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INTEGRATION OF THE NONLINEAR SCHRÖDINGER EQUATION WITH A SELF-CONSISTENT SOURCE AND NONZERO BOUNDARY CONDITIONS

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Abstract: This paper is devoted to the study of the defocusing nonlinear Schrödinger equation with a self-consistent source and nonzero boundary conditions by the method of the inverse scattering problem. In cases where the source consists of a combination of eigenfunctions of the corresponding spectral problem for the Zakharov-Shabat system, the complete integrability of the nonlinear Schrödinger equation is investigated. Namely, the evolutions of the scattering data of the self-adjoint Zakharov-Shabat system, whose potential is a solution of the defocusing nonlinear Schrödinger equation with a self-consistent source, are obtained.

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1. Introduction

Nonlinear Schrödinger (NLS) equation

$$iu_t - 2\chi |u|^2 u + u_{xx} = 0 \tag{1}$$

with various boundary conditions models a wide class of nonlinear phenomena in physics. In the work [16], V. Zakharov and A. Shabat showed that NLS

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equation can be applied in the study of optical self-focusing and splitting of optical beams. The sign of the coupling constant corresponds to the attraction ($\chi < 0$) and repulsion ($\chi > 0$) of particles. In the case of attraction, the problem of a finite number of particles and their bound states has a physical meaning. In the classical limit, this is modeled by rapidly decreasing boundary conditions. In the case of repulsion, the problem corresponding to a gas of particles with a finite density is of interest.

It is well known that inverse scattering method for integration of the NLS equation is based on the so-called Zakharov-Shabat and Ablowitz-Kaup-Newell-Segur scattering problem (see [1, 16, 8]).

In 1971, V. Zakharov and A. Shabat [16] showed that the NLS equation can be solved by means of the inverse scattering transform (IST) technique. The IST as a method to solve the initial-value problem for the NLS equation has been extensively studied in many literatures, both in the focusing ($\chi = -1$) and in the defocusing ($\chi = 1$) dispersion regimes. The IST for the defocusing NLS equation with nonzero boundary conditions was first studied in 1973 by V. Zakharov and A. Shabat [17] and a detailed study can be found in the monograph [8].

In the work [4] a rigorous theory of the IST for the defocusing NLS equation with nonvanishing boundary values $u_\pm = u_0 e^{i\theta_\pm}$ as $x \to \pm\infty$ is presented. The IST theory for the defocusing NLS equation with nonzero boundary conditions was studied by B. Gino, F. Emily and B. Prinari [2] and the focusing case has been studied by G. Biondini, G. Kovačić [3], F. Demontis, B. Prinari, C. van der Mee, F. Vitale [5].

V.K. Melnikov [10, 11] showed that the NLS equation remains its integrability by the inverse scattering method, if a source is added to them in the form of a combination of eigenfunctions of the corresponding spectral problem. Namely, the term “self-consistent source” was introduced in the works of V.K. Melnikov.

The NLS equation with the self-consistent sources in various classes of functions were considered by A.B. Khasanov, I.D. Rakhimov [13], A.B. Yakhshimuratov [15].

In the matrix case, the inverse scattering theory for the matrix Zakharov-Shabat system was investigated by P. Barbara, F. Demontis and C. Van der Mee [12, 6, 7] and applied for the integration of the matrix NLS equation. In [14], the matrix NLS equation with the self-consistent source was considered in the class of rapidly decreasing matrix functions.

In the case when the self-consistent source are corresponding to one Zakharov-Shabat spectral problem, NLS was studied by the present author [9]. In the present paper we study the case, when self-consistent source corresponds to general form.

2. Formulation of the problem

We consider the following system of equations

$$iu_t - 2u\,|u|^2 + u_{xx} = -2i\sum_{n=1}^{N}(f_{1,n}^* g_{2,n}^* + f_{2,n}g_{1,n}), \tag{2}$$

$$\begin{aligned} \frac{\partial f_{1,n}}{\partial x} &= -i\xi_n f_{1,n} + u^* f_{2,n}, \\ \frac{\partial f_{2,n}}{\partial x} &= uf_{1,n} + i\xi_n f_{2,n}, \;\; n = 1, 2, ..., N, \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{\partial g_{1,n}}{\partial x} &= i\xi_n g_{1,n} + ug_{2,n}, \\ \frac{\partial g_{2,n}}{\partial x} &= u^* g_{1,n} - i\xi_n g_{2,n}, \;\; n = 1, 2, ..., N \end{aligned} \tag{4}$$

under the initial condition

$$u(x, 0) = u_0(x), \quad x \in R. \tag{5}$$

Here the initial function $u_0(x)$, $x \in R$ satisfies the following properties:

1)

$$\begin{aligned} &\int_{-\infty}^{0} (1 - x) \left|u_0(x) - \rho e^{i\alpha_-}\right| dx \\ &+ \int_{0}^{\infty} (1 + x) \left|u_0(x) - \rho e^{i\alpha_+}\right| dx < \infty, \end{aligned} \tag{6}$$

where $\rho > 0$ and $0 \le \alpha_\pm < 2\pi$ are arbitrary constants.

2) The system of equations (3) with coefficient $u_0(x)$ possesses exactly N eigenvalues $\xi_1(0)$, $\xi_2(0)$,..., $\xi_N(0)$.

We assume that the solution $u(x, t)$ of the equation (2) is sufficiently smooth and tends to its limits sufficiently rapidly as $x \to \pm\infty$, i.e., for all $t \ge 0$ satisfies the condition

$$\begin{aligned} &\int_{-\infty}^{0} (1 - x) \left|u(x, t) - \rho e^{i\alpha_- - 2i\rho^2 t}\right| dx \\ &+ \int_{0}^{\infty} (1 + x) \left|u(x, t) - \rho e^{i\alpha_+ - 2i\rho^2 t}\right| dx \end{aligned}$$

$$+\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left|\frac{\partial^k u(x,t)}{\partial x^k}\right|dx<\infty. \tag{7}$$

It follows from this condition that the left-hand side of equation (2) for all $t \geq 0$ tends to zero as $x \to \pm\infty$. Taking this into account, the solutions $F_n(x,t) = (f_{1,n}, f_{2,n})^T$ and $G_n(x,t) = (g_{1,n}, g_{2,n})^T$ of equations (3) and (4), respectively, are to be chosen so that the expression in the right-hand side of equation (2) for all $t \geq 0$ should tend to zero rapidly enough as $x \to \pm\infty$. This can be done in the following two ways:

(A) Let the functions $F_n(x,t)$ and $G_n(x,t)$ be the eigenfunctions of equations (3) and (4) respectively, corresponding to the eigenvalues $\xi_n(t)$. In this case, the functions $F_n(x,t)$ and $G_n(x,t)$ belong to the $L^2(R)$ for all $t \geq 0$. We assume that

$$\int_{-\infty}^{\infty} G_n^T(s,t)F_n(s,t)ds = A_n(t), \quad t \geq 0, \quad n = 1,2,...,N, \tag{8}$$

where $A_n(t)$ are given and the continuous functions of t.

(B) Let the function $F_n(x,t)$ be the eigenfunctions of equations (3) corresponding to the eigenvalues $\xi_n(t)$ and let the function $G_n(x,t)$ be unbounded solution of the equation (4), satisfying equalities

$$f_{1,n}g_{1,n} - f_{2,n}g_{2,n} = B_n(t), \quad t \geq 0, \quad n = 1,2,...,N, \tag{9}$$

where the functions $B_n(t)$ are given continuous real valued functions of t.

Our main goal is to obtain representations for solutions $u(x,t)$, $F_n(x,t)$, $G_n(x,t)$, $n = 1,2,...,N$ of problem (2)-(7), within the framework of the inverse scattering method.

3. Preliminaries

Let a function $u(x,t)$ belong to the class of functions (7). In this section, we give well known [8], necessary information concerning the theory of direct and inverse scattering problems for the operator

$$L(t) = i\begin{pmatrix} \frac{\partial}{\partial x} & -u^*(x,t) \\ u(x,t) & -\frac{\partial}{\partial x} \end{pmatrix}, \quad t \geq 0.$$

Consider the equation

$$(L(t) - \xi I)f = 0 \tag{10}$$

with respect to an unknown 2×2 square matrix function $f(x, \xi, t)$. Here ξ is a spectral parameter.

We introduce a new spectral parameter p as follows $p = \sqrt{\xi^2 - \rho^2}$. The variable p is then thought of as belonging to a Riemann surface Γ consisting of a sheet Γ_+ and a sheet Γ_- which both coincide with the complex plane cut along the semi lines $\Sigma = (-\infty, -\rho] \cup [\rho, \infty)$ with its edges glued in such a way that $p(\xi)$ is continuous through the cut. The variable p is thought of as belonging to the complex plane consisting of the upper half complex plane Γ_+ and the lower half complex plane Γ_- glued together along the whole real line. For all $\xi \in \Sigma$, the branch of the square root is fixed by the condition $sign\, p(\xi) = sign\, \xi$.

For $\xi \in \Sigma$, we define matrix Jost solutions $f^-(x, \xi, t)$ and $f^+(x, \xi, t)$ from the right and the left, respectively as those square matrix solutions to (10) satisfying asymptotics

$$f^{\pm} \sim E^{\pm}(x, \xi, t) \quad \text{as} \quad x \to \pm\infty, \tag{11}$$

where

$$E^{\pm}(x, \xi, t) = \begin{pmatrix} 1 & -\frac{i(\xi - p)}{\rho} e^{-i\alpha_{\pm} + 2i\rho^2 t} \\ \frac{i(\xi - p)}{\rho} e^{i\alpha_{\pm} - 2i\rho^2 t} & 1 \end{pmatrix} e^{-ip\sigma_3 x}.$$

Here and everywhere below, we will use the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If a function $u(x, t)$ belongs to the class of functions (7), then such a solution to equations (10) exists and is unique.

It can be shown that

$$\frac{d}{dx} \det f^{\pm}(x, \xi, t) = 0. \tag{12}$$

From (12) and (11) it follows that

$$\det f^{\pm}(x, \xi, t) = \frac{2p(\xi - p)}{\rho^2}. \tag{13}$$

The system (10) is invariant with respect to the involution

$$\bar{f}^{\pm}(x, \xi, t) = \sigma_1 f^{\pm}(x, \xi, t) \sigma_1, \quad \xi \in \Sigma. \tag{14}$$

We now call the columns of

$$f^{+}(x, \xi, t) = \left(\bar{\psi}(x, \xi, t) \;\; \psi(x, \xi, t) \right), \tag{15}$$

$$f^-(x,\xi,t) = \left(\varphi(x,\xi,t) \;\; \bar{\varphi}(x,\xi,t)\right), \tag{16}$$

the Jost solutions from the right and the left, respectively. For the Jost solutions we get the following asymptotic estimates

$$\psi(x,\xi,t) \sim \begin{pmatrix} -\frac{i(\xi-p)}{\rho} e^{-i\alpha_+ + 2i\rho^2 t} \\ 1 \end{pmatrix} e^{ipx}, \quad x \to \infty, \tag{17}$$

$$\bar{\psi}(x,\xi,t) \sim \begin{pmatrix} 1 \\ \frac{i(\xi-p)}{\rho} e^{i\alpha_+ - 2i\rho^2 t} \end{pmatrix} e^{-ipx}, \quad x \to \infty,$$

$$\varphi(x,\xi,t) \sim \begin{pmatrix} 1 \\ \frac{i(\xi-p)}{\rho} e^{i\alpha_- - 2i\rho^2 t} \end{pmatrix} e^{-ipx}, \quad x \to -\infty, \tag{18}$$

$$\bar{\varphi}(x,\xi,t) \sim \begin{pmatrix} -\frac{i(\xi-p)}{\rho} e^{-i\alpha_- + 2i\rho^2 t} \\ 1 \end{pmatrix} e^{ipx}, \quad x \to -\infty.$$

Since $f^-(x,\xi,t)$ and $f^+(x,\xi,t)$ are square matrix solutions of the homogeneous first order equation (10), we necessarily have for $\xi \in \Sigma$

$$f^-(x,\xi,t) = f^+(x,\xi,t) S(\xi,t), \tag{19}$$

where $S(\xi,t)$ is the transition coefficient matrix.

From the involution property (14) for $\xi \in \Sigma$, it follows that

$$\bar{S}(\xi,t) = \sigma_1 S(\xi,t) \sigma_1.$$

Hence, we have

$$S(\xi,t) = \begin{pmatrix} a(\xi,t) & \bar{b}(\xi,t) \\ b(\xi,t) & \bar{a}(\xi,t) \end{pmatrix}. \tag{20}$$

Coefficients $a(\xi,t)$ and $b(\xi,t)$ are called scattering coefficients. From relations (13) and (20) we obtain

$$a(\xi,t)\bar{a}(\xi,t) - b(\xi,t)\bar{b}(\xi,t) = 1.$$

By using (19) we can represent the scattering coefficients as

$$a(\xi,t) = \frac{\rho^2}{2p(\xi-p)} \det(\varphi(x,\xi,t),\, \psi(x,\xi,t)) \tag{21}$$

and

$$b(\xi,t) = \frac{\rho^2}{2p(\xi-p)} \det(\bar{\psi}(x,\xi,t), \varphi(x,\xi,t)).$$

If a function $u(x,t)$ belongs to the class of functions (7), then for each $x \in R$ the Jost solutions $\psi(x,\xi,t)e^{-ipx}$ and $\varphi(x,\xi,t)e^{ipx}$ are analytic for $\xi \in \Gamma_+$ excluding branch points $\xi = \pm\rho$, there are asymptotes for $|\xi| \to \infty$

$$\varphi(x,\xi,t)e^{ipx} = \begin{pmatrix} 1 \\ \frac{i(\xi-p)}{\rho}e^{i\alpha_- - 2ip^2 t} \end{pmatrix} + O\left(\frac{|1+\xi-p|}{|\xi|}\right), \tag{22}$$

$$\psi(x,\xi,t)e^{-ipx} = \begin{pmatrix} -\frac{i(\xi-p)}{\rho}e^{-i\alpha_+ + 2ip^2 t} \\ 1 \end{pmatrix} + O\left(\frac{|1+\xi-p|}{|\xi|}\right). \tag{23}$$

It follows from the analyticity properties of the Jost solutions and equality (21) that the function $a(\xi,t)$ can be analytically continued to the sheet Γ_+ excluding branch points $\xi = \pm\rho$.

From (22) and (23) we obtain that for $|\xi| \to \infty$, the function $a(\xi,t)$ has the asymptotics

$$a(\xi,t) = 1 + O\left(\frac{1}{|\xi|}\right) \quad \text{as } Im\,\xi > 0 \tag{24}$$

and

$$a(\xi,t) = e^{-i\theta} + O\left(\frac{1}{|\xi|}\right) \quad \text{as } Im\,\xi < 0, \tag{25}$$

where we recall $\theta = \alpha_+ - \alpha_-$.

Similarly, the function $\bar{a}(\xi,t)$ can be analytically continued to the sheet Γ_-, excluding branch points $\xi = \pm\rho$.

It follows from the analyticity with respect to the function $a(\xi,t)$ on Γ_+ and from the asymptotics (24), (25) that the function $a(\xi,t)$ can have only a finite number of zeros on the sheet Γ_+. These zeros will be denoted by $\xi_1, \xi_2, ..., \xi_N$. In [4] it is shown that all zeros are simple and all belong to the $(-\rho,\rho)$.

It is seen from representation (21) that $\xi = \xi_n$ the functions $\varphi(x,\xi,t)$ and $\psi(x,\xi,t)$ are proportional to each other

$$\begin{aligned} \varphi_n(x,t) &= c_n(t)\psi_n(x,t), \\ \bar{\varphi}_n(x,t) &= c_n^*(t)\bar{\psi}_n(x,t), \;\; n = 1,2,...,N. \end{aligned} \tag{26}$$

Where $\varphi_n(x,t) = \varphi(x,\xi_n,t)$, $\psi_n(x,t) = \psi(x,\xi_n,t)$.

The zeros of $a(\xi,t)$ correspond to the eigenvalues of the equation (10). The equation (10) is self-adjoint, so its eigenvalues and thus the zeros of the function $a(\xi,t)$ are real.

Note, the vector functions

$$h_n(x,t) = \frac{\frac{\partial}{\partial\xi}\left(\varphi(x,\xi,t) - c_n\psi(x,\xi,t)\right)|_{\xi=\xi_n}}{\dot{a}(\xi_n,t)}, \;\; n = 1,2,...,N \tag{27}$$

are a solution to the equations $(L(t)-\xi_n I)h_n = 0$. Where $\dot{a}(\xi_n,t) = \frac{\partial}{\partial\xi}\, a(\xi,t)|_{\xi=\xi_n}$.

From equality (27) it follows that

$$h_n(x,t) \sim -c_n(t)\begin{pmatrix} -\frac{i(\xi_n-p_n)}{\rho}e^{-i\alpha_-+2ip^2t} \\ 1 \end{pmatrix} e^{ip_nx} \text{ as } x\to-\infty,$$

$$h_n(x,t) \sim \begin{pmatrix} 1 \\ \frac{i(\xi_n-p_n)}{\rho}e^{i\alpha_+-2ip^2t} \end{pmatrix} e^{-ip_nx} \text{ as } x\to\infty, \tag{28}$$

where $p_n = i\sqrt{\rho^2-\xi_n^2}$. In particular, we have

$$\det(\varphi_n,h_n) = -\frac{2p_n(\xi_n-p_n)}{\rho^2}c_n,\ n=1,2,...,N. \tag{29}$$

Definition 1. The set $\left\{a(\xi,t),\, b(\xi,t),\, \{\xi_n(t),\, c_n(t)\}_{n=1}^N\right\}$ is called the scattering data for equation (10). The direct scattering problem is to find the scattering data via the given potentials $u(x,t)$ and the inverse scattering problem is to find the potentials $u(x,t)$ of the equation (10) via the given scattering data.

Before we proceed further solving the inverse problem, it is convenient to introduce a uniformization variable z (see [4, 8]) defined by the conformal mapping: $z = z(\xi) = \xi + p(\xi)$. The inverse mapping given by

$$\xi = \frac{1}{2}\left(z+\frac{\rho^2}{z}\right),\ \ p = z-\xi = \frac{1}{2}\left(z-\frac{\rho^2}{z}\right).$$

With this mapping the sheets Γ_+ and Γ_- of the Riemann surface Γ are, respectively, mapped onto the upper and lower complex half-planes $Im\, z > 0$ and $Im\, z < 0$ of the complex z-plane. The cut Σ on the Riemann surface is mapped onto the real z axis. The segments $[-\rho,\rho]$ on Γ_+ and Γ_- are mapped onto the upper and lower semicircles of radius ρ and center at the origin of the z-plane. The neighborhood of the point $\xi=\infty$ on $\Gamma_\pm$ with the condition $\pm Im\xi > 0$ is mapped into the neighborhood of the point $z=\infty$, and the neighborhood of the point $\xi=\infty$ on $\Gamma_\pm$ with the condition $\pm Im\xi < 0$ is mapped into the neighborhood of the point $z=0$.

In terms of the variable z, relation (19) can be written when $Imz = 0$ following form

$$f^-(x,z,t) = f^+(x,z,t)S(z,t), \tag{30}$$

where $f^{\pm}(x,z,t) = f^{\pm}(x,\xi(z),t)$, $S(z,t) = S(\xi(z),t)$ and one can obtain the symmetries of the scattering coefficients:

$$a(z,t) = \bar{a}\left(\frac{\rho^2}{z},t\right), \quad Imz \geq 0, \tag{31}$$

$$b(z,t) = -\bar{b}\left(\frac{\rho^2}{z},t\right), \quad Imz = 0.$$

Equality (31), together with the self-adjointness of the equation (10), ensure that the scattering coefficient $a(z,t)$ ($\bar{a}(z,t)$) can only have zeros at $z_n = \xi_n + iv_n$ ($\bar{z}_n = \xi_n - iv_n$), with $-\rho < \xi_n < \rho$ and $v_n = \sqrt{\rho^2 - \xi_n^2} > 0$.

Taking into account the analyticity properties of $a(z,t)$ in the upper half plane $Imz > 0$ we can obtain the following representation

$$a(z,t) = \prod_{n=1}^{N} \frac{z - z_n}{z - z_n^*} \exp\left[-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1 - |r(\zeta,t)|^2)}{\zeta - z} d\zeta\right].$$

Where $r(z,t) \equiv \frac{b(z,t)}{a(z,t)}$ is called reflection coefficient. According to (25), for $z \to 0$ we obtain that

$$e^{-i\theta} = \prod_{n=1}^{N} \frac{z_n}{z_n^*} \exp\left[-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1 - |r(\zeta,t)|^2)}{\zeta} d\zeta\right].$$

If $\varphi_n = \begin{pmatrix} \varphi_{1,n} \\ \varphi_{2,n} \end{pmatrix}$ is an eigenfunction of the equation (10) corresponding to z_n, then we define $\bar{\varphi}_n = \begin{pmatrix} \varphi_{2,n}^* \\ \varphi_{1,n}^* \end{pmatrix}$ to be the eigenfunction of the equation (10) corresponding to $\bar{z}_n$.

It is well known that the inverse scattering theory of (10) can be formulated in terms of the Gelfand-Levitan-Marchenko equations. The Jost solution $\psi(x,z,t)$ of the equation (10) can be represented in the following form

$$\psi(x,z,t) = \left[e(x,z) + \int_x^{\infty} \mathcal{K}(x,y,t) e(y,z) dy\right] \begin{pmatrix} -\frac{i\rho}{z} e^{-i\alpha_+ + 2i\rho^2 t} \\ 1 \end{pmatrix}, \tag{32}$$

here $e(x,z) = e^{\frac{i}{2}\left(z - \frac{\rho^2}{z}\right)x}$ and $\mathcal{K}(x,y,t)$ is a 2×2 matrix function which has to satisfy the following Gelfand-Levitan-Marchenko equation:

$$\mathcal{K}(x,y,t) + \mathcal{F}(x+y,t) + \int_x^{\infty} \mathcal{K}(x,s,t)\mathcal{F}(s+y,t) ds = 0, \quad y \geq x,$$

where $\mathcal{F}(x,t)$ is defined as

$$\mathcal{F}(x,t)=\begin{pmatrix} F_1(x,t) & F_2^*(x,t) \\ F_2(x,t) & F_1(x,t) \end{pmatrix}$$

with

$$F_1(x,t)=\frac{\rho e^{i\alpha_+-2i\rho^2t}}{4\pi i}\int_{-\infty}^{\infty}\frac{r(z,t)}{z}\cdot e(x,z)dz \\ -\frac{1}{2}\sum_{n=1}^{N}\frac{c_n(t)\rho e^{-i\alpha_++2i\rho^2t}}{\dot{a}(z_n,t)z_n}\cdot e(x,z_n),$$

$$F_2(x,t)=\frac{1}{4\pi}\int_{-\infty}^{\infty}r(z,t)\cdot e(x,z)dz-\frac{1}{2}\sum_{n=1}^{N}\frac{ic_n(t)}{\dot{a}(z_n,t)}\cdot e(x,z_n).$$

In representations (32), the component $K_{21}(x,x,t)$ of the matrix $\mathcal{K}(x,y,t)$ have relations with the potential

$$2K_{21}(x,x,t)=\rho e^{i\alpha_+-2i\rho^2t}-u(x,t).$$

In the work [8], it was proven the uniquely determining of the potential $u(x,t)$ by the scattering data.

4. Time evolution

The use of the inverse scattering method for integration of the problem (2)-(7) is based on the following. Let the function $u(x,t)$ be a solution of equation (2), from the class of functions (6). Consider equation (10) with a potential $u(x,t)$ and find the evolution from the scattering data.

Assuming that

$$F_{N+n}=\begin{pmatrix} f_{2,n}^* \\ f_{1,n}^* \end{pmatrix},\quad G_{N+n}=\begin{pmatrix} g_{2,n}^* \\ g_{1,n}^* \end{pmatrix},\quad n=1,2,...,N, \tag{33}$$

equation (2) can be represented as an equality of operators in the class of smooth functions $f(x,\xi,t)$ satisfying the equation (10):

$$\frac{\partial L}{\partial t}+[L,A]=i\sum_{n=1}^{2N}[\sigma_3,\ F_nG_n^T].$$

Where $[L, A] = LA - AL$ and

$$A = \begin{pmatrix} i|u|^2 + 2i\xi^2 & -iu_x^* - 2\xi u^* \\ iu_x - 2\xi u & -i|u|^2 - 2i\xi^2 \end{pmatrix}.$$

Let $f(x, \xi, t)$ be solution of the equation (10) and let $\phi_n(x, \xi, t)$, $n = 1, 2, ..., 2N$ be any functions, which satisfy the conditions

$$\frac{\partial \phi_n}{\partial x} = G_n^T f, \quad n = 1,\ 2,\ ...,\ 2N. \tag{34}$$

Then, the function $G_n(x, t)$ satisfy the equalities

$$G_n^T \sigma_3 f + i(\xi - \xi_n)\phi_n = 0\,,\ \ n = 1,\ 2,\ ...,\ 2N \tag{35}$$

and the function

$$H = \frac{\partial f}{\partial t} - Af + \sum_{n=1}^{2N} F_n \phi_n \tag{36}$$

satisfies the equation (10) for any $\xi \in \Sigma$.

4.1. Evolution equation for the scattering data in the case of a source satisfying the conditions (A)

Let us take matrix Jost solutions $f^-(x, \xi, t)$ and $f^+(x, \xi, t)$ for $\xi \in \Sigma$ as the solution $f(x, \xi, t)$ and $\xi = \xi_n$, $n = 1, 2, ..., N$ are eigenvalues of the equation (10). According to the definition of eigenfunctions, there are $\alpha_n(t)$ and $\beta_n(t)$ such that the relations hold

$$\begin{aligned} &F_n(x, t) = \alpha_n(t)\psi_n(x, t), \\ &G_n(x, t) = \beta_n(t)\sigma_1\varphi_n(x, t),\ \ n = 1, 2, ..., N. \end{aligned} \tag{37}$$

According to these relations, due to the assumptions (33), we obtain

$$\begin{aligned} &F_{N+n}(x, t) = \alpha_n^*(t)\bar{\psi}_n(x, t), \\ &G_{N+n}(x, t) = \beta_n^*(t)\sigma_1\bar{\varphi}_n(x, t),\ \ \ n = 1, 2, ..., N. \end{aligned} \tag{38}$$

By definition functions $G_n(x, t)$, belong to the $L^2(R)$ for all $t \geq 0$ and matrix Jost solutions $f^-(x, \xi, t)$, $f^+(x, \xi, t)$ are bounded for all $\xi \in \Sigma$. Therefore $\phi_n^- \in L^2(R)$ and $\phi_n^+ \in L^2(R)$ for all $t \geq 0$ and $\xi \in \Sigma$. Hence, by virtue of (35) it follows that at any $\xi \in \Sigma$ and $n = 1, 2, ..., 2N$ the asymptotics

$$\begin{aligned} &\phi_n^-(x, \xi, t) \to 0 \text{ as } x \to -\infty, \\ &\phi_n^+(x, \xi, t) \to 0 \text{ as } x \to \infty \end{aligned} \tag{39}$$

are valid. So, from (34) for $n = 1, 2, ..., 2N$ we obtain the following expressions

$$\begin{aligned} \phi_n^- &= \int_{-\infty}^{x} G_n^T(s,t) f^-(s,\xi,t)ds, \\ \phi_n^+ &= -\int_{x}^{\infty} G_n^T(s,t) f^+(s,\xi,t)ds. \end{aligned} \tag{40}$$

Using the matrix Jost solutions f^+ and f^- of equation (10), we rewrite equality (36) in the form

$$H^- = \frac{\partial f^-}{\partial t} - Af^- + \sum_{n=1}^{2N} F_n \phi_n^- \tag{41}$$

and

$$H^+ = \frac{\partial f^+}{\partial t} - Af^+ + \sum_{n=1}^{2N} F_n \phi_n^+. \tag{42}$$

These functions satisfy the equation (10) for any $\xi \in \Sigma$. Therefore, H^+ and H^- are linearly dependent on f^+ and f^-, respectively, i.e., there exist such $C_0^-(\xi,t)$ and $C_0^+(\xi,t)$ that the following identities hold

$$H^-(x,\xi,t) = f^-(x,\xi,t)C_0^-(\xi,t), \;\; H^+(x,\xi,t) = f^+(x,\xi,t)C_0^+(\xi,t).$$

By virtue of the definition of the matrix A, from relations (41), (42) and from asymptotics (11), (39) we obtain

$$H^-(x,\xi,t) \to -(i\rho^2 + 2i\xi p)E^-(x,\xi,t)\sigma_3, \;\; x \to -\infty, \tag{43}$$

$$H^+(x,\xi,t) \to -(i\rho^2 + 2i\xi p)E^+(x,\xi,t)\sigma_3, \;\; x \to \infty. \tag{44}$$

By the uniqueness of the Jost solutions we get

$$\begin{aligned} H^-(x,\xi,t) &= -(i\rho^2 + 2i\xi p) f^-(x,\xi,t)\sigma_3, \\ H^+(x,\xi,t) &= -(i\rho^2 + 2i\xi p) f^+(x,\xi,t)\sigma_3. \end{aligned} \tag{45}$$

We introduce the function $\mathcal{H}$ in the following form

$$\mathcal{H} = H^-(x,\xi,t) - H^+(x,\xi,t)S(\xi,t).$$

Based on equalities (19) and (45), the function $\mathcal{H}$ can be rewritten in the form

$$\mathcal{H} = (i\rho^2 + 2i\xi p) f^+(x,\xi,t)[\sigma_3, S(\xi,t)]. \tag{46}$$

On the other hand, by virtue of (19), (41) and (42) the equality

$$\mathcal{H} = H^-(x,\xi,t) - H^+(x,\xi,t)S(\xi,t) = f^+(x,\xi,t)S_t(x,\xi,t) + \\ + \sum_{n=1}^{2N}[F_n(x,t)\phi_n^-(x,\xi,t) - F_n(x,t)\phi_n^+(x,\xi,t)S(\xi,t)] \tag{47}$$

holds.

Based on equality (35) this relation becomes

$$\mathcal{H} = f^+(x,\xi,t)S_t(\xi,t) + \sum_{n=1}^{2N}\frac{i}{\xi-\xi_n}[F_n(x,t)G_n^T(x,t)\sigma_3 f^-(x,\xi,t) - \\ - F_n(x,t)G_n^T(x,t)\sigma_3 f^+(x,\xi,t)S(\xi,t)].$$

Finally, based on (19), we obtain

$$\mathcal{H} = f^+(x,\xi,t)S_t(\xi,t). \tag{48}$$

Comparing equalities (46) and (48) we have

$$(i\rho^2 + 2i\xi p)f^+(x,\xi,t)[\sigma_3, S(\xi,t)] = f^+(x,\xi,t)S_t(\xi,t).$$

Therefore, for all $\xi \in \Sigma$ we have the relation

$$S_t(\xi,t) - (i\rho^2 + 2i\xi p)[\sigma_3, S(\xi,t)] = 0,$$

i.e.

$$\frac{d}{dt}a(\xi,t) = 0, \quad \frac{d}{dt}b(\xi,t) = -2(i\rho^2 + 2i\xi p)b(\xi,t).$$

Since, the function $a(\xi,t)$ does not depend on t, hence we conclude that its zeros ξ_n also do not depend on t.

Based on identities (41) and (42), we write the following equalities

$$H_1^-(x,\xi_m,t) = \frac{\partial\varphi_m(x,t)}{\partial t} - A(x,\xi_m,t)\varphi_m(x,t) + \\ + \sum_{n=1}^{2N}F_n(x,t)\phi_{1,n}^-(x,\xi_m,t) \tag{49}$$

$$H_2^+(x,\xi_m,t) = \frac{\partial\psi_m(x,t)}{\partial t} - A(x,\xi_m,t)\psi_m(x,t) + \\ + \sum_{n=1}^{2N}F_n(x,t)\phi_{2,n}^+(x,\xi_m,t) \tag{50}$$

By virtue of the definition of the matrix A, from relations (49), (50) and from asymptotics (17), (18) we obtain

$$\begin{aligned} H_1^-(x,\xi_m,t) &= -(i\rho^2 + 2i\xi_m p_m)\varphi_m(x,t), \\ H_2^+(x,\xi_m,t) &= (i\rho^2 + 2i\xi_m p_m)\psi_m(x,t). \end{aligned} \tag{51}$$

We now introduce the following functions

$$\mathcal{H}_m = H_1^-(x,\xi_m,t) - c_m(t)H_2^+(x,\xi_m,t), \;\; m = 1,2,...,2N.$$

Using equalities (26), (51) the function $\mathcal{H}_m$ can be rewritten in the form

$$\mathcal{H}_m = (-2i\rho^2 - 4i\xi_m p_m)\varphi_m(x,t). \tag{52}$$

Substituting instead of $\phi_{1,n}^-(x,\xi,t)$ and $\phi_{2,n}^+(x,\xi,t)$ the expressions from (40) into equalities (49), (50) and using (26), we obtain

$$\begin{aligned} H_1^-(x,\xi_m,t) - c_m(t)H_2^+(x,\xi_m,t) = \frac{dc_m(t)}{dt}\psi_m(x,t) + \\ + \sum_{n=1}^{2N} F_n(x,t)\int_{-\infty}^{\infty} G_n^T(s,t)\varphi_m(s,t)ds. \end{aligned} \tag{53}$$

If $\xi_m \neq \xi_n$, according to equation (35) we get

$$\int_{-\infty}^{\infty} G_n^T(s,t)\varphi_m(s,t)ds = 0.$$

According to (37) and (38), equality (53) can be rewritten in the form

$$\begin{aligned} H_1^-(x,\xi_m,t) - c_m(t)H_2^+(x,\xi_m,t) = \frac{dc_m(t)}{dt}\psi_m(x,t) \\ + \left(\int_{-\infty}^{\infty} G_m^T(s,t)F_m(s,t)ds + \int_{-\infty}^{\infty} G_{N+m}^T(s,t)F_{N+m}(s,t)ds\right)\varphi_m(x,t). \end{aligned} \tag{54}$$

Comparing equalities (52) and (54) we obtain

$$\begin{aligned} (-2i\rho^2 - 4i\xi_m p_m)\varphi_m(x,t) = \frac{dc_m(t)}{dt}\psi_m(x,t) \\ + \left(\int_{-\infty}^{\infty} G_m^T(s,t)F_m(s,t)ds + \int_{-\infty}^{\infty} G_{N+m}^T(s,t)F_{N+m}(s,t)ds\right)\varphi_m(x,t). \end{aligned}$$

Finally, using these equalities and taking into account (8) and (26) we determine

$$\frac{dc_m(t)}{dt} = (-2i\rho^2 - 4i\xi_m p_m - A_m(t) - A_m^*(t))c_m(t).$$

Thus, we have proved the following theorem.

Theorem 2. *If functions $u(x,t)$, $F_k(x,t)$, $G_k(x,t)$, $k = 1, 2, ..., N$ are the solutions of the problem* (2)-(7) *in the case of a source satisfying the conditions (A), then the scattering data for the equation* (10) *satisfy the following relations*

$$a(\xi,t) = a(\xi,0),$$

$$b(\xi,t) = b(\xi,0)\exp(-2i\rho^2 t - 4i\xi pt) \quad for \quad \xi \in \Sigma,$$

$$\xi_k(t) = \xi_k(0),$$

$$c_k(t) = c_k(0)\exp(-2i\rho^2 t - 4i\xi_k p_k t - \int_0^t (A_k(\tau) + A_k^*(\tau))d\tau),$$

$$k = 1,\ 2,\ ...,\ N.$$

4.2. Evolution equation for the scattering data in the case of a source satisfying the conditions (B)

Let us take matrix Jost solutions $f^-(x,\xi,t)$ and $f^+(x,\xi,t)$ for $\xi \in \Sigma$ as the solution $f(x,\xi,t)$ and $\xi = \xi_n$, $n = 1, 2, ..., N$, are eigenvalues of the equation (10). According to the definition of eigenfunctions of equation (10), there are $\alpha_n(t)$ such that the relations

$$\begin{aligned} &F_n(x,t) = \alpha_n(t)\psi_n(x,t), \\ &F_{N+n}(x,t) = \alpha_n^*(t)\bar{\psi}_n(x,t), \quad n = 1, 2, ..., N. \end{aligned} \tag{55}$$

Due to the assumptions (B) the functions $G_n(x,t)$ are unbounded functions. So, there are $\beta_n(t)$ such that which follow the equalities

$$\begin{aligned} &G_n(x,t) = \frac{\beta_n(t)}{\dot{a}(\xi_n,t)}\sigma_1\varphi_n(x,t) + \sigma_1 h_n(x,t), \\ &G_{N+n}(x,t) = \frac{\beta_n^*(t)}{\dot{\bar{a}}(\xi_n,t)}\sigma_1\bar{\varphi}_n(x,t) + \sigma_1\bar{h}_n(x,t),\ n = 1, 2, ..., N. \end{aligned} \tag{56}$$

One can easily see from (9) and (29), that the quantities $\alpha_n(t)$ satisfy the following equalities

$$\begin{aligned} &\alpha_n(t) = -\frac{\rho^2}{2p_n(\xi_n - p_n)}B_n(t), \\ &\alpha_n^*(t) = \frac{\rho^2}{2p_n(\xi_n + p_n)}B_n(t),\ n = 1, 2, ..., N. \end{aligned} \tag{57}$$

Using equalities (26), (35), (55), (56) and asymptotics (17), (28) we can verify that at any $\xi \in \Sigma$ and when $x \to \infty$ the following asymptotics are valid:

$$F_n\phi_n^+ \sim \frac{i\alpha_n(t)}{\xi-\xi_n}\begin{pmatrix} \frac{(\xi_n-p_n)^2}{\rho^2} & -\frac{i(\xi_n-p_n)}{\rho}e^{-i\alpha_+ + 2i\rho^2 t} \\ \frac{i(\xi_n-p_n)}{\rho}e^{i\alpha_+ - 2i\rho^2 t} & 1 \end{pmatrix}\sigma_3 E^+,$$

$$F_{N+n}\phi_{N+n}^+ \sim \frac{i\alpha_n^*(t)}{\xi-\xi_n}\begin{pmatrix} 1 & -\frac{i(\xi_n+p_n)}{\rho}e^{-i\alpha_+ + 2i\rho^2 t} \\ \frac{i(\xi_n+p_n)}{\rho}e^{i\alpha_+ - 2i\rho^2 t} & \frac{(\xi_n+p_n)^2}{\rho^2} \end{pmatrix}\sigma_3 E^+.$$

Taking into account of equalities (26), (35), (55), (56) and asymptotics (17), (28) we are convinced that at any $\xi \in \Sigma$ and $x \to -\infty$ there hold the asymptotics

$$F_n\phi_n^- \sim -\frac{i\alpha_n(t)}{\xi-\xi_n}\begin{pmatrix} 1 & -\frac{i(\xi_n-p_n)}{\rho}e^{-i\alpha_- + 2i\rho^2 t} \\ \frac{i(\xi_n-p_n)}{\rho}e^{i\alpha_- - 2i\rho^2 t} & \frac{(\xi_n-p_n)^2}{\rho^2} \end{pmatrix}\sigma_3 E^-,$$

$$F_{N+n}\phi_{N+n}^- \sim -\frac{i\alpha_n^*(t)}{\xi-\xi_n}\begin{pmatrix} \frac{(\xi_n+p_n)^2}{\rho^2} & -\frac{i(\xi_n+p_n)}{\rho}e^{-i\alpha_- + 2i\rho^2 t} \\ \frac{i(\xi_n+p_n)}{\rho}e^{i\alpha_- - 2i\rho^2 t} & 1 \end{pmatrix}\sigma_3 E^-.$$

Using equalities (57) one can easily verify that at any $\xi \in \Sigma$ the asymptotic

$$F_n\phi_n^+ + F_{N+n}\phi_{N+n}^+ \to 0 \quad \text{for} \quad x \to \infty,$$

and

$$F_n\phi_n^- + F_{N+n}\phi_{N+n}^- \to 0 \quad \text{for} \quad x \to -\infty$$

are valid.

Hence, it follows that the quantities $H^-(x,\xi,t)$ and $H^+(x,\xi,t)$ determined by (41) and (42) satisfy equalities

$$\begin{aligned} H^-(x,\xi,t) &= f^-(x,\xi,t)(-i\rho^2 - 2i\xi p)\sigma_3, \\ H^+(x,\xi,t) &= f^+(x,\xi,t)(-i\rho^2 - 2i\xi p)\sigma_3. \end{aligned} \tag{58}$$

Now, consider the function $\mathcal{H}_m$ of the form

$$\mathcal{H}_m = H^-(x,\xi,t) - H^+(x,\xi,t)S(\xi,t).$$

Taking into account (58) we find that

$$\mathcal{H}_m = (i\rho^2 + 2i\xi p)f^+(x,\xi,t)[\sigma_3, S(\xi,t)]. \tag{59}$$

From equalities (41), (42) and (19) it is easy to get that

$$H^-(x,\xi,t) - H^+(x,\xi,t)S(\xi,t) = f^+(x,\xi,t)S_t(x,\xi,t)+ \\ + \sum_{n=1}^{2N}[F_n(x,t)\phi_n^-(x,\xi,t) - F_n(x,t)\phi_n^+(x,\xi,t)S(\xi,t)]. \tag{60}$$

Using equalities (35), we obtain

$$H^-(x,\xi,t) - H^+(x,\xi,t)S(\xi,t) = f^+(x,\xi,t)S_t(\xi,t) \\ + \sum_{n=1}^{2N}\frac{i}{\xi-\xi_n}[F_n(x,t)G_n^T(x,t)\sigma_3 f^-(x,\xi,t) \\ - F_n(x,t)G_n^T(x,t)\sigma_3 f^+(x,\xi,t)S(\xi,t)].$$

By virtue of (19), it follows that

$$H^-(x,\xi,t) - H^+(x,\xi,t)S(\xi,t) = f^+(x,\xi,t)S_t(\xi,t). \tag{61}$$

Comparing equalities (59) and (61) we have

$$(i\rho^2 + 2i\xi p)f^+(x,\xi,t)[\sigma_3, S(\xi,t)] = f^+(x,\xi,t)S_t(\xi,t).$$

Therefore, for all $\xi \in \Sigma$ we have

$$S_t(\xi,t) - (i\rho^2 + 2i\xi p)[\sigma_3, S(\xi,t)] = 0,$$

i.e.

$$\frac{d}{dt}a(\xi,t) = 0, \quad \frac{d}{dt}b(\xi,t) = -2(i\rho^2 + 2i\xi p)b(\xi,t).$$

Thus, we conclude that the function $a(\xi,t)$ does not depend on t, so the zeros $\xi_n(t)$ of function $a(\xi,t)$ do not depend on t.

Let us now find the evolution of the normalizing constants $c_m(t)$, $m = 1,2,...,N$. We now introduce the following functions

$$\mathcal{H}_m = H_1^-(x,\xi_m,t) - c_m(t)H_2^+(x,\xi_m,t), \ m = 1,2,...,N, \tag{62}$$

where

$$H_1^-(x,\xi_m,t) = \frac{\partial\varphi_m(x,t)}{\partial t} - A(x,\xi_m,t)\varphi_m(x,t) \\ + \sum_{n=1}^{2N}\Phi_n(x,t)\varphi_{1,n}^-(x,\xi_m,t), \tag{63}$$

$$H_2^+(x,\xi_m,t) = \frac{\partial \psi_m(x,t)}{\partial t} - A(x,\xi_m,t)\psi_m(x,t)$$
$$+ \sum_{n=1}^{2N} \Phi_n(x,t)\varphi_{2,n}^+(x,\xi_m,t). \tag{64}$$

It is easy to show that

$$\begin{aligned} H_1^-(x,\xi_m,t) &= (-i\rho^2 - 2i\xi_m p_m)\varphi_m(x,t), \\ H_2^+(x,\xi_m,t) &= (i\rho^2 + 2i\xi_m p_m)\psi_m(x,t). \end{aligned} \tag{65}$$

Substituting (65) into (62) and using equalities (26), we get for $m = 1, 2, ..., N$

$$\mathcal{H}_m = (-2i\rho^2 - 4i\xi_m p_m)\varphi_m(x,t). \tag{66}$$

On the other hand, using equalities (63), (64) and (26) we obtain

$$H_1^-(x,\xi_m,t) - c_m(t)H_2^+(x,\xi_m,t) = \frac{dc_m(t)}{dt}\psi_m(x,t)$$

$$+ \sum_{\substack{n=1 \\ n\neq m}}^{2N} \frac{i}{\xi_m - \xi_n}[F_n(x,t)G_n^T(x,t)\sigma_3\varphi_m(x,t)$$

$$-c_m(t)F_n^+(x,t)G_n^T(x,t)\sigma_3\psi_m(x,t)]$$

$$+iF_m(x,t)G_m^T(x,t)\sigma_3\frac{\partial}{\partial \xi}\left(\varphi(x,\xi,t) - c_m(t)\psi(x,\xi,t)\right)\big|_{\xi=\xi_m}$$

$$+iF_{N+m}(x,t)G_{N+m}^T(x,t)\sigma_3\frac{\partial}{\partial \xi}\left(\varphi(x,\xi,t) - c_m(t)\psi(x,\xi,t)\right)\big|_{\xi=\xi_m}.$$

According to (26) and (27), this equation can be rewritten in the following form

$$H_1^-(x,\xi_m,t) - c_m(t)H_2^+(x,\xi_m,t) = \frac{dc_m(t)}{dt}\psi_m(x,t)$$

$$+i\dot{a}(\xi_n,t)\left(F_m(x,t)G_m^T(x,t) + F_{N+m}(x,t)G_{N+m}^T(x,t)\right)\sigma_3 h_m(x,t).$$

Further, by virtue (9), (55) and (56), we obtain the equality

$$\begin{aligned} H_1^-(x,\xi_m,t) - c_m(t)H_2^+(x,\xi_m,t) &= \frac{dc_m(t)}{dt}\psi_m(x,t) \\ - i\beta_m(t)B_m(t)\varphi_m(x,t) &+ i\beta_m^*(t)B_m(t)\varphi_m(x,t). \end{aligned} \tag{67}$$

Comparing equalities (66) and (67), we obtain

$$(-2i\rho^2 - 4i\xi_m p_m)\varphi_m(x,t)$$

$$= \frac{dc_m(t)}{dt}\psi_m(x,t) - i\beta_m(t)B_m(t)\varphi_m(x,t) + i\beta_m^*(t)B_m(t)\varphi_m(x,t)$$

hence, taking into account equalities (26), we find

$$\frac{dc_m(t)}{dt} = (-2i\rho^2 - 4i\xi_m p_m + i(\beta_m(t) - \beta_m^*(t))B_m(t))c_m(t).$$

Thus, we have proved the following theorem.

Theorem 3. *If functions $u(x,t)$, $F_k(x,t)$, $G_k(x,t)$, $k = 1,\ 2,\ ...,N$ are the solutions of the problem* (2)-(7) *in the case of a source satisfying the conditions (B), then the scattering data for the equation* (10) *satisfy the following relations*

$$a(\xi,t) = a(\xi,0),$$

$$b(\xi,t) = b(\xi,0)\exp(-2i\rho^2 t - 4i\xi p t) \quad \textit{for} \quad \xi \in \Sigma,$$

$$\xi_k(t) = \xi_k(0),$$

$$c_k(t) = c_k(0)\exp(-2i\rho^2 - 4i\xi_k p_k + i\int_0^t (\beta_k(\tau) - \beta_k^*(\tau))\, B_k(\tau)d\tau,$$

$$k = 1,\ 2,\ ...,\ N.$$

We will illustrate inverse scattering method of constructing exact solutions to the NLS equation with concrete example.

Example 4. Let the initial function $u_0(x)$ have the form

$$u_0(x) = \rho \cdot \frac{e^{i\alpha_+}e^{\nu x} + e^{i\alpha_-}e^{-\nu x}}{e^{\nu x} + ce^{-\nu x}}.$$

Where α_+, α_-, ρ, ν, c are positive real numbers and $\rho > \nu$.

In this case, solving the direct scattering problem for the equation (10), we obtain

$$a(\xi,0) = \frac{\xi + p - \zeta - i\nu}{\xi + p - \zeta + i\nu},\ \zeta = \sqrt{\rho^2 - \nu^2},$$

$$b(\xi,0) = 0, \quad \xi_1(0) = \zeta, \quad c_1(0) = \frac{i(\zeta - i\nu)}{\rho}ce^{i\alpha_-}.$$

Based on Theorem 2, we can show the evolution of the scattering data in the following form

$$a(\xi,t)=\frac{\xi+p-\zeta-i\nu}{\xi+p-\zeta+i\nu},\ \zeta=\sqrt{\rho^2-\nu^2},$$

$$b(\xi,t)=0,\quad \xi_1(t)=\zeta,$$

$$c_1(t)=\frac{i(\zeta-i\nu)}{\rho}\cdot c\cdot\exp(i\alpha_- - 2i\rho^2 t+4\zeta\nu t-\int_0^t(A_k(\tau)+A_k^*(\tau))d\tau).$$

Applying the procedure of the inverse scattering problem, we find

$$u(x,t)=\rho e^{-2i\rho^2 t}\cdot\frac{e^{i\alpha_+}e^{\nu x}+e^{i\alpha_-}ce^{-\nu x+4\zeta\nu t-g(t)}}{e^{\nu x}+ce^{-\nu x+4\zeta\nu t-g(t)}},$$

where $g(t)=\int_0^t(A_1(\tau)+A_1^*(\tau))\,d\tau$.

Using representation (32) and conditions (8), we obtain

$$F_1=\alpha_1(t)\cdot\begin{pmatrix}-\frac{i(\zeta-i\nu)}{\rho}\cdot e^{-i\alpha_+ +2i\rho^2 t}\\ 1\end{pmatrix}\cdot\frac{1}{e^{\nu x}+ce^{-\nu x+4\zeta\nu t-g(t)}},$$

$$G_1=\frac{\nu A_1(t)}{\alpha_1(t)}\cdot\begin{pmatrix}\frac{i(\zeta-i\nu)}{\rho}\cdot e^{i\alpha_- -2i\rho^2 t}\\ 1\end{pmatrix}\cdot\frac{ce^{4\zeta\nu t-g(t)}}{e^{\nu x}+ce^{-\nu x+4\zeta\nu t-g(t)}}.$$

Analogously, in the case (B), using results of Theorem 3, we obtain

$$u(x,t)=\rho e^{-2i\rho^2 t}\cdot\frac{e^{i\alpha_+}e^{\nu x}+e^{i\alpha_-}ce^{-\nu x+4\zeta\nu t-g(t)}}{e^{\nu x}+ce^{-\nu x+4\zeta\nu t-g(t)}},$$

$$F_1=-\frac{\rho^2 B_1(t)}{2i\nu(\zeta-i\nu)}\cdot\begin{pmatrix}-\frac{i(\zeta-i\nu)}{\rho}\cdot e^{-i\alpha_+ +2i\rho^2 t}\\ 1\end{pmatrix}\cdot\frac{1}{e^{\nu x}+ce^{-\nu x+4\zeta\nu t-g(t)}},$$

$$G_1=\frac{-2\nu}{\zeta+i\nu}\cdot(\nu\beta_1(t)-2x\zeta+i\sigma_3)\cdot\begin{pmatrix}\frac{i(\zeta-i\nu)}{\rho}\cdot e^{i\alpha_- -2i\rho^2 t}\\ 1\end{pmatrix}$$

$$\times\frac{ce^{4\zeta\nu t-g(t)}}{e^{\nu x}+ce^{-\nu x+4\zeta\nu t-g(t)}}$$

$$+\begin{pmatrix}\frac{i(\zeta-i\nu)}{\rho}\cdot e^{i\alpha_+ -2i\rho^2 t}\\ 1\end{pmatrix}\cdot\frac{e^{2\nu x}}{e^{\nu x}+ce^{-\nu x+4\zeta\nu t-g(t)}}$$

$$+\begin{pmatrix}-\frac{i(\zeta-i\nu)}{\rho}\cdot e^{i\alpha_- -2i\rho^2 t}\\ e^{i\theta}\end{pmatrix}\cdot\frac{c^2e^{-2\nu x+8\zeta\nu t-2g(t)}}{e^{\nu x}+ce^{-\nu x+4\zeta\nu t-g(t)}},$$

where $g(t)=-i\int_0^t(\beta_1(\tau)-\beta_1^*(\tau))\,B_1(\tau)d\tau$.

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