International Journal of Applied Mathematics

Volume 36 No. 3 2023, 391-403
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v36i3.6

ON THE OSCILLATORY BEHAVIOR OF A CLASS OF EVEN ORDER NONLINEAR DAMPED DELAY DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS

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Abstract: The present study concerns the oscillation of a class of even-order nonlinear damped delay differential equations with distributed deviating arguments. We offer a new description of oscillation of the even-order equations in terms of oscillation of a related well studied second-order linear differential equation without damping. Some new oscillatory criteria are obtained by using the generalized Riccati transformation, integral averaging technique and comparison principles. The effectiveness of the obtained criteria is illustrated via examples.

AMS Subject Classification: 34C10, 34K11
Key Words: oscillation; even-order differential equations; damped delay; distributed deviation

Received: February 12, 2023

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1. Introduction

We consider the even order delay differential equation of the form

$$\big(l_2(r)\big(l_1(r)\big(x^{(n-2)}(r)\big)^{\alpha}\big)'\big)' + p(r)\Big(x^{(n-2)}(\delta(r))\Big)^{\alpha} + \int_c^d q(r,\varrho)f(r,x(g(r,\varrho)))d\varrho = 0, \tag{E_1}$$

where $\alpha \geq 1$ is a quotient of odd positive integers and $c < d$. Throughout this paper, we use the following assumptions:

$$\begin{cases} l_1, l_2, p, \delta \in C\big(I,[0,\infty)\big) \text{ and } l_1, l_2 > 0, \text{ where } I = [r_0, +\infty); \\ q, g \in C[I \times [c,d],[0,\infty)), \delta(r) \leq r, \lim_{r\to+\infty}\delta(r) = \infty, \\ \qquad g(r,\varrho) \text{ is a nondecreasing} \\ \text{function for } \varrho \in [c,d] \text{ satisfying } g(r,\varrho) \leq r, \lim_{r\to+\infty} g(r,\varrho) = \infty; \\ f \in C(\mathbb{R},\mathbb{R}), \text{there exists a constant } k_1 > 0 \text{ such that } f(r,\, x(r))/x^{\beta} \geq k_1. \end{cases}$$

We define the operators

$$N(x(r)) = l_1(r)\big(x^{(n-2)}(r)\big)^{\alpha}, \quad l(x(r)) = l_2(r)(N(x(r)))'.$$

By a solution to (E_1), we mean a function $x(r)$ in $C^2[r_x,\infty)$ for which $N(x(r))$, $l(x(r))$ is in $C^1[r_x,\infty)$ and (E_1) is satisfied on some interval $[r_x,\infty)$, where $r_x \geq r_0$. We consider only solutions $x(r)$ for which $\sup\{|x(r)| : r \geq r\} > 0$ for all $r \geq r_x$. A solution of (E_1) is called oscillatory if it is neither eventually positive nor eventually negative on $[r_x,\infty)$ and otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

We define

$$\Omega_1(r_1,r) = \int_{r_1}^{r} l_1^{-1/\alpha}(s)ds, \quad \Omega_2(r_1,r) = \int_{r_1}^{r} l_2^{-1}(s)ds,$$
$$\Omega_3(r_1,r) = \int_{r_1}^{r} \frac{(r-x)^{n-4}}{(n-4)!}\Big(\frac{\Omega_2(r_1,x)}{l_1(x)}\Big)^{1/\alpha}du,$$
$$\Omega_3^*(r_1,r) = \int_{r_1}^{r} \frac{(r-x)^{n-3}}{(n-3)!}\Big(\frac{\Omega_2(r_1,x)}{l_1(x)}\Big)^{1/\alpha}du,$$

for $r_0 \leq r_1 \leq r < \infty$ and assume that

$$\Omega_1(r_1,r) = \infty, \quad \Omega_2(r_1,r) = \infty \quad \text{as} \quad r \to \infty. \tag{1}$$

Fourth/Higher-order differential equations are often used to model a wide range of physical, chemical, and biological processes in a mathematical way [1, 3]. For example, it could be used to solve problems with elasticity, structure deformation, or soil settlement. In mechanical and engineering problems, questions about whether or not there are oscillatory and non-oscillatory solutions are very important [5]. Many authors have done a lot of research on the problem of oscillation in fourth (or higher) order differential equations. They have come up with many ways to get oscillatory criteria for fourth (or higher) order differential equations. Several studies have had very interesting results related to oscillatory properties of solutions of neutral differential equations and damped delay differential equations with/without distributed deviating arguments [4, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20].

In this paper, using suitable Riccati type transformation, integral averaging condition, and comparison method, we present some sufficient conditions which insure that any solution of Eq. (E_1) oscillates when the associated second order equation

$$(l_2(r)z'(r))' + \frac{p(r)}{l_1(\delta(r))}z(r) = 0, \qquad (E_2)$$

is oscillatory or nonoscillatory.

2. Basic lemmas

In this section, we state and prove some lemmas that are frequently used in the remainder of this paper.

Lemma 1. *[21] Assume that Eq. (E_2) is nonoscillatory. If Eq.(E_1) has a nonoscillatory solution $x(r)$ on I, for $r_1 \geq r_0$, then there exists a $r_2 \in I$ such that $x(r)N(x(r)) > 0$ or $x(r)N(x(r)) < 0$ for $r \geq r_2$.*

Lemma 2. *If Eq.(E_1) has a nonoscillatory solution $x(r)$ which satisfies $x(r)N(x(r)) > 0$, in Lemma 1 for $r \geq r_1 \geq r_0$. Then,*

$$N(x(r)) > \Omega_2(r_1, r)\, l(x(r)), \quad r \geq r_1, \qquad (2)$$

$$x'(r) > \Omega_3(r_1, r)\, l^{1/\alpha}(x(r)), \quad r \geq r_1, \qquad (3)$$

and

$$x(r) > \Omega_3^*(r_1, r)\, l^{1/\alpha}(x(r)), \quad r \geq r_1. \qquad (4)$$

Proof. Let Eq. (E_1) have a non-oscillatory solution x. Suppose that there exists a $r_1 \geq r_0$ such that $x(r) > 0$ and $x(g(r,\varrho))$ for $r \geq r_1$. From Eq. (E_1), we have

$$l'(x(r)) = -\Big(\frac{p(r)}{l_1(\delta(r))}\Big) N(x(\delta(r))) - k_1 \int_c^d q(r,\varrho) x^{\beta}(g(r,\varrho)) d\varrho \leq 0,$$

and $l(x(r))$ is non increasing on I, we get

$$N(x(r)) \geq \int_{r_1}^{r} \big(N(x(s))\big)' \, ds = \int_{r_1}^{r} (l_2(s))^{-1} l(x(s)) \, ds \geq \Omega_2(r_1, r) \, l(x(r)),$$

this implies that

$$x^{(n-2)}(r) \geq l^{1/\alpha}(x(r)) \Big((l_1(r))^{-1} \Omega_2(r_1, r) \Big)^{1/\alpha}.$$

Now, integrating above inequality repeatedly from r_1 to r and using $l(x(r)) \leq 0$, we find

$$\begin{aligned} x'(r) &\geq l^{1/\alpha}(x(r)) \left[\int_{r_1}^{r} \frac{(r-x)^{n-4}}{(n-4)!} \Big(\frac{\Omega_2(r_1,x)}{l_1(x)} \Big)^{1/\alpha} du \right] \\ &= l^{1/\alpha}(x(r)) \Omega_3(r_1, r), \end{aligned}$$

and

$$\begin{aligned} x(r) &\geq l^{1/\alpha}(x(r)) \left[\int_{r_1}^{r} \frac{(r-x)^{n-3}}{(n-3)!} \Big(\frac{\Omega_2(r_1,x)}{l_1(x)} \Big)^{1/\alpha} du \right] \\ &= l^{1/\alpha}(x(r)) \Omega_3^*(r_1, r), \quad \text{for } r \leq r_1. \end{aligned}$$

□

Lemma 3. *[21] Let $\xi \in C^1(I, \mathbb{R}^+)$, $\xi(r) \leq r$, $\xi'(r) \geq 0$ and $G(r) \in C(I, \mathbb{R}^+)$ for $r \geq r_0$. Let $y(r)$ be a bounded solution of 2^{nd}-order delay differential equation*

$$\big(l_2(r) \, y'(r)\big)' - \Theta(r) \, y(\xi(r)) = 0. \tag{E_3}$$

If

$$\limsup_{r \to \infty} \int_{\xi(r)}^{r} \Theta(s) \, \Omega_2(\xi(r), \, \xi(s)) \; ds > 1 \tag{5}$$

or

$$\limsup_{r\to\infty} \int_{\xi(r)}^{r} \Big[\big(l_2(r)\big)^{-1} \int_{x}^{r} \Theta(s)\ ds\Big] du > 1, \tag{6}$$

where $l_2(r)$ is as in (E_1), then the solutions of (E_3) are oscillatory.

3. Main results

In this section, we establish some oscillation criteria for Eq. (E_1) by comparison Principle Method. For convenience, we denote

$$\widetilde{q}(r,\varrho) = \int_c^d q(r,\varrho)\,d\varrho, \quad \psi(r) = \exp\Big(\int_{r_1}^{r} Q(s)ds\Big),$$
$$Q(r) = \Big(\frac{p(r)}{l_1(\delta(r))}\Big)\Omega(r_1,\delta(r)), \quad \Theta^*(r) = k_1\,\widetilde{q}(r,\varrho)\Big(\Omega_3^*(r_1,g(r,d))\Big)^{\beta}.$$

Theorem 4. *Suppose that $\alpha \geq \beta$, conditions (1) hold, Eq. (E_2) is nonoscillatory. Suppose there exists a $\xi \in C^1(I,\mathbb{R})$ such that $g(r,\varrho) \leq \xi(r) \leq \delta(r) \leq r$, $\xi'(r) \geq 0$ for $r \geq r_1$, and (5) or (6) holds with*

$$\Theta(r) = \ell_*\,k_1\big[g^{n-3}(r,d)\big]^{\beta}\widetilde{q}(r,\varrho)\big(\Omega_1(\xi(r),g(r,d))\big)^{\beta} - \frac{p(r)}{l_1(\delta(r))} \geq 0, \ \ r \geq r_1,$$

for constant $\ell_* > 0$. Moreover, suppose that every solution of the first-order delay equation have the following form

$$z'(r) + \psi^{1-\frac{\beta}{\alpha}}(g(r,d))\,\Theta^*(r)\,z^{\frac{\beta}{\alpha}}(g(r,d)) = 0, \tag{7}$$

then every solution of Eq.(E_1) is oscillatory.

Proof. Let Eq. (E_1) have a nonoscillatory solution $x(r)$. Suppose, there exists a $r \geq r_1$ such that $x(r) > 0$ and $x(g(r,\varrho)) > 0$ for some $r \geq r_0$. From Lemma 1, $x(r)$ has the conditions either $N(x(r)) > 0$ or $N(x(r)) < 0$ for $r \geq r_1$.

Assume that $x(r)$ has the condition $N(x(r)) > 0$, for $r \geq r_1$, then one can easily see that $l(x(r)) > 0$ for $r \geq r_1$. Take $r_2 \geq r_1$ such that $g(r,\varrho) \geq r_1$ for $r \geq r_2$, $g(r,\varrho) \to \infty$ as $r \to \infty$ and we have (4),

$$x(g(r,d)) > \Omega_3^*(r_1,g(r,d))\,\big(l(x(g(r,d)))\big)^{1/\alpha}, \quad r \geq r_2. \tag{8}$$

By substituting (2), (8) in Eq.(E_1) and $l(x(r))$ is decreasing, then

$$\Big(l(x(r))\Big)' + \Big(\frac{p(r)}{l_1(\delta(r))}\Big) l(x(r))\,\Omega(r_1,\delta(r)) \\ + k_1\,\widetilde{q}(r,\varrho)\Big(\Omega_3^*(r_1,g(r,d))\Big)^{\beta}\Big(l(x(g(r,d)))\Big)^{\beta/\alpha} \le 0. \tag{9}$$

Take $\phi = l(x(r))$, we have

$$\phi'(r) + Q(r)\phi(r) + \Theta^*(r)\phi^{\frac{\beta}{\alpha}}(g(r,d)) \le 0, \tag{10}$$

or

$$\Big(\psi(r)\,\phi(r)\Big)' + \psi(r)\Theta^*(r)\phi^{\frac{\beta}{\alpha}}(g(r,d)) \le 0, \quad \text{for } r \ge r_2. \tag{11}$$

Next, assume $z = \psi\,\phi > 0$ and $\psi(g(r,d)) \le \phi(r)$, thus we have

$$z'(r) + \psi^{1-\frac{\beta}{\alpha}}(g(r,d))\Theta^*(r)z^{\frac{\beta}{\alpha}}(g(r,d)) \le 0. \tag{12}$$

This means (12) is a positive for this inequality. Also, by [2, Corollary 2.3.5], we get a contradiction of positivity of Eq. (E_1).

Next, assume $x(r)$ satisfies the condition $N(x(r)) < 0$, for $r \ge r_1$, we get $x^{(n-3)}(r) > 0$, $l(x(r)) \ge 0$ for $r \ge r_3 (\ge r_2)$. By [16, Lemma 2], one can now deduce that there exists a constant $\theta \in (0,1)$ such that

$$x(r) \ge \theta\, r^{n-3}\, x^{(n-3)}(r), \qquad \text{for } r \ge r_3. \tag{13}$$

Set $w(r) = x^{(n-3)}(r)$, then $w'(r) = x^{(n-2)}(r) < 0$. Using (13) in Eq.(E_1) we get

$$\big(l_2(r)\big(l_1(r)\big[w'(r)\big]^{\alpha}\big)'\big)' + p(r)(w'(\delta(r)))^{\alpha} \\ + k_1\left[\theta g^{n-3}(r,d)\right]^{\beta}\widetilde{q}(r,\varrho)w^{\beta}(g(r,d)) \le 0,$$

and so $l_1(r)\big[w'(r)\big]^{\alpha} < 0$, we have $\big(l_1(r)\big[w'(r)\big]^{\alpha}\big)' > 0$ for $r \ge r_3$. Now, for $v \ge x \ge r_3$, we get

$$\begin{aligned} w(x) > w(x) - w(v) &= -\int_x^v l_1^{-1/\alpha}(s)(l_1(s)(w'(s))^{\alpha})^{1/\alpha} ds \\ &\ge l_1^{1/\alpha}(v)(-w'(v))\left(\int_x^v l_1^{-1/\alpha}(s) ds\right) \\ &= l_1^{1/\alpha}(v)(-w'(v))\Omega_1(x,v). \end{aligned}$$

Taking $x = \xi(r)$ and $v = g(r,d)$, we obtain

$$w(g(r,d)) > \Omega_1(g(r,d),\xi(r)) l_1^{1/\alpha}(v)(-w'(v)) = \Omega_1(g(r,d),\xi(r))\ y(\xi(r)),$$

where $y(r) = l_1^{1/\alpha}(v)(-w'(v)) > 0$ for $r \geq r_3$. From Eq.(E_1), we get $y(r)$ is decreasing and $g(r,d) \leq \xi(r) \leq \delta(r) \leq r$, and

$$\begin{aligned}(l_2(r)z'(r))' + \tfrac{p(r)}{l_1(\delta(r))} z(\delta(r))\\ \geq k_1 \left[\theta g^{n-3}(r,d)\right]^{\beta} \widetilde{q}(r,\varrho)\Omega_1(g(r,d),\xi(r)) z^{\frac{\beta}{\alpha}-1}(\xi(r)) z(\xi(r)).\end{aligned}$$

Since z is decreasing and $\alpha \geq \beta$, there exists a constant ℓ such that $z^{\frac{\beta}{\alpha}-1}(r) \geq \ell$ for $r \geq r_3$. Thus, we obtain

$$(l_2(r)z'(r))' \geq \left(\ell\, k_1 \left[\theta g^{n-3}(r,d)\right]^{\beta} \widetilde{q}(r,\varrho)\Omega_1(g(r,d),\xi(r)) - \frac{p(r)}{l_1(\delta(r))}\right) z(\xi(r)).$$

Proceeding similarly to the proof of Lemma 3, we get the required conclusion. We omit the details. □

Theorem 5. *If $\alpha \geq \beta$ and (1) hold, Eq. (E_2) is nonoscillatory. Suppose there exists η, $\xi \in C^1(I,\mathbb{R})$ such that $g(r,\varrho) \leq \xi(r) \leq \delta(r) \leq r$, $\xi'(r) \geq 0$ and $\eta > 0$ for $r \geq r_1$ with*

$$\limsup_{r\to\infty} \int_{r_4}^{r} \left(k_1\, \eta(s)\, \widetilde{q}(s,\varrho) - \frac{A^2(s)}{4B(s)} \right) ds = \infty \text{ for all } r_1 \in I, \tag{14}$$

where, for $r \geq r_1$,

$$A(r) = \frac{\eta'(r)}{\eta(r)} - \frac{p(r)}{l_1(\delta(r))}\Omega_2(r_1,\delta(r)) \tag{15}$$

and

$$B(r) = \beta\, \ell_1^{\beta-\alpha} \eta^{-1}(r) g'(r,d) \Big(\Omega_3^*(r_1, g(r,d))\Big)^{\beta-1} \Big(\Omega_3(r_1, g(r,d))\Big)^{1/\alpha}, \tag{16}$$

also (5) or (6) holds with $\Theta(r)$ as in Theorem 4. Then every solution of Eq.(E_1) is oscillatory.

Proof. Let Eq.(E_1) have a nonoscillatory solution $x(r)$. Assume that, there exists a $r \geq r_1$ such that $x(r) > 0$ and $x(g(r,\varrho)) > 0$ for some $r \geq r_0$. From

Lemma 1, $x(r)$ satisfies the conditions either $N(x(r)) > 0$ or $N(x(r)) < 0$ for $r \geq r_1$. If condition $N(x(r)) < 0$ holds, the proof follows from Theorem 4.

Next, if condition $N(x(r)) > 0$ holds, define

$$w(r) = \eta(r)\frac{l(x(r))}{x^{\beta}(g(r,d))}, \quad r \in I, \tag{17}$$

then $w(r) > 0$ for $r \geq r_1$. From (4) and $l'(x(r)) < 0$, we have

$$\begin{aligned} w(r) = \eta(r)\frac{l(x(r))}{x^{\beta}(g(r,d))} &\leq \eta(r)\frac{l(x(g(r,d)))}{x^{\beta}(g(r,d))} \\ &\leq \eta(r)(\Omega_3^*(r_1, g(r,d)))^{-\alpha} x^{\alpha-\beta}(g(r,d)), \end{aligned} \tag{18}$$

for $r \geq r_1$. From (3) and definition of $N(x(r))$, we find

$$\begin{aligned} x'(g(r,d)) = x'(g(r,d)) &\geq \Omega_3(r_1, g(r,d))(l(x(r)))^{1/\alpha} \\ &\geq \Omega_3(r_1, g(r,d))(l(x(g(r,d))))^{1/\alpha}. \end{aligned}$$

Then

$$\begin{aligned} \frac{x'(g(r,d))}{x(g(r,d))} &\geq \left(\frac{\Omega_3(r_1, g(r,d))}{\eta(\delta(r))}\right)^{1/\alpha} \frac{\eta^{1/\alpha}(\delta(r))(l(x(r)))^{1/\alpha}}{x^{\beta/\alpha}(g(\delta(r),d))} x^{\beta/\alpha-1}(g(\delta(r),d)) \\ &= \left(\frac{\Omega_3(r_1, g(r,d))}{\eta(r)}\right)^{1/\alpha} w^{1/\alpha}(r) x^{\beta/\alpha-1}(g(\delta(r),d)). \end{aligned} \tag{19}$$

Also, since there exists a ℓ_1 (constant) and $r_2 \geq r_1$ such that for $l(x(r)) \leq l(x(r_2)) = \ell_1^{\alpha}$, it follows that

$$x^{(n-2)}(r) \leq \ell_1\Big(\frac{1}{l_1(r)}\int_{r_2}^{r}\frac{1}{l_2(s)}ds\Big)^{1/\alpha} ds = \ell_1\Big(\frac{\Omega_2(r_2, r)}{l_1(r)}\Big)^{1/\alpha}, \tag{20}$$

and hence

$$x(r) \leq \ell_1 \Omega_3^*(r_2, r), \quad r \geq r_2. \tag{21}$$

Further,

$$x^{\beta/\alpha-1}(g(r,d)) \geq \big(\ell_1^*\big)^{\beta/\alpha-1}\big(\Omega_3^*(r_3, g(r,d))\big)^{\beta/\alpha-1}, \quad r \geq r_3. \tag{22}$$

By using (21) in (18), we obtain

$$w(r) \leq \big(\ell_1^*\big)^{\alpha-\beta}\eta(r)\,(\Omega_3^*(r_1, g(r,d)))^{-\beta}, \tag{23}$$

and hence

$$w^{\frac{1}{\alpha}-1}(r) \le \left(\ell_1^*\right)^{(\alpha-\beta)(\frac{1}{\alpha}-1)} \eta^{\frac{1}{\alpha}-1}(r) \left(\Omega_3^*(r_1, g(r,d))\right)^{-\beta(\frac{1}{\alpha}-1)}. \tag{24}$$

Now differentiating (17), we get

$$w'(r) = \frac{\eta'(r)}{\eta(r)} w(r) + \frac{L^{[4]}x(r)}{L^{[3]}x(r)} w(r) - \beta g'(r,d) \frac{x'(g(r,d))}{x(g(r,d))} w(r). \tag{25}$$

Using Eq. (E_1), (2) in (25), we have

$$\begin{aligned} w'(r) &\le \left[\frac{\eta'(r)}{\eta(r)} - \frac{p(r)}{l_1(g(r,d))}\,\Omega_2(r_3, g(r,d))\right] w(r) - k_1 \eta(r) \widetilde{q}(r,\varrho) \\ &\qquad -\beta g'(r) \frac{x'(g(r,d))}{x(g(r,d))} w(r) \\ &\le A(r) w(r) - k_1 \eta(r) \widetilde{q}(r,\varrho) - \beta g'(r) \frac{x'(g(r,d))}{x(g(r,d))} w(r). \end{aligned} \tag{26}$$

By using (19), (22) and (25) in (26), we have

$$\begin{aligned} w'(r) &\le A(r) w(r) - k_1 \eta(r) \widetilde{q}(r,\varrho) \\ &\qquad - \frac{\beta\, \ell_1^{\beta-\alpha}\, g'(r)}{\eta(r)} \Big(\Omega_3^*(r_1, g(r,d))\Big)^{\beta-1} \Big(\Omega_3(r_1, g(r,d))\Big)^{1/\alpha} w^2(r) \\ &= A(r) w(r) - k_1 \eta(r) \widetilde{q}(r,\varrho) + B(r) w^2(r) \qquad (27) \\ &= -k_1 \eta(r) \widetilde{q}(r,\varrho) + \left[\sqrt{B(r)}\, w(r) - \frac{1}{2} \frac{A(r)}{\sqrt{B(r)}}\right]^2 + \frac{1}{4} \frac{A^2(r)}{B(r)} \\ &\le -k_1 \eta(r) \widetilde{q}(r,\varrho) + \frac{1}{4} \frac{A^2(r)}{B(r)}. \qquad (28) \end{aligned}$$

Integrating (28) from $r_4(> r_3)$ to r gives

$$\int_{r_4}^{r} \left(k_1\, \eta(s)\, \widetilde{q}(s,\varrho) - \frac{1}{4} \frac{A^2(s)}{B(s)}\right) ds \le w(r_4), \tag{29}$$

which contradicts (14). ∎

Corollary 6. *Assume $\alpha \ge \beta$ and the conditions (1) hold, Eq.(E_2) is nonoscillatory. Suppose there exist η, $\xi \in C^1(I, \mathbb{R})$ such that $g(r,\varrho) \le \xi(r) \le \delta(r) \le r$, $\xi'(r) \ge 0$ and $\eta > 0$ for $r \ge r_1$ such that the function $A(r) \le 0$,*

$$\limsup_{r\to\infty} \int_{r_1}^{r} \Big(\eta(s)\, \widetilde{q}(s,\varrho)\Big) ds = \infty \text{ for all } r_1 \in I, \tag{30}$$

where $A(r)$ is defined in (15)*, also* (5) *or* (6) *holds with $\Theta(r)$ as in Theorem 4. Then every solution of Eq.(E_1) is oscillatory.*

Next, we examine the oscillation results of solutions of (E_1) of Philos-type. Let $\mathbb{D}_0 = \{(r,s) : a \leq s < r < +\infty\}$, $\mathbb{D} = \{(r,s) : a \leq s \leq r < +\infty\}$ the continuous function $H(r,s)$, $H : \mathbb{D} \to \mathbb{R}$ belong to the class function $\mathfrak{R}$, then

(i) $H(r,r) = 0$ for $r \geq r_0$ and $H(r,s) > 0$ for $(r,s) \in \mathbb{D}_0$,

(ii) H has a continuous and non-positive partial derivative on $\mathbb{D}_0$ with respect to the second variable such that

$$-\frac{\partial H(r,s)}{\partial s} = h(r,s)[H(r,s)]^{1/2},$$

for all $(r,s) \in \mathbb{D}_0$.

Theorem 7. *Assume $\alpha \geq 1$ and the conditions* (1) *hold, Eq. (E_2) is nonoscillatory. Suppose there exists η, $\xi \in C^1(I,\mathbb{R})$ such that $g(r,\varrho) \leq \xi(r) \leq \delta(r) \leq r$, $\xi'(r) \geq 0$, $\eta > 0$ and $H(r,s) \in \mathfrak{R}$ for $r \geq r_1$ with*

$$\limsup_{r\to\infty} \frac{1}{H(r,r_4)} \int_{r_4}^{r} \Bigg(k_1\, \eta(s)\, \widetilde{q}(s,\varrho) H(r,s) - \frac{\Big[h(r,s) - A(s)\sqrt{H(r,s)}A(s)\Big]^2}{4\,B(s)} \Bigg) ds = \infty, \tag{31}$$

for all $r_4 \in I$, where $A(r)$, $B(r)$ is defined in Theorem 5, also (5) *or* (6) *holds with $\Theta(r)$ as in Theorem 4. Then every solution of Eq.(E_1) is oscillatory.*

Proof. Let Eq.(E_1) have a nonoscillatory solution $x(r)$. Assume that, there exists a $r \geq r_1$ such that $x(r) > 0$ and $x(g(r,\varrho)) > 0$ for some $r \geq r_0$. Proceeding as in the proof of Theorem 5, we obtain the inequality (27), i.e.,

$$w'(r) \;\leq\; A(r)w(r) - k_1\eta(r)\widetilde{q}(r,\varrho) + B(r)w^2(r),$$

and so,

$$\int_{r_4}^{r} H(r,s)\eta(s)\widetilde{q}(s,\varrho)ds \;\leq\; \int_{r_4}^{r} H(r,s)[-w'(s)+A(s)w(s)-B(s)w^2(s)]ds$$

$$\begin{aligned}
&= -H(r,s)\Big[w(s)\Big]_{r_4}^{r} + \int_{r_4}^{r} \Big[\frac{\partial H(r,s)}{\partial s} w(s) \\
&\quad + H(r,s)\Big[A(s)w(s) - B(s)w^2(s)\Big]\Big] ds \\
&= H(r,r_4)w(r_4) - \int_{r_4}^{r} \Big[w^2(s)B(s)H(r,s) \\
&\quad + w(s)\Big(h(r,s)\sqrt{H(r,s)} - H(r,s)A(s)\Big)\Big] ds \\
&\le H(r,r_4)w(r_4) + \int_{r_4}^{r} \frac{P^2(r,s)}{4B(s)} ds,
\end{aligned}$$

which contradicts to (31). The rest of the proof is similar to that of Theorem 5 and hence is omitted. □

Corollary 8. *Suppose that all conditions of Theorem 7 are satisfied with (31) replaced by*

$$\limsup_{r\to\infty} \frac{1}{H(r,r_4)} \int_{r_4}^{r} k_1\, H(r,s)\eta(s)\, \widetilde{q}(s,\varrho) ds = \infty$$

and

$$\limsup_{r\to\infty} \frac{1}{H(r,r_4)} \int_{r_4}^{r} \frac{\Big[h(r,s) - A(s)\sqrt{H(r,s)}A(s)\Big]^2}{4\,B(s)} ds < \infty,$$

then Eq.(E_1) is oscillatory.

Below, we present an example to show application of the main results.

Example 9. For $r \ge 1/2$, consider even order differential equation

$$\big(1/2r\big(9e^{-r}(r)\big(x''(r)\big)\big)'\big)' + 36e^{-s/2}x^{(ii)}(r/2) + \int_1^2 \frac{r}{3}x(\varrho, 36e^{r/3})d\varrho = 0. \qquad (32)$$

Here $l_1 = \frac{1}{(2r-1)^{3/2}}$, $l_2 = \frac{1}{2r-1}$, $\alpha = 3/2$, $\beta = 1$, $p(r) = \frac{(2r-1)^{3/2}}{t^2}$, $q(r,\varrho) = r/3$ and $\delta(r) = r$, $g(r,\varrho) = r/3$. Now pick $\eta(r) = r$, we obtain $\Omega_1(r_1,r) = r(r-1)$, $\Omega_2(r_1,r) = r(r-1)$, $\Omega_2(r_1,r) = r(r-1)$, $A(r) = \frac{2-r}{r}$ and $B(r) = \frac{(t-3)^{n-3}}{3^{n-2}(n-3)(n-4)!}$ Take $r_2 = 4$, we get

$$\limsup_{r\to\infty} \int_4^r \left(k_1\, \eta(s)\,(s/3) - \frac{A^2(s)}{4B(s)}\right) ds \to \infty \text{ as } r \to \infty,$$

and all hypotheses of Theorem 5 are satisfied, so every solution of (32) is oscillatory.

4. Conclusions

It is clear that the form of the problem Eq. (E_1) is more general than all the problems considered in this study. In this paper, using the suitable Riccati type transformation, integral averaging condition, and comparison method, we offer some oscillatory properties which ensure that any solution of Eq. (E_1) oscillates under assumption of $\Omega_1(r_1, r) = \infty$, $\Omega_2(r_1, r) = \infty$ as $r \to \infty$. Also, it would be useful to extend oscillation criteria of Eq.(E_1) under the condition of $\Omega_1(r_1, r) < \infty$, $\Omega_2(r_1, r) < \infty$ as $r \to \infty$.

References

[1] R.P. Agarwal, S.R. Grace, The oscillation of higher-order differential equations with deviating arguments. *Comput. Math. Appl.*, **38** (1999), 185-199; https://doi.org/10.1016/S0898-1221(99)00193-5.

[2] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation theory for second order dynamic equations. In: *Mathematical Analysis Applications*, Vol. 5. Taylor and Francis, London (2003).

[3] R.P. Agarwal, S.R. Grace, D. O'Regan, Oscillation of certain fourth order functional differential equations, *Ukrain. Mat. Zh.*, **59** (2007), 315-342; https://doi.org/10.1007/s11253-007-0021-4.

[4] R.P. Agarwal, S.R. Grace, P.J. Wong, On the bounded oscillation of certain fourth order functional differential equations, *Nonlinear Dyn. Syst. Theory*, **5** (2005), 215-227.

[5] M. Bartušek, M. Cecchi, Z. Došlá, M. Marini, Fourth-order differential equation with deviating argument, *Abstr. Appl. Anal.*, **2012**, ID 185242; https://doi.org/10.1155/2012/185242.

[6] M. Bartušek, Z. Došlá, Oscillatory solutions of nonlinear fourth order differential equations with a middle term. *Electron. J. Qual. Theory Differ. Equ.*, **2014**, No. 55, 1-9.

[7] M. Sathish Kumar, V. Ganesan, Oscillatory behavior of solutions of certain third-order neutral differential equation with continuously distributed delay, *Journal of Physics: Conference Series*, **1850**, No 1 (2021), ID 012091; https://doi.org/10.1088/1742-6596/1850/1/012091.

[8] J. Dzurina, B. Baculíková, I. Jadlovská, New oscillation results to fourth order delay differential equations with damping, *Electron. J. Qual. Theory Differ. Equ.*, **2016** (7), 1-15.

[9] S.R. Grace, S. Sun, E. Akin, Oscillation criteria for certain fourth-order nonlinear delay differential equations, *Mediterr. J. Math.*, **13** (2016), 2383-2396; https://doi.org/10.1007/s00009-015-0630-3.

[10] B. Almarri, S. Janaki, V. Ganesan, AH Ali, K. Nonlaopon, O. Bazighifan, Novel oscillation theorems and symmetric properties of nonlinear delay differential equations of fourth-order with a middle term, *Symmetry*, **14**, No 3 (2022), 585; https://doi.org/10.3390/sym14030585.

[11] S.R. Grace, Oscillation criteria for third-order nonlinear delay differential equations with damping, *Opuscula Math.*, **35**, No 4 (2015), 485-497; http://dx.doi.org/10.7494/OpMath.2015.35.4.485.

[12] S.R. Grace, J. Džurina, I. Jadlovská, T. Li, On the oscillation of fourth-order delay differential equations, *Adv. Difference Equ.*, **2019**, No 1, p. 118; https://doi.org/10.1186/s13662-019-2060-1.

[13] V. Ganesan, S. Janaki, M. Sathish Kumar, Oscillation of the even-order nonlinear neutral differential equations, *International Journal of Applied Mathematics*, **31**, No 6 (2018), 821-831; http://dx.doi.org/10.12732/ijam.v31i6.11.

[14] C. Hou, SS. Cheng, Asymptotic dichotomy in a class of fourth-order nonlinear delay differential equations with damping, *Abstr. Appl. Anal.*, **2009**, Art. ID 484158; https://doi.org/10.1155/2009/484158.

[15] M. Sathish Kumar, O. Bazighifan, K. Al-Shaqsi, F. Wannalookkhee, K. Nonlaopon, Symmetry and its role in oscillation of solutions of third-order differential equations, *Symmetry*, **13**, No 8, ID 1485; https://doi.org/10.3390/sym13081485.

[16] C.G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math. (Basel)*, **36** (1981), 168-178.

[17] M. Sathish Kumar, S. Janaki, V. Ganesan, Some new oscillatory behavior of certain third-order nonlinear neutral differential equations of mixed type, *International Journal of Applied and Computational Mathematics*, **78** (2018), 1-14; https://doi.org/10.1007/s40819-018-0508-8.

[18] M. Sathish Kumar, V. Ganesan, Asymptotic behavior of solutions of third-order neutral differential equations with discrete and distributed delay, *AIMS Mathematics*, **5**, No 4, (2020), 3851-3874; https://doi.org/10.3934/math.2020250.

[19] M. Sathish Kumar, O. Bazighifan, A. Almutairi, D.N. Chalishajar, Philos-type oscillation results for third-order differential equation with mixed neutral terms, *Mathematics*, **9** (2021), ID 1021; https://doi.org/10.3390/math9091021.

[20] A. Tiryaki, M.F. Aktas, Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping, *J. Math. Anal. Appl.*, **325** (2007), 54-68; https://doi.org/10.1016/j.jmaa.2006.01.001.

[21] S.R. Grace, A. Zafer, Oscillation criteria for nth-order nonlinear delay differential equations with a middle term, *Math. Methods Appl. Sci.*, **5**, No 39, 1150-1158; https://doi.org/10.1002/mma.3559.

[22] S.R. Grace, I. Jadlovska, A. Zafer, Oscillation criteria for odd-order nonlinear delay differential equations with a middle term, *Math. Methods Appl. Sci.*, **40**, No 14, 5147-5160; http://dx.doi.org/10.1002/mma.4377.

[23] C.S. Varun Bose, R. Udhayakumar, A.M. Elshenhab, M.S. Kumar, J.S. Ro, Discussion on the approximate controllability of Hilfer fractional neutral Integro-Differential inclusions via almost sectorial operators, *Fractal and Fractional*, **6**, No 10 (2022), ID 607; https://doi.org/10.3390/fractalfract6100607.