $R(\Upsilon_1) \odot \Upsilon_2$ is acquired by taking one replica of distinct vertex graph $R(\Upsilon_1)$ and $p_{\Upsilon_1}$ replicas of $\Upsilon_2$ and linking a node of $V(\Upsilon_1)$ on the $i_{th}$ location in $R(\Upsilon_1)$ to every node in the $i_{th}$ replica of $\Upsilon_2$. $|V(R(\Upsilon_1) \odot \Upsilon_2)| = p_{\Upsilon_1} + q_{\Upsilon_1} + p_{\Upsilon_1}p_{\Upsilon_2}$ and $|E(R(\Upsilon_1) \odot \Upsilon_2)| = 3q_{\Upsilon_1} + p_{\Upsilon_1}q_{\Upsilon_2} + p_{\Upsilon_1}p_{\Upsilon_2}$, [15].

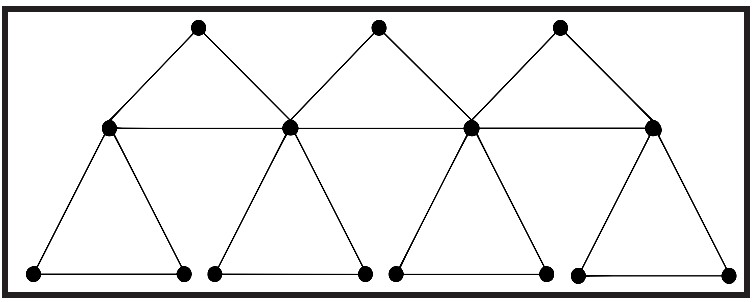

Figure 1: R-Vertex Corona Product $R(P_4) \odot P_2$

Suppose $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ & $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$ are two graphs; the respective degrees of the nodes in graph $\Upsilon_1 \odot \Upsilon_2$:

$$d_{\Upsilon_1 \odot \Upsilon_2}(w) = \begin{cases} 2d_{\Upsilon_1}(w) + p_{\Upsilon_2} & ;\text{if } w \in V(\Upsilon_1) \\ 2 & ;\text{if } w \in I(\Upsilon_1) \\ d_{\Upsilon_2}(w) + 1 & ;\text{if } w \in V(\Upsilon_2) \end{cases} .$$

**Theorem 1.** *Let $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$ represent two arbitrary graphs. Then:*

$$\begin{aligned}HM_2(\Upsilon_1 \odot \Upsilon_2) &= 4\Big[p_{\Upsilon_2}^2 HM_1(\Upsilon_1) + 4HM_2(\Upsilon_1)\Big] + p_{\Upsilon_1}\Big[HM_1(\Upsilon_2)\\ &+ HM_2(\Upsilon_2) + 2\big(ReZG_3(\Upsilon_2) + M_1(\Upsilon_2) + M_2(\Upsilon_2)\big)\Big]\\ &+ 4p_{\Upsilon_2}\Big[4ReZG_3(\Upsilon_1) + (p_{\Upsilon_2}^2 + 5)M_1(\Upsilon_1) + 2p_{\Upsilon_2}M_2(\Upsilon_1)\\ &+ M_1(\Upsilon_2)\Big(2q_{\Upsilon_1} + \frac{p_{\Upsilon_1}p_{\Upsilon_2}}{4}\Big)\Big] + 4\Big[M_1(\Upsilon_1)\big(M_1(\Upsilon_2) + 4q_{\Upsilon_2}\big)\\ &+ 4F(\Upsilon_1)\Big] + q_{\Upsilon_1}p_{\Upsilon_2}\Big[p_{\Upsilon_2}\big(p_{\Upsilon_2}^2 + 16\big) + 32q_{\Upsilon_2}\Big] + p_{\Upsilon_1}\Big[q_{\Upsilon_2}\\ &+ p_{\Upsilon_2}^2\big(p_{\Upsilon_2} + 4q_{\Upsilon_2}\big)\Big].\end{aligned}$$

*Proof.*

$$\begin{aligned}HM_2(\Upsilon_1 \odot \Upsilon_2) &= \sum_{uv\in E(\Upsilon_1\odot\Upsilon_2)} \big(d_{\Upsilon_1\odot\Upsilon_2}(u).d_{\Upsilon_1\odot\Upsilon_2}(v)\big)^2\\ &= \sum_{uv\in E(R(\Upsilon_1))} \big(d_{R(\Upsilon_1)}(u).d_{R(\Upsilon_1)}(v)\big)^2 + \sum_{i=1}^{p_{\Upsilon_1}} \sum_{uv\in V(\Upsilon_2)} \big(d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\big)^2\\ &+ \sum_{u\in V(R(\Upsilon_1))} \sum_{v\in V(\Upsilon_2)} \big(d_{R(\Upsilon_1)}(u).d_{\Upsilon_2}(v)\big)^2\\ &= \sum f + \sum g + \sum h.\end{aligned}$$

For the computation of $\sum f$, we have

$$\begin{aligned}\sum f &= \sum_{uv\in E(R(\Upsilon_1))} \big(d_{R(\Upsilon_1)}(u).d_{R(\Upsilon_1)}(v)\big)^2\\ &= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big[\big(2d_{\Upsilon_1}(u) + p_{\Upsilon_2}\big).\big(2d_{\Upsilon_1}(v) + p_{\Upsilon_2}\big)\Big]^2\\ &+ \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1), v\in I(\Upsilon_1)}} \Big[\big(2d_{\Upsilon_1}(u) + p_{\Upsilon_2}\big).2\Big]^2\\ &= \sum f_1 + \sum f_2.\end{aligned}$$

For the computation of $\sum f_1$,

$$\sum f_1 = \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big[\big(2d_{\Upsilon_1}(u)+p_{\Upsilon_2}\big).\big(2d_{\Upsilon_1}(v)+p_{\Upsilon_2}\big)\Big]^2$$

$$= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big[4d_{\Upsilon_1}(u)d_{\Upsilon_1}(v)+2p_{\Upsilon_2}\big(d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v)\big)+p^2_{\Upsilon_2}\Big]^2$$

$$= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} 16\Big(d_{\Upsilon_1}(u)d_{\Upsilon_1}(v)\Big)^2+4p^2_{\Upsilon_2}\Big(d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v)\Big)^2+p^4_{\Upsilon_2}$$

$$+16p_{\Upsilon_2}d_{\Upsilon_1}(u)d_{\Upsilon_1}(v)\Big(d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v)\Big)+8p^2_{\Upsilon_2}\Big(d_{\Upsilon_1}(u)d_{\Upsilon_1}(v)\Big)$$

$$+4p^3_{\Upsilon_2}\Big(d_{\Upsilon_1}(u)+d_{\Upsilon_1}(v)\Big)$$

$$\begin{aligned}\Rightarrow \sum f_1 \quad &= \quad 4p^2_{\Upsilon_2}HM_1(\Upsilon_1)+16HM_2(\Upsilon_1)+4p^3_{\Upsilon_2}M_1(\Upsilon_1)+8p^2_{\Upsilon_2}M_2(\Upsilon_1)\\ &\quad +16p_{\Upsilon_2}ReZG_3(\Upsilon_1)+p^4_{\Upsilon_2}q_{\Upsilon_1}.\end{aligned}$$

Similarly for $\sum f_2$,

$$\sum f_2 = \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1),v\in I(\Upsilon_1)}} \Big[\big(2d_{\Upsilon_1}(u)+p_{\Upsilon_2}\big).2\Big]^2$$

$$= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1),v\in I(\Upsilon_1)}} 16d_{\Upsilon_1}(u)^2+16p_{\Upsilon_2}d_{\Upsilon_1}(u)+4p^2_{\Upsilon_2}$$

$$\Rightarrow \sum f_2 \quad = \quad 16\Big[F(\Upsilon_1)+p_{\Upsilon_2}M_1(\Upsilon_1)\Big]+8q_{\Upsilon_1}p^2_{\Upsilon_2}.$$

Now, for the determination of $\sum g$,

$$\begin{aligned}\sum g &= \sum_{i=1}^{p_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \big(d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\big)^2 \\ &= \sum_{i=1}^{p_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big[\big(d_{\Upsilon_2}(u)+1\big).\big(d_{\Upsilon_2}(v)+1\big)\Big]^2 \\ &= \sum_{i=1}^{p_{\Upsilon_1}} \sum_{uv\in V(\Upsilon_2)} \Big(d_{\Upsilon_2}(u)d_{\Upsilon_2}(v)\Big)^2 + \Big(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v)\Big)^2 + 1\end{aligned}$$

$$+\, 2d_{\Upsilon_2}(u)d_{\Upsilon_2}(v)\Big(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v)\Big) + 2\Big(d_{\Upsilon_2}(u)d_{\Upsilon_2}(v)\Big) + 2\Big(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v)\Big)$$

$$\begin{aligned}\Rightarrow \sum g \quad &= \quad p_{\Upsilon_1}\Big[HM_1(\Upsilon_2)+HM_2(\Upsilon_2)+2\big(M_1(\Upsilon_2)+M_2(\Upsilon_2) \\ &+ ReZG_3(\Upsilon_2)\big) + q_{\Upsilon_2}\Big].\end{aligned}$$

Also, to determine $\sum h$,

$$\begin{aligned}\sum h &= \sum_{u\in V(R(\Upsilon_1))} \sum_{v\in V(\Upsilon_2)} \big(d_{R(\Upsilon_1)}(u).d_{\Upsilon_2}(v)\big)^2 \\ &= \sum_{u\in V(\Upsilon_1)} \sum_{v\in V(\Upsilon_2)} \Big[\big(2d_{\Upsilon_1}(u)+p_{\Upsilon_2}\big).\big(d_{\Upsilon_2}(v)+1\big)\Big]^2\end{aligned}$$

$$\begin{aligned}\Rightarrow \sum h \quad &= \quad 4M_1(\Upsilon_1)\big(p_{\Upsilon_2}+4q_{\Upsilon_2}\big) + p_{\Upsilon_2}M_1(\Upsilon_2)\big(p_{\Upsilon_1}p_{\Upsilon_2}+8q_{\Upsilon_1}\big) \\ &+ 4M_1(\Upsilon_1)M_2(\Upsilon_2) + p_{\Upsilon_2}\Big[p_{\Upsilon_2}\big(p_{\Upsilon_1}p_{\Upsilon_2}+4p_{\Upsilon_1}q_{\Upsilon_2}+8q_{\Upsilon_1}\big)+32q_{\Upsilon_1}q_{\Upsilon_2}\Big].\end{aligned}$$

Hence from all the computations,

$$HM_2(\Upsilon_1 \odot \Upsilon_2) = 4\Big[p^2_{\Upsilon_2} HM_1(\Upsilon_1) + 4HM_2(\Upsilon_1)\Big] + p_{\Upsilon_1}\Big[HM_1(\Upsilon_2) + HM_2(\Upsilon_2) + 2\big(ReZG_3(\Upsilon_2) + M_1(\Upsilon_2) + M_2(\Upsilon_2)\big)\Big] + 4p_{\Upsilon_2}\Big[4ReZG_3(\Upsilon_1) + (p^2_{\Upsilon_2} + 5)M_1(\Upsilon_1) + 2p_{\Upsilon_2}M_2(\Upsilon_1) + M_1(\Upsilon_2)\Big(2q_{\Upsilon_1} + \frac{p_{\Upsilon_1}p_{\Upsilon_2}}{4}\Big)\Big] + 4\Big[M_1(\Upsilon_1)\big(M_1(\Upsilon_2) + 4q_{\Upsilon_2}\big) + 4F(\Upsilon_1)\Big] + q_{\Upsilon_1}p_{\Upsilon_2}\Big[p_{\Upsilon_2}(p^2_{\Upsilon_2} + 16) + 32q_{\Upsilon_2}\Big] + p_{\Upsilon_1}\Big[q_{\Upsilon_2} + p^2_{\Upsilon_2}(p_{\Upsilon_2} + 4q_{\Upsilon_2})\Big].$$

□

**Corollary 2.** *([1, 3, 4, 2]) The Second Hyper Zagreb Index of R-Vertex Corona Product of two path graphs $P_n$,$P_m$ and cycle graphs $C_n$,$C_m$:*

$$HM_2(P_n \odot P_m) = m^4n - m^4 + 25m^3n - 24m^3 + 166m^2n - 264m^2 + 465mn - 808m + 53n - 624.$$

$$HM_2(C_n \odot C_m) = m^4n + 25m^3n + 176m^2n + 545mn + 384n.$$

### 2.2. R-Edge Corona Product

For $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$, the $R$-Edge Corona Product indicated as $R(\Upsilon_1) \ominus \Upsilon_2$ is acquired by taking a replica of distinct vertex graph $R(\Upsilon_1)$ and $q_{\Upsilon_1}$ replicas of $\Upsilon_2$ and linking a node of $I(\Upsilon_1)$ on $i_{th}$ location in $R(\Upsilon_1)$ to each node in the $i_{th}$ replica of $\Upsilon_2$.$|V(R(\Upsilon_1) \ominus \Upsilon_2)| = p_{\Upsilon_1} + q_{\Upsilon_1} + q_{\Upsilon_1}p_{\Upsilon_2}$ and $|E(R(\Upsilon_1) \ominus \Upsilon_2)| = 3q_{\Upsilon_1} + q_{\Upsilon_1}q_{\Upsilon_2} + q_{\Upsilon_1}p_{\Upsilon_2}$, [15].

Suppose $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ & $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$ are two graphs; the respective degrees of the nodes in graph $\Upsilon_1 \ominus \Upsilon_2$:

$$d_{\Upsilon_1 \ominus \Upsilon_2}(w) = \begin{cases} 2d_{\Upsilon_1}(w) & ; \text{if } w \in V(\Upsilon_1) \\ p_{\Upsilon_2} + 2 & ; \text{if } w \in I(\Upsilon_1) \\ d_{\Upsilon_2}(w) + 1 & ; \text{if } w \in V(\Upsilon_2) \end{cases}.$$

**Theorem 3.** *Let $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$ represent two arbitrary graphs. Then:*

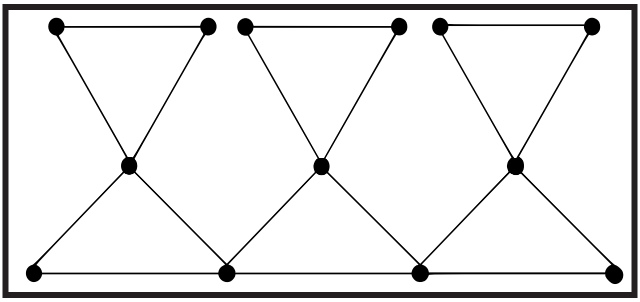

Figure 2: R-Edge Corona Product $R(P_4) \ominus P_2$

$$\begin{aligned} HM_2(\Upsilon_1 \ominus \Upsilon_2) &= 16HM_2(\Upsilon_1) + q_{\Upsilon_1} HM_2(\Upsilon_2) + 4(p_{\Upsilon_2}+2)^2 F(\Upsilon_1) \\ &\quad + q_{\Upsilon_1} F(\Upsilon_2) + q_{\Upsilon_1}\Big[2\big(M_1(\Upsilon_1) + 2M_2(\Upsilon_2) + ReZG_3(\Upsilon_2)\big) \\ &\quad + (p_{\Upsilon_2}+2)^2\big(M_1(\Upsilon_2) + 4q_{\Upsilon_2} + p_{\Upsilon_2}\big) + q_{\Upsilon_2}\Big]. \end{aligned}$$

*Proof.*

$$\begin{aligned} HM_2(\Upsilon_1 \ominus \Upsilon_2) &= \sum_{uv\in E(\Upsilon_1\ominus\Upsilon_2)} \Big(d_{\Upsilon_1\ominus\Upsilon_2}(u).d_{\Upsilon_1\ominus\Upsilon_2}(v)\Big)^2 \\ &= \sum_{uv\in E(R(\Upsilon_1))} \Big(d_{R(\Upsilon_1)}(u).d_{R(\Upsilon_1)}(v)\Big)^2 + \sum_{l=1}^{q_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big(d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\Big)^2 \\ &\quad + \sum_{u\in V(R(\Upsilon_1))} \sum_{v\in V(\Upsilon_2)} \Big(d_{R(\Upsilon_1)}(u).d_{\Upsilon_2}(v)\Big)^2 \\ &= \sum f + \sum g + \sum h. \end{aligned}$$

For the computation of $\sum f$,

$$\begin{aligned} \sum f &= \sum_{uv\in E(R(\Upsilon_1))} \Big(d_{R(\Upsilon_1)}(u).d_{R(\Upsilon_1)}(v)\Big)^2 \\ &= \sum_{\substack{uv\in E(R(\Upsilon_1)) \\ u,v\in V(\Upsilon_1)}} \Big(2d_{\Upsilon_1}(u).2d_{\Upsilon_1}(v)\Big)^2 + \sum_{\substack{uv\in E(R(\Upsilon_1)) \\ u\in V(\Upsilon_1), v\in I(\Upsilon_1)}} \Big(2d_{\Upsilon_1}(u).(p_{\Upsilon_2}+2)\Big)^2 \\ &= \sum f_1 + \sum f_2. \end{aligned}$$

For computing $\sum f_1$,

$$\begin{aligned}\sum f_1 &= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big(2d_{\Upsilon_1}(u).2d_{\Upsilon_1}(v)\Big)^2\\ &= 16 \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big(d_{\Upsilon_1}(u).d_{\Upsilon_1}(v)\Big)^2\\ \Rightarrow \sum f_1 &= 16HM_2(\Upsilon_1).\end{aligned}$$

Similarly for $\sum f_2$,

$$\begin{aligned}\sum f_2 &= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1),v\in I(\Upsilon_1)}} \Big(2d_{\Upsilon_1}(u).(p_{\Upsilon_2}+2)\Big)^2\\ &= 4(p_{\Upsilon_2}+2)^2 \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1),v\in I(\Upsilon_1)}} \Big(d_{\Upsilon_1}(u)\Big)^2\\ \Rightarrow \sum f_2 &= 4(p_{\Upsilon_2}+2)^2 F(\Upsilon_1).\end{aligned}$$

Now, for the determination of $\sum g$,

$$\begin{aligned}\sum g &= \sum_{l=1}^{q_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big(d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\Big)^2\\ &= q_{\Upsilon_1} \sum_{uv\in E(\Upsilon_2)} \Big((d_{\Upsilon_2}(u)+1).(d_{\Upsilon_2}(v)+1)\Big)^2\\ &= q_{\Upsilon_1} \sum_{uv\in E(\Upsilon_2)} \Bigg[\Big(d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\Big)^2 + 2d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\Big(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v)\Big)\\ &\quad + \Big(d_{\Upsilon_2}(u)^2 + d_{\Upsilon_2}(v)^2\Big) + 4d_{\Upsilon_2}(u).d_{\Upsilon_2}(v) + 2\Big(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v)\Big) + 1\Bigg]\\ \Rightarrow \sum g &= q_{\Upsilon_1}\Bigg[2\Big(M_1(\Upsilon_1) + 2M_2(\Upsilon_2) + ReZG_3(\Upsilon_2)\Big) + F(\Upsilon_2)\\ &\quad + HM_2(\Upsilon_2) + q_{\Upsilon_2}\Bigg].\end{aligned}$$

Also, to determine $\sum h$,

$$\sum h = \sum_{u\in V(R(\Upsilon_1))}\sum_{v\in V(\Upsilon_2)}\Big(d_{R(\Upsilon_1)}(u).d_{\Upsilon_2}(v)\Big)^2$$
$$= \sum_{\substack{u\in V(R(\Upsilon_1))\\ u\in I(\Upsilon_1)}}\sum_{v\in V(\Upsilon_2)}\Big((p_{\Upsilon_2}+2).(d_{\Upsilon_2}(v)+1)\Big)^2$$
$$\Rightarrow \sum h = q_{\Upsilon_1}(p_{\Upsilon_2}+2)^2\Big[M_1(\Upsilon_2)+4q_{\Upsilon_2}+p_{\Upsilon_2}\Big].$$

Hence from all the computations,

$$HM_2(\Upsilon_1\ominus\Upsilon_2) = 16HM_2(\Upsilon_1)+q_{\Upsilon_1}HM_2(\Upsilon_2)+4(p_{\Upsilon_2}+2)^2F(\Upsilon_1)$$
$$+q_{\Upsilon_1}F(\Upsilon_2)+q_{\Upsilon_1}\Big[2\big(M_1(\Upsilon_1)+2M_2(\Upsilon_2)+ReZG_3(\Upsilon_2)\big)$$
$$+(p_{\Upsilon_2}+2)^2\big(M_1(\Upsilon_2)+4q_{\Upsilon_2}+p_{\Upsilon_2}\big)+q_{\Upsilon_2}\Big].$$

□

**Corollary 4.** *The Second Hyper Zagreb Index of R-Edge Corona Product of two path graphs $P_n$,$P_m$ and cycle graphs $C_n$,$C_m$:*

$$HM_2(P_n\ominus P_m) = 9m^3n-9m^3+58m^2n-82m^2+197mn$$
$$+8n^2-293m+165n-653.$$
$$HM_2(C_n\ominus C_m) = 9m^3n+68m^2n+237mn+8n^2+384n.$$

### 2.3. R-Vertex Neighborhood Corona Product

For $\Upsilon_1(p_{\Upsilon_1},q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2},q_{\Upsilon_2})$, the $R$-Vertex Neighborhood Corona Product indicated as $R(\Upsilon_1)\boxdot\Upsilon_2$ is acquired by taking one replica of distinct vertex graph $R(\Upsilon_1)$ and $p_{\Upsilon_1}$ replicas of $\Upsilon_2$ and linking the adjacent or neighboring node of $\Upsilon_1$ in $R(\Upsilon_1)$ on the $i_{th}$ location in $R(\Upsilon_1)$ to each node in the $i_{th}$ replica of $\Upsilon_2$. $|V(R(\Upsilon_1)\boxdot\Upsilon_2)| = p_{\Upsilon_1}+q_{\Upsilon_1}+p_{\Upsilon_1}p_{\Upsilon_2}$ and $|E(R(\Upsilon_1)\boxdot\Upsilon_2)| = 3q_{\Upsilon_1}+p_{\Upsilon_1}q_{\Upsilon_2}+4q_{\Upsilon_1}q_{\Upsilon_2}$, [15].

Suppose $\Upsilon_1(p_{\Upsilon_1},q_{\Upsilon_1})$ & $\Upsilon_2(p_{\Upsilon_2},q_{\Upsilon_2})$ are two graphs; the respective degrees of the nodes in graph $\Upsilon_1\boxdot\Upsilon_2$:

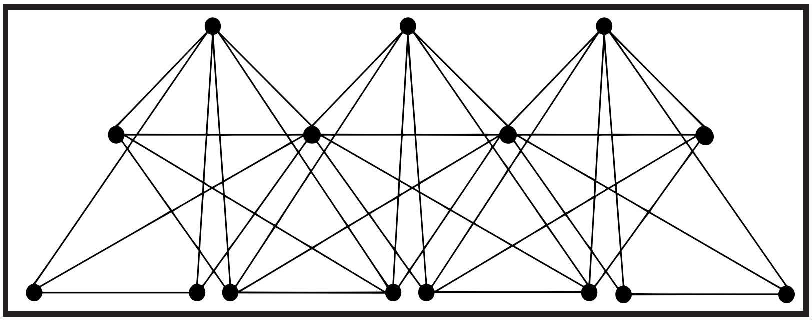

Figure 3: R-Vertex Neighborhood Corona Product $R(P_4) \boxdot P_2$

$$d_{\Upsilon_1 \boxdot \Upsilon_2}(w) = \begin{cases} (p_{\Upsilon_2}+2)d_{\Upsilon_1}(w) & ;\text{if } w \in V(\Upsilon_1) \\ 2(p_{\Upsilon_2}+1) & ;\text{if } w \in I(\Upsilon_1) \\ d_{\Upsilon_2}(w) + 2d_{\Upsilon_1}(v) & ;\text{if } w \in V(\Upsilon_2),\, v \in V(\Upsilon_1) \end{cases}.$$

**Theorem 5.** *Let $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$ represent two arbitrary graphs. Then:*

$$\begin{aligned} HM_2(\Upsilon_1 \boxdot \Upsilon_2) = &\Big[(p_{\Upsilon_2}+2)^2\Big((p_{\Upsilon_2}+2)^2 + 8p_{\Upsilon_2}\Big)\Big]HM_2(\Upsilon_1) \\ &+ p_{\Upsilon_1}HM_2(\Upsilon_2) + F(\Upsilon_1)\Big[4(p_{\Upsilon_2}+1)^2\Big((p_{\Upsilon_2}+2)^2 + 4p_{\Upsilon_2}\Big) \\ &+ M_1(\Upsilon_2)\Big((p_{\Upsilon_2}+2)^2 + 16\Big)\Big] + 4M_1(\Upsilon_1)\Big[4M_2(\Upsilon_2) + F(\Upsilon_2) \\ &+ 8q_{\Upsilon_2}(p_{\Upsilon_2}+1)^2\Big] + 8\Big[q_{\Upsilon_1}\Big((p_{\Upsilon_2}+1)^2M_1(\Upsilon_2) + ReZG_3(\Upsilon_2)\Big) \\ &+ q_{\Upsilon_2}\Big(2Y(\Upsilon_1) + (p_{\Upsilon_2}+2)^2ReZG_3(\Upsilon_1)\Big)\Big]. \end{aligned}$$

*Proof.*

$$HM_2(\Upsilon_1 \boxdot \Upsilon_2) = \sum_{uv\in E(\Upsilon_1\boxdot\Upsilon_2)} \Big(d_{\Upsilon_1\boxdot\Upsilon_2}(u).d_{\Upsilon_1\boxdot\Upsilon_2}(v)\Big)^2$$

$$= \sum_{uv\in E(R(\Upsilon_1)} \Big(d_{R(\Upsilon_1)}(u).d_{R(\Upsilon_1)}(v)\Big)^2 + p_{\Upsilon_1} \sum_{uv\in E(\Upsilon_2)} \Big(d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\Big)^2$$

$$+ \sum_{u\in V(R(\Upsilon_1))} \sum_{v\in V(\Upsilon_2)} \Big(d_{R(\Upsilon_1)}(u).d_{\Upsilon_2}(v)\Big)^2$$

$$= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big((p_{\Upsilon_2}+2)d_{\Upsilon_1}(u).(p_{\Upsilon_2}+2)d_{\Upsilon_1}(v)\Big)^2$$

$$+ \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1)\\ v\in I(\Upsilon_1)}} \Big((p_{\Upsilon_2}+2)d_{\Upsilon_1}(u).2(p_{\Upsilon_2}+1)\Big)^2$$

$$+ \sum_{i=1}^{p_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big((d_{\Upsilon_2}(u)+2d_{\Upsilon_1}(w_i)).(d_{\Upsilon_2}(v)+2d_{\Upsilon_1}(w_i))\Big)^2$$

$$+ \sum_{\substack{u\in V(\Upsilon_1)\\ w_i\in N_{\Upsilon_1}(u)\\ w_i\in V(\Upsilon_1)}} \sum_{v\in V(\Upsilon_2)} \Big((p_{\Upsilon_2}+2)d_{\Upsilon_1}(u).(d_{\Upsilon_2}(v)+2d_{\Upsilon_1}(w_i))\Big)^2$$

$$+ \sum_{\substack{u\in V(\Upsilon_1)\\ w_i\in N_{\Upsilon_1}(u)\\ w_i\in I(\Upsilon_1)}} \sum_{v\in V(\Upsilon_2)} \Big(2(p_{\Upsilon_2}+1).(d_{\Upsilon_2}(v)+2d_{\Upsilon_1}(u))\Big)^2$$

$$= \sum f_1 + \sum f_2 + \sum g + \sum h_1 + \sum h_2.$$

For the computation of $\sum f_1$,

$$\sum f_1 = \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big((p_{\Upsilon_2}+2)d_{\Upsilon_1}(u).(p_{\Upsilon_2}+2)d_{\Upsilon_1}(v)\Big)^2$$

$$= (p_{\Upsilon_2}+2)^4 \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big(d_{\Upsilon_1}(u).d_{\Upsilon_1}(v)\Big)^2$$

$$\Rightarrow \sum f_1 = (p_{\Upsilon_2}+2)^4 HM_2(\Upsilon_1).$$

Similarly for $\sum f_2$,

$$\sum f_2 = \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1)\\ v\in I(\Upsilon_1)}} \Big((p_{\Upsilon_2}+2)d_{\Upsilon_1}(u).2(p_{\Upsilon_2}+1)\Big)^2$$

$$\Rightarrow \sum f_2 \quad = \quad 4(p_{\Upsilon_2}+1)^2(p_{\Upsilon_2}+2)^2F(\Upsilon_1).$$

Now to determine $\sum g$,

$$\sum g = \sum_{i=1}^{p_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big((d_{\Upsilon_2}(u)+2d_{\Upsilon_1}(w_i)).(d_{\Upsilon_2}(v)+2d_{\Upsilon_1}(w_i))\Big)^2$$

$$= \sum_{i=1}^{p_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big(4d_{\Upsilon_1}(w_i)^2 + 4d_{\Upsilon_1}(w_i)d_{\Upsilon_2}(u) + d_{\Upsilon_2}(u)^2\Big)$$
$$\Big(4d_{\Upsilon_1}(w_i)^2 + 4d_{\Upsilon_1}(w_i)d_{\Upsilon_2}(v) + d_{\Upsilon_2}(v)^2\Big)$$
$$= \sum_{i=1}^{p_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} 4d_{\Upsilon_1}(w_i)^2\Big[d_{\Upsilon_2}(u)^2 + d_{\Upsilon_2}(v)^2\Big] + 16d_{\Upsilon_1}(w_i)^3\Big[d_{\Upsilon_2}(u) + d_{\Upsilon_2}(v)\Big]$$
$$+ 16d_{\Upsilon_1}(w_i)^4 + \Big[d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\Big]^2 + 4d_{\Upsilon_1}(w_i)d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\Big[d_{\Upsilon_2}(u) + d_{\Upsilon_2}(v)\Big]$$
$$+ 16d_{\Upsilon_1}(w_i)^2d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)$$

$$\Rightarrow \sum g = 4M_1(\Upsilon_1)F(\Upsilon_2) + 16F(\Upsilon_1)M_1(\Upsilon_2) + 16q_{\Upsilon_2}Y(\Upsilon_1)$$
$$+ p_{\Upsilon_1}HM_2(\Upsilon_2) + 8q_{\Upsilon_1}ReZG_3(\Upsilon_2) + 16M_1(\Upsilon_1)M_2(\Upsilon_2).$$

To determine $\sum h_1$,

$$\sum h_1 = \sum_{\substack{u\in V(\Upsilon_1)\\ w_i\in N_{\Upsilon_1}(u)\\ w_i\in V(\Upsilon_1)}} \sum_{v\in V(\Upsilon_2)} \Big((p_{\Upsilon_2}+2)d_{\Upsilon_1}(u).(d_{\Upsilon_2}(v)+2d_{\Upsilon_1}(w_i))\Big)^2$$

$$= \sum_{\substack{u\in V(\Upsilon_1)\\ w_i\in N_{\Upsilon_1}(u)\\ w_i\in V(\Upsilon_1)}} \sum_{v\in V(\Upsilon_2)} (p_{\Upsilon_2}+2)^2\Big[d_{\Upsilon_1}(w_i)^2 d_{\Upsilon_2}(v)^2 + 4d_{\Upsilon_1}(u)^2 d_{\Upsilon_1}(w_i)^2$$

$$+ 4d_{\Upsilon_1}(u)d_{\Upsilon_2}(v)d_{\Upsilon_1}(w_i)^2\Big]$$

$$\Rightarrow \sum h_1 = (p_{\Upsilon_2}+2)^2\Big[F(\Upsilon_1)M_1(\Upsilon_2) + 8p_{\Upsilon_2}HM_2(\Upsilon_1) + 8q_{\Upsilon_2}ReZG_3(\Upsilon_1)\Big].$$

Similarly for the determination of $\sum h_2$,

$$\sum h_2 = \sum_{\substack{u\in V(\Upsilon_1)\\ w_i\in N_{\Upsilon_1}(u)\\ w_i\in I(\Upsilon_1)}} \sum_{v\in V(\Upsilon_2)} \Big(2(p_{\Upsilon_2}+1).(d_{\Upsilon_2}(v)+2d_{\Upsilon_1}(u))\Big)^2$$

$$= \sum_{u\in V(\Upsilon_1)} \sum_{v\in V(\Upsilon_2)} 4(p_{\Upsilon_2}+1)^2\Big[d_{\Upsilon_2}(v)+2d_{\Upsilon_1}(u)\Big]^2 d_{\Upsilon_1}(u)$$

$$\Rightarrow \sum h_2 = 4(p_{\Upsilon_2}+1)^2\Big[2q_{\Upsilon_1}M_1(\Upsilon_2) + 4p_{\Upsilon_2}F(\Upsilon_1) + 8q_{\Upsilon_2}M_1(\Upsilon_1)\Big].$$

Hence from all the computations,

$$\begin{aligned}HM_2(\Upsilon_1 \boxdot \Upsilon_2) = {} & \Big[(p_{\Upsilon_2}+2)^2\Big((p_{\Upsilon_2}+2)^2 + 8p_{\Upsilon_2}\Big)\Big]HM_2(\Upsilon_1)\\ & + p_{\Upsilon_1}HM_2(\Upsilon_2) + F(\Upsilon_1)\Big[4(p_{\Upsilon_2}+1)^2\Big((p_{\Upsilon_2}+2)^2 + 4p_{\Upsilon_2}\Big)\\ & + M_1(\Upsilon_2)\Big((p_{\Upsilon_2}+2)^2+16\Big)\Big] + 4M_1(\Upsilon_1)\Big[4M_2(\Upsilon_2) + F(\Upsilon_2)\\ & + 8q_{\Upsilon_2}(p_{\Upsilon_2}+1)^2\Big] + 8\Big[q_{\Upsilon_1}\Big((p_{\Upsilon_2}+1)^2M_1(\Upsilon_2) + ReZG_3(\Upsilon_2)\Big)\\ & + q_{\Upsilon_2}\Big(2Y(\Upsilon_1) + (p_{\Upsilon_2}+2)^2ReZG_3(\Upsilon_1)\Big)\Big].\end{aligned}$$

This concludes the result. □

**Corollary 6.** *The Second Hyper Zagreb Index of R-Vertex Neighborhood Corona Product of two path graphs $P_n$,$P_m$ and cycle graphs $C_n$,$C_m$:*

$$HM_2(P_n \boxdot P_m) = 48m^4n - 96m^4 + 896m^3n - 1768m^3 + 2176m^2n - 4628m^2 + 2576mn - 5168m - 2584n + 4080.$$
$$HM_2(C_n \boxdot C_m) = 48m^4n + 896m^3n + 2528m^2n + 3632mn + 384n.$$

### 2.4. R-Edge Neighborhood Corona Product

For $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$, the $R$-Edge Neighborhood Corona Product indicated as $R(\Upsilon_1) \boxminus \Upsilon_2$ is acquired by taking replica of distinct vertex graph $R(\Upsilon_1)$ and $q_{\Upsilon_1}$ replicas of $\Upsilon_2$ and linking the adjacencies or neighboring vertices of $I(\Upsilon_1)$ in $R(\Upsilon_1)$ on the $i_{th}$ location in $R(\Upsilon_1)$ to each node in $i_{th}$ replica of $\Upsilon_2$. $|V(R(\Upsilon_1) \boxminus \Upsilon_2)| = p_{\Upsilon_1} + q_{\Upsilon_1} + q_{\Upsilon_1}q_{\Upsilon_2}$ and $|E(R(\Upsilon_1) \boxminus \Upsilon_2)| = 3q_{\Upsilon_1} + q_{\Upsilon_1}q_{\Upsilon_2} + 2q_{\Upsilon_1}p_{\Upsilon_2}$, [15].

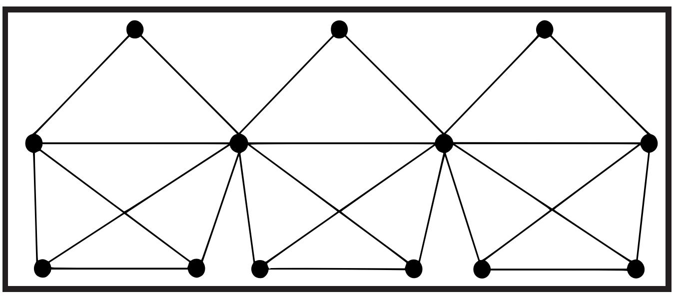

Figure 4: R-Edge Neighborhood Corona Product $R(P_4) \boxminus P_2$

Suppose $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ & $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$ are two graphs; the respective degrees of the nodes in graph $\Upsilon_1 \boxminus \Upsilon_2$:

$$d_{\Upsilon_1 \boxminus \Upsilon_2}(w) = \begin{cases} (p_{\Upsilon_2} + 2)d_{\Upsilon_1}(w) & ;\text{if } w \in V(\Upsilon_1) \\ 2 & ;\text{if } w \in I(\Upsilon_1) \\ d_{\Upsilon_2}(w) + 2 & ;\text{if } w \in V(\Upsilon_2) \end{cases}.$$

**Theorem 7.** *Let $\Upsilon_1(p_{\Upsilon_1}, q_{\Upsilon_1})$ and $\Upsilon_2(p_{\Upsilon_2}, q_{\Upsilon_2})$ represent two arbitrary graphs. Then;*

$$HM_2(\Upsilon_1 \boxminus \Upsilon_2) = (p_{\Upsilon_2}+2)^4 HM_2(\Upsilon_1) + q_{\Upsilon_1} HM_2(\Upsilon_2) + (p_{\Upsilon_2}+2)^2 F(\Upsilon_1)$$
$$\Big[M_1(\Upsilon_2) + 4(2q_{\Upsilon_2} + p_{\Upsilon_2} + 1)\Big] + 4q_{\Upsilon_1}\Big[F(\Upsilon_2) + 4\big(M_1(\Upsilon_2) + M_2(\Upsilon_2)$$
$$+ q_{\Upsilon_2}\big) + ReZG_3(\Upsilon_2)\Big].$$

*Proof.*

$$\begin{aligned}
HM_2(\Upsilon_1 \boxminus \Upsilon_2) &= \sum_{uv\in E(\Upsilon_1\boxminus\Upsilon_2)} \Big(d_{\Upsilon_1\boxminus\Upsilon_2}(u).d_{\Upsilon_1\boxminus\Upsilon_2}(v)\Big)^2 \\
&= \sum_{uv\in E(R(\Upsilon_1))} \Big(d_{R(\Upsilon_1)}(u).d_{R(\Upsilon_1)}(v)\Big)^2 + \sum_{i=1}^{q_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big(d_{\Upsilon_2}(u).d_{\Upsilon_2}(v)\Big)^2 \\
&+ \sum_{u\in V(R(\Upsilon_1))} \sum_{v\in V(\Upsilon_2)} \Big(d_{R(\Upsilon_1)}(u).d_{\Upsilon_2}(v)\Big)^2 \\
&= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big(d_{\Upsilon_1}(u)(p_{\Upsilon_2}+2).d_{\Upsilon_1}(v)(p_{\Upsilon_2}+2)\Big)^2 \\
&+ \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1)\\ v\in I(\Upsilon_1)}} \Big(d_{\Upsilon_1}(u)(p_{\Upsilon_2}+2).2\Big)^2 \\
&+ \sum_{i=1}^{q_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big((d_{\Upsilon_2}(u)+2).(d_{\Upsilon_2}(v)+2)\Big)^2 \\
&+ \sum_{\substack{u\in I(\Upsilon_1)\\ w_i\in N_{\Upsilon_1}(u)\\ w_i\in V(\Upsilon_1)}} \sum_{v\in V(\Upsilon_2)} \Big(\big((p_{\Upsilon_2}+2)d_{\Upsilon_1}(w_i)\big).\big(d_{\Upsilon_2}(v)+2\big)\Big)^2 \\
&= \sum f_1 + \sum f_2 + \sum g + \sum h.
\end{aligned}$$

Now for the computation of $\sum f_1$,

$$\begin{aligned}\sum f_1 &= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} \Big(d_{\Upsilon_1}(u)(p_{\Upsilon_2}+2).d_{\Upsilon_1}(v)(p_{\Upsilon_2}+2)\Big)^2\\ &= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u,v\in V(\Upsilon_1)}} (p_{\Upsilon_2}+2)^4\Big(d_{\Upsilon_1}(u).d_{\Upsilon_1}(v)\Big)^2\\ \Rightarrow \sum f_1 &= (p_{\Upsilon_2}+2)^4 HM_2(\Upsilon_1).\end{aligned}$$

Similarly for $\sum f_2$,

$$\begin{aligned}\sum f_2 &= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1)\\ v\in I(\Upsilon_1)}} \Big(d_{\Upsilon_1}(u)(p_{\Upsilon_2}+2).2\Big)^2\\ &= \sum_{\substack{uv\in E(R(\Upsilon_1))\\ u\in V(\Upsilon_1)\\ v\in I(\Upsilon_1)}} 4(p_{\Upsilon_2}+2)^2\Big(d_{\Upsilon_1}(u)\Big)^2\\ \Rightarrow \sum f_2 &= 4(p_{\Upsilon_2}+2)^2 F(\Upsilon_1).\end{aligned}$$

To determine $\sum g$,

$$\begin{aligned}\sum g &= \sum_{i=1}^{q_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big((d_{\Upsilon_2}(u)+2).(d_{\Upsilon_2}(v)+2)\Big)^2\\ &= \sum_{i=1}^{q_{\Upsilon_1}} \sum_{uv\in E(\Upsilon_2)} \Big[\big(d_{\Upsilon_2}(u)d_{\Upsilon_2}(v)\big)^2 + 4\big(d_{\Upsilon_2}(u)^2 + d_{\Upsilon_2}(v)^2\big)\\ &\quad + 4d_{\Upsilon_2}(u)d_{\Upsilon_2}(v)\big(d_{\Upsilon_2}(u)+d_{\Upsilon_2}(v)\big) + 16\big(d_{\Upsilon_2}(u)\\ &\quad + d_{\Upsilon_2}(v) + d_{\Upsilon_2}(u)d_{\Upsilon_2}(v)\big) + 16\Big]\end{aligned}$$

$$\begin{aligned}\Rightarrow \sum g = q_{\Upsilon_1}\Big[&HM_2(\Upsilon_2) + 4F(\Upsilon_2) + 4ReZG_3(\Upsilon_2) + 16\big(M_1(\Upsilon_2)\\ &+ M_2(\Upsilon_2)\big) + 16q_{\Upsilon_2}\Big].\end{aligned}$$

Also, to determine $\sum h$,

$$\sum h = \sum_{\substack{u\in I(\Upsilon_1)\\ w_i\in N_{\Upsilon_1}(u)\\ w_i\in V(\Upsilon_1)}} \sum_{v\in V(\Upsilon_2)} \Big(\big((p_{\Upsilon_2}+2)d_{\Upsilon_1}(w_i)\big).\big(d_{\Upsilon_2}(v)+2\big)\Big)^2$$

$$= \sum_{u\in V(\Upsilon_1)} \sum_{v\in V(\Upsilon_2)} (p_{\Upsilon_2}+2)^2\Big[d_{\Upsilon_1}(u).\big(d_{\Upsilon_2}(v)+2\big)\Big]^2 d_{\Upsilon_1}(u)$$

$$= (p_{\Upsilon_2}+2)^2 \sum_{u\in V(\Upsilon_1)} \sum_{v\in V(\Upsilon_2)} d_{\Upsilon_1}(u)^3\Big[d_{\Upsilon_2}(v)^2+4d_{\Upsilon_2}(v)+4\Big]$$

$$\Rightarrow \sum h \quad = \quad (p_{\Upsilon_2}+2)^2F(\Upsilon_1)\Big[M_1(\Upsilon_2)+4\big(p_{\Upsilon_2}+2q_{\Upsilon_2}\big)\Big].$$

Hence from all the computations,

$$HM_2(\Upsilon_1 \boxminus \Upsilon_2) = (p_{\Upsilon_2}+2)^4HM_2(\Upsilon_1) + q_{\Upsilon_1}HM_2(\Upsilon_2) + (p_{\Upsilon_2}+2)^2F(\Upsilon_1)$$
$$\Big[M_1(\Upsilon_2)+4(2q_{\Upsilon_2}+p_{\Upsilon_2}+1)\Big] + 4q_{\Upsilon_1}\Big[F(\Upsilon_2)+4\big(M_1(\Upsilon_2)+M_2(\Upsilon_2)$$
$$+ q_{\Upsilon_2}\big) + ReZG_3(\Upsilon_2)\Big].$$

□

**Corollary 8.** *The Second Hyper Zagreb Index of R-Edge Neighborhood Corona Product of two path graphs $P_n$,$P_m$ and cycle graphs $C_n$,$C_m$:*

$$HM_2(P_n \boxminus P_m) = 16m^4n - 40m^4 + 256m^3n - 544m^3 + 816m^2n$$
$$- 1716m^2 + 960mn - 1872m - 544n + 400.$$
$$HM_2(C_n \boxminus C_m) = 16m^4n + 256m^3n + 928m^2n + 1408mn + 384n.$$

## 3. Conclusion

In this paper, some specific precise formulations for the Second Hyper-Zagreb Index of certain varieties of corona product related to the semi-total point graph particularly the R-Vertex, R-Edge, R-Vertex Neighborhood and R-Edge Neighborhood Corona Product variants of two graphs are investigated and explored. Further, our outcomes have been utilised to determine the Second Hyper Zagreb

Index of some standard appealing molecular graphs. The results have beneficial perspectives towards extensive analysis of degree and distance dependent descriptors and series of graph operational procedures.

## References

[1] G. Rajasekharaiah, U. Murthy, Hyper-Zagreb indices of graphs and its applications, *Journal of Algebra Combinatorics Discrete Structures And Applications*, **8**, No 1 (2021), 9-22.

[2] V. Kulli, The Gourava indices and coindices of graphs, *Annals of Pure and Applied Mathematics*, **14**, No 1 (2017), 33-38.

[3] I. Cangul, A. Yurttas, M. Togan, A. Cevik, Some formulae for the zagreb indices of graphs, In: *AIP Conference Proceedings*, **1479**, No 1 (2012), 365-367.

[4] S. Akhter, M. Imran, Computing the forgotten topological index of four operations on graphs, *AKCE International Journal of Graphs and Combinatorics*, **14**, No 1 (2017), 70-79.

[5] A. Alameri, N. Al-Naggar, M. Al-Rumaima, M. Alsharafi, $Y$-index of some graph operations, *International Journal of Applied Engineering Research*, **15**, No 2 (2020), 173-179.

[6] H. Wiener, Structural Determination of Paraffin Boiling Points, *Journal Of The American Chemical Society*, **69**, No 1 (1947), 17-20.

[7] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, Total $\varphi$-electron energy of alternant hydrocarbons, *Chemical Physics Letters*, **17**, No 4 (1972), 535-538.

[8] I. Gutman, E. Milovanović, I. Milovanović, Beyond the Zagreb indices, *AKCE International Journal of Graphs and Combinatorics*, **17**, No 1 (2020), 74-85.

[9] M. Randić, On characterization of molecular branching, *J. Am. Chem. Soc.*, **97**, No 23 (1975), 6609-6615.

[10] I. Gutman, Degree-based topological indices, *Croatica Chemica Acta*, **86**, No 4 (2013), 351-361.

[11] B. Bollobás, P. Erdös, Graphs of extremal weights, *Ars Combinatoria*, **50** (1998), 225-233.

[12] G. Shirdel, H. Rezapour, A. Sayadi, The hyper-Zagreb index of graph operations, *Iranian Journal of Mathematical Chemistry*, **4**, No 2 (2013), 213-220.

[13] P. Ranjini, V. Lokesha, A. Usha, Relation between phenylene and hexagonal squeeze using harmonic index, *Int. J. Graph Theory*, **1**, No 4 (2013), 116-121.

[14] A. Usha, P. Ranjini, V. Lokesha, Zagreb co-indices, augmented Zagreb index, redefined Zagreb indices and their polynomials for phenylene and hexagonal squeeze, In: *Proc. of Internat. Congress in Honour of Dr. Ravi P. Agarwal*, Uludag University, Bursa, Turkey (2014).

[15] J. Lan, B. Zhou, Spectra of graph operations based on $R$-graph, *Linear and Multilinear Algebra*, **63**, No 7 (2015), 1401-1422.

[16] S. Akhter, M. Imran, Z. Raza, Bounds for the general sum-connectivity index of composite graphs, *Journal of Inequalities and Applications*, **2017**, No 1 (2017), 1-12.

[17] B. Zhou, N. Trinajstić, On a novel connectivity index, *Journal of Mathematical Chemistry*, **46** (2009), 1252-1270.

[18] B. Zhou, N. Trinajstić, On general sum-connectivity index, *Journal of Mathematical Chemistry*, **47** (2010), 210-218.

[19] T. Mansour, C. Song, The $a$ and $(a, b)$-analogs of Zagreb indices and coindices of graphs, *International J. of Combinatorics*, **2012** (2012).

[20] B. Furtula, I. Gutman, A forgotten topological index, *J. of Mathematical Chemistry*, **53**, No 4 (2015), 1184-1190.

[21] M. Farahani, M. Kanna, R. Kumar, On the Hyper-Zagreb indices of nano-structures, *Asian Academic Research J. of Multidisciplinary*, **3**, No 1 (2016), 115-123.

[22] A. Alameri, Second hyper-Zagreb index of titania nanotubes and their applications, *IEEE Access*, **9** (2021), 9567-9571.

[23] W. Gao, M. Farahani, M. Siddiqui, M. Jamil, On the first and second Zagreb and first and second hyper-Zagreb indices of carbon nanocones $CNC_k[n]$, *J. of Computational and Theoretical Nanoscience*, **13**, No 10 (2016), 7475-7482.

[24] N. De, $F$-index and coindex of some derived graphs, *Bull. of the International Mathematical Virtual Institute*, **8** (2018), 81-88.

[25] D. Maji, G. Ghorai, The first entire Zagreb index of various corona products and their bounds, *J. Math. Comput. Sci.*, **11**, No 5 (2021), 6018-6044.

[26] N. De, S. Nayeem, A. Pal, $F$-index of some graph operations, *Discrete Mathematics, Algorithms and Applications*, **8**, No 2 (2016), 1650025.