International Journal of Applied Mathematics

Volume 36 No. 3 2023, 405-424
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v36i3.7

PROPERTIES OF SOLUTIONS FOR A NONLINEAR DIFFUSION PROBLEM WITH A GRADIENT NONLINEARITY

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Abstract: This paper studies the properties of solutions for a nonlinear diffusion problem with a gradient nonlinearity. The problem is formulated as a partial differential equation with a nonlinear term that depends on both the solution and its gradient. The main results are: existence and uniqueness of weak solutions in suitable function spaces; regularity and positivity of solutions; asymptotic behavior of solutions as time goes to infinity; comparison principles and maximum principles for solutions. The proofs are based on variational methods, fixed point arguments, energy estimates, and comparison techniques. Some examples and applications are also given to illustrate the features of the problem.

AMS Subject Classification: 35A01, 35B44, 35K57, 35K65
Key Words: critical global existence; degenerate parabolic equation; critical Fujita exponent; nonlinear boundary flux; blow-up

1. Introduction

In this article, we deal with the following doubly degenerate parabolic equations

Received: March 23, 2023

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with the damping

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial u^k}{\partial x}\right|^{m-1}\frac{\partial u^k}{\partial x}\right)+\left|\frac{\partial u}{\partial x}\right|^p, \qquad (t,x)\in Q, \tag{1}$$

coupled through nonlinear boundary flux and the initial data

$$-\left|\frac{\partial u^k}{\partial x}\right|^{m-1}\frac{\partial u^k}{\partial x}\Bigg|_{x=0}=u^q(0,t), \qquad t\in Q_t, \tag{2}$$

$$u|_{t=0}=u_0(x)\geq 0, \qquad x\in Q_x, \tag{3}$$

where $m>1$, $k\geq 1$ and $q>0$ are given parameters and $Q_t=\{t|t>0\}$, $Q_x=\{x|x\in R_+\}$, $Q=Q_t\times Q_x$.

The problem (1) arises in different applications (see [9]-[16] and references therein). Equation (1) is of degenerate type. Therefore, in the domain Q, where $u=0, \nabla u=0$ it is a degenerate type. Therefore, in this case, we need to consider a weak solution from having a physical sense class.

The problem (1), for the particular values of numerical parameters, is intensively studied by many authors (see [15]-[16] and literature therein). Self-similar solutions to this problem are based on investigating qualitative properties of the problem such as Fujita type global solvability, asymptotic solution, localization of solution, finite speed propagation of distribution, blow-up solution, and so on by many authors (for example, see [1]-[11] and literature therein).

The problem (1) has been intensively studied by many authors (see [4]-[17] and references therein) for various values of numerical parameters. In particular, Keng Deng and H.A. Levine studied (1) the p-Laplacian case and they investigated the local and global existence, also the global nonexistence of a solution to the Cauchy problem [2].

V.A. Galaktionov and H.A. Levine studied (1) in the cases: $k=m=1$, the p-Laplacian and $k=1$. They have proved that if $\frac{2m}{m+1}<q<2m$, then all solutions of the problem (1) become unbounded in finite time [3]. They also found that the solutions of problem (1) have the following properties:

- if $0<q\leq\frac{2m}{m+1}$, then global solution of problem (1) exists;
- if $q>2m$, then problem (1) admits nontrivial global solutions with small initial,

where $q_c = 2m$ is the critical Fujita exponent and $q_0 = \dfrac{2m}{m+1}$ is the critical global existence exponent.

The author of the work [9] studied the following problem

$$\begin{cases} (1+|x|)^n u_t = \nabla\left(|\nabla u^m|^{p-2}\nabla u^m\right), \qquad (x,t) \in R_+^N \times (0,\infty), \\ -|\nabla u^m|^{p-2}\dfrac{\partial u^m}{\partial x_1}(0,t) = u^q,\ x_i = 0,\ i = \overline{2,n},\ t > 0, \\ u(x,0) = u_0(x) \ge 0, \quad x \in R_+^N, \end{cases} \tag{4}$$

where $R_+^N = \{(x_1, x') : x' \in R^{N-1},\ x_1 > 0\}$, $n > -p$, $m > 1$, $q > 0$, $1 < p < 1 + \dfrac{1}{m}$.

It is shown that when $0 \le q \le \dfrac{(m(n+1)+1)(p-1)}{p+n}$, each solution of problem (4) is global in time and when $q > m(p-1) + \dfrac{p-1}{N+n}$, (4) has nontrivial global solutions with small initial data.

Yongsheng Mi, Chunlai Mu, and Rong Zeng investigated [13] the equation below

$$u_t = div\left(|\nabla u|^p \nabla u^m\right) + u^q, \tag{5}$$

where $p > 0$, $m, q > 1$.

They proved that, if $p > 0$, $N \ge 2$, $m > 1$, $q > q_c = m + p + \dfrac{p+2}{N}$, then of the Cauchy problem (5) blows up in finite time and investigated the large time behavior and the life spans of solutions and the secondary critical exponent to Cauchy problem.

The authors of the work [14] studied the following problem

$$\begin{cases} |x|^{-n}\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u}{\partial x}\right|^{p-2}\dfrac{\partial u}{\partial x}\right) + |x|^{-n}u^\beta, \ \ (x,t) \in \mathbb{R}_+ \times (0,+\infty), \\ -\left|\dfrac{\partial u}{\partial x}\right|^{p-2}\dfrac{\partial u}{\partial x}(0,t) \ = \ u^q(0,t), \ \ t > 0, \\ u(x,0) = u_0(x) \ge 0, \ \ x \in \mathbb{R}_+, \end{cases} \tag{6}$$

where $p > 2$, $\beta, q > 0$, $n \in \mathbb{R}_+$, $u_0(x)$ - is a bounded, continuous, nonnegative and nontrivial initial data. They showed that when $0 < \beta \le 1$ and $0 < q \le \dfrac{(2-n)(p-1)}{p-n}$, then each solution of problem (6) is global in time.

Furthermore, these authors established the critical Fujita exponent and analyzed the self-similar solution to Neumann problem (6).

Zhiyong Wang and Jingxue Yin studied in [17] the Hamilton–Jacobi equation (1) in the case $k = 1$, and they established a gradient blow-up solution with a small L^1 initial datum when $q > m - 1 > 2$.

2. The main results

The main aim of this paper is to establish the conditions of blow-up, global existence, and nonexistence of solutions to the Neumann problem. Also, to analyze the asymptotics of the solution under some conditions.

Theorem 1. *If* $k \geq 1 + \dfrac{2}{m}$ *and* $q \leq \dfrac{m(k+1)}{m+1}$ *inequalities hold, then each solution of problem* (1)-(3) *is global in time.*

Proof. We look for a globally defined in time supersolution of the following self-similar form

$$\bar{u}(x,t) = e^{\lambda_1 t}\left(M + e^{-Lxe^{-\lambda_2 t}}\right)^{\frac{1}{k}}, \ x,t \in R_+, \tag{7}$$

where $\lambda_i > 0$, $L > 0$, $i = 1,2$; $M = \|u_0\|_\infty^k + 1$.

After computation we have

$$\frac{\partial \bar{u}}{\partial t} = \lambda_1 e^{\lambda_1 t}\left(M + e^{-Lxe^{-\lambda_2 t}}\right)^{\frac{1}{k}} + \frac{\lambda_2 L}{k} x e^{(\lambda_1-\lambda_2)t}\left(M + e^{-Lxe^{-\lambda_2 t}}\right)^{\frac{1}{k}-1}$$
$$\times e^{-Lxe^{-\lambda_2 t}} \geq \lambda_1 e^{\lambda_1 t}\left(M + e^{-Lxe^{-\lambda_2 t}}\right)^{\frac{1}{k}} \geq \lambda_1 M^{\frac{1}{k}} e^{\lambda_1 t}.$$

Hence

$$\frac{\partial \bar{u}}{\partial t} \geq \lambda_1 M^{\frac{1}{k}} e^{\lambda_1 t}, \tag{8}$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}^k}{\partial x}\right|^{m-1}\frac{\partial \bar{u}^k}{\partial x}\right) = mL^{m+1} e^{(k\lambda_1 m - \lambda_2(m+1))t} e^{-mLxe^{-\lambda_2 t}}$$
$$\leq mL^{m+1} e^{(k\lambda_1 m - \lambda_2(m+1))t}, \tag{9}$$

$$\begin{aligned}\left|\frac{\partial\bar{u}}{\partial x}\right|^p &= e^{p\lambda_1 t}\left|-\frac{L}{k}e^{-\lambda_2 t}\left(M+e^{-Lxe^{-\lambda_2 t}}\right)^{\frac{1}{k}}\right|^p \\ &\le \left(\frac{L}{k}\right)^p (M+1)^{\frac{p}{k}}\, e^{(\lambda_1-\lambda_2)pt},\end{aligned} \tag{10}$$

$$-\left|\frac{\partial\bar{u}^k}{\partial x}\right|^{m-1}\frac{\partial\bar{u}^k}{\partial x}\Bigg|_{x=0} = L^m e^{(k\lambda_1-\lambda_2)mt} = \bar{u}^q(0,t) = (M+1)^{\frac{q}{k}}\, e^{q\lambda_1 t}. \tag{11}$$

Now, we will show that the function $\bar{u}(x,t)$ is a supersolution of problem (1)-(3). According to the comparison principle, it must satisfy the following inequality:

$$\frac{\partial\bar{u}}{\partial t} \ge \frac{\partial}{\partial x}\left(\left|\frac{\partial\bar{u}^k}{\partial x}\right|^{m-1}\frac{\partial\bar{u}^k}{\partial x}\right) + \left|\frac{\partial\bar{u}}{\partial x}\right|^p. \tag{12}$$

From (8)-(11), we obtain the following system

$$\begin{cases}\lambda_1 M^{\frac{1}{k}} e^{\lambda_1 t} \ge mL^{m+1}e^{(k\lambda_1 m-\lambda_2(m+1))t} + \left(\frac{L}{k}\right)^p (M+1)^{\frac{p}{k}}\, e^{(\lambda_1-\lambda_2)pt}, \\ L^m e^{(k\lambda_1-\lambda_2)mt} = (M+1)^{\frac{q}{k}}\, e^{q\lambda_1 t}.\end{cases} \tag{13}$$

The last expression brings the following:

$$\lambda_2 = \frac{\lambda_1(km-2)}{m}, \;\; L = (M+1)^{\frac{q}{mk}}.$$

Substituting the above into inequality in (13), we achieve:

$$\lambda_1 \ge k\lambda_1 m - (m+1)\lambda_2 + (\lambda_1-\lambda_2)\,p = \lambda_1\,(mk+p) - \lambda_2\,(m+p+1)\,.$$

Computation of this inequality gives us:

$$q \le \frac{m(k(p+1)+1-p)}{m+p+1},$$

$$\lambda_1 = M^{-\frac{1}{k}}\left(mL^{m+1} + \frac{L^{p\left(\frac{m}{q}+1\right)}}{k^p}\right).$$

Hence, $\bar{u}\,(x,0) \ge u_0\,(x)$. Thus, by the comparison principle, Theorem 1 is proved. □

Remark 1. Theorem 1 shows that the critical global existence exponent of the problem (1)-(3) is $q = \dfrac{m(k(p+1)+1-p)}{m+p+1}$.

Theorem 2. *If* $0 < p < \dfrac{m+1}{2} < \dfrac{k}{m}$ *and* $q \geq \dfrac{m(k(p+1)+1-p)}{m+p+1}$ *inequality holds, then every solution of problem* (1)-(3) *blows up in finite time.*

Proof. We will seek a blow up subsolution of the self-similar form:

$$\underline{u}(x,t) = t^{\alpha_1}\varphi(\eta), \quad \eta = xt^{-\alpha_2}. \tag{14}$$

And we need to evaluate the following derivatives:

$$\frac{\partial \underline{u}}{\partial t} = t^{\alpha_1 - 1}\left[\alpha_1\varphi - \alpha_2\eta\varphi_\eta\right],$$

$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \underline{u}^k}{\partial x}\right|^{m-1}\frac{\partial \underline{u}^k}{\partial x}\right) = t^{\alpha_1 mk-(m+1)\alpha_2}\left(\left|\varphi_\eta^k\right|^{m-1}\varphi_\eta^k\right)_\eta,$$

$$\left|\frac{\partial \underline{u}}{\partial x}\right|^p = t^{(\alpha_1-\alpha_2)p}\left|\varphi_\eta\right|^p.$$

We choose $\alpha_{1,2}$ as follows:

$$\alpha_1 - 1 = \alpha_1 mk - (m+1), \quad \alpha_2 = (\alpha_1 - \alpha_2)\,p.$$

Hence, we find $\alpha_{1,2}$ the following form

$$\begin{aligned} \alpha_1 &= \frac{m+1-p}{mk-1+(p-1)(m(k-1)-2}, \\ \alpha_2 &= \frac{mk-p}{mk-1+(p-1)(m(k-1)-2)}. \end{aligned} \tag{15}$$

Also, we have a boundary flux as follows:

$$t^{\alpha_1 mk - m\alpha_2}\left|\varphi_\eta^k\right|^{m-1}\varphi_\eta^k\Big|_{\eta=0} = t^{\alpha_1 q}\varphi^q(0),$$

$$0 \leq \alpha_1 q - \alpha_1 mk + m\alpha_2 = \frac{(m+1-p)(q-mk)+m(mk-p)}{mk-1+(p-1)(m(k-1)-2)}.$$

It is easy to see that, $1+\dfrac{2}{m}-\dfrac{m+1}{mp}<k<1+\dfrac{2}{m}$, $0<p<\dfrac{m+1}{2}$ and

$$q \geq \frac{m(k(p+1)+1-p)}{m+p+1} \geq \frac{m[1-(k-1)(p-1)]}{m+1-p}.$$

Hence, $\forall t \geq 1$: $\left|\varphi_\eta^k\right|^{m-1}\varphi_\eta^k\Big|_{\eta=0} \leq \varphi^q(0)$

$$\left(\left|\varphi_\eta^k\right|^{m-1}\varphi_\eta^k\right)_\eta + \alpha_2\eta\varphi_\eta - \alpha_1\varphi + |\varphi_\eta|^p \geq 0. \tag{16}$$

We set:

$$\varphi(\eta) = A\left(a-\eta^{\frac{m+1}{m}}\right)_+^{\frac{m}{mk-1}}, \tag{17}$$

where A, a are constants to be determined. It is easy to see that

$$\left(\left|\varphi_\eta^k\right|^{m-1}\varphi_\eta^k\right)_\eta + \alpha_2\eta\varphi_\eta - \alpha_1\varphi + |\varphi_\eta|^p = A\left(a-\eta^{\frac{m+1}{m}}\right)_+^{\frac{m}{mk-1}-1}$$
$$\times\left[A^{mk-1}\left(\frac{k(m+1)}{mk-1}\right)^m\frac{m+1}{mk-1}\eta^{\frac{m+1}{m}} - A^{mk-1}\left(\frac{k(m+1)}{mk-1}\right)^m\left(a-\eta^{\frac{m+1}{m}}\right)\right.$$
$$-\alpha_2\frac{m+1}{mk-1}\eta^{\frac{m+1}{m}} - \alpha_1\left(a-\eta^{\frac{m+1}{m}}\right)$$
$$\left.+A^{p-1}\left(\frac{m+1}{mk-1}\right)^p\eta^{\frac{p}{m}}\left(a-\eta^{\frac{m+1}{m}}\right)_+^{\left(\frac{m}{mk-1}-1\right)(p-1)}\right] \geq 0,$$

$$A^{mk-1}\left(\frac{k(m+1)}{mk-1}\right)^m\frac{m(k+1)}{mk-1}\eta^{\frac{m+1}{m}} - A^{mk-1}\left(\frac{k(m+1)}{mk-1}\right)^m a$$
$$+\left(\alpha_1-\frac{(m+1)\,\alpha_2}{mk-1}\right)\eta^{\frac{m+1}{m}} - a\alpha_1$$
$$+A^{p-1}\left(\frac{m+1}{mk-1}\right)^p\eta^{\frac{p}{m}}\left(a-\eta^{\frac{m+1}{m}}\right)_+^{\left(\frac{m}{mk-1}-1\right)(p-1)} \geq 0,$$

$$\left[A^{mk-1}\left(\frac{k(m+1)}{mk-1}\right)^m\frac{m(k+1)}{mk-1}+\alpha_1-\frac{(m+1)\,\alpha_2}{mk-1}\right]\eta^{\frac{m+1}{m}}$$

$$+ A^{p-1}\left(\frac{m+1}{mk-1}\right)^{p}\eta^{\frac{p}{m}}\times\left(a-\eta^{\frac{m+1}{m}}\right)_{+}^{\left(\frac{m}{mk-1}-1\right)(p-1)}$$
$$\geq\left[A^{mk-1}\left(\frac{k(m+1)}{mk-1}\right)^{m}\frac{m(k+1)}{mk-1}+\alpha_1-\frac{(m+1)\,\alpha_2}{mk-1}\right]\eta^{\frac{m+1}{m}}$$
$$\geq\left(A^{mk-1}\left(\frac{k(m+1)}{mk-1}\right)^{m}\frac{m(k+1)}{mk-1}+\alpha_1\right)a.$$

Since, $0<p<\dfrac{m+1}{2}<\dfrac{k}{m}$, the last inequality holds. Thus $\underline{u}(x,t)$ is a subsolution of the problem (1)-(3) with every nontrivial initial data. □

Theorem 3. *If* $p>\dfrac{m+1}{2}$ *and* $q\leq\dfrac{m\,(k\,(p+1)+1-p)}{m+p+1}$, *then every solution of problem* (1)-(3) *blows up in time.*

Proof. In this case, we prove that the flux condition makes the solution large enough to be in the set of initial data for which the reaction term alone is enough to cause blows up. We consider the self-similar subsolution of the problem (1)-(3) without a source:

$$u_b\,(x,t)=t^{\mu_1}g\,(\xi)\,,\ \ \xi=xt^{-\mu_2}. \tag{18}$$

And we need to evaluate the following derivatives

$$\frac{\partial u_b}{\partial t}=t^{\mu_1-1}\left[\mu_1 g-\mu_2\xi g_\xi\right],$$
$$\frac{\partial}{\partial x}\left(\left|\frac{\partial u_b^k}{\partial x}\right|^{m-1}\frac{\partial u_b^k}{\partial x}\right)=t^{mk\mu_1-(m+1)\mu_2}\left(\left|g_\xi^k\right|^{m-1}g_\xi^k\right)_\xi,$$
$$-\left|\frac{\partial u_b^k}{\partial x}\right|^{m-1}\frac{\partial u_b^k}{\partial x}=t^{mk\mu_1-m\mu_2}\left|g_\xi^k\right|^{m-1}g_\xi^k\,(0)\leq u_b^q\,(0,t)=t^{\mu_1 q}g^q\,(0)\,.$$

And we choose $\mu_{1,2}$ as follows:

$$\begin{cases}\mu_1-1=mk\mu_1-(m+1)\,\mu_2\\ mk\mu_1-m\mu_2=\mu_1 q=\mu_1+\mu_2-1\end{cases}\Rightarrow\begin{cases}\mu_1=\dfrac{m}{m\,(k+1-q)-q}\\ \mu_2=\dfrac{mk-q}{m\,(k+1-q)-q}\end{cases}$$

$$\left(\left|g_\xi^k\right|^{m-1} g_\xi^k\right)_\xi + \mu_2 \xi g_\xi - \mu_1 g \geq 0, \tag{19}$$

$$-\left|g_\xi^k\right|^{m-1} g_\xi^k (0) \leq g^q (0) . \tag{20}$$

Let us construct

$$g(\xi) = B (b - \xi)_+^{\frac{m}{mk-1}},$$

where b and B are positive constants to be determined. After some evaluations, we have

$$\left(\left|g_\xi^k\right|^{m-1} g_\xi^k\right)_\xi = B^{mk} \left(\frac{mk}{mk-1}\right)^m \frac{m}{mk-1} (b-\xi)_+^{\frac{m}{mk-1}-1} .$$

And we note that

$$\mu_2 \xi g_\xi - \mu_1 g = -\frac{\mu_2 m}{mk-1} B \xi (b-\xi)_+^{\frac{m}{mk-1}-1} - \mu_1 B (b-\xi)_+^{\frac{m}{mk-1}}$$
$$= -B (b-\xi)_+^{\frac{m}{mk-1}} \left[\frac{\mu_2 m}{mk-1} b + \mu_1 b\right] = -bB \left[\mu_1 + \frac{\mu_2 m}{mk-1}\right] (b-\xi)_+^{\frac{m}{mk-1}} .$$

Take

$$B^{mk-1} k^m \left(\frac{m}{mk-1}\right)^{m+1} \geq b \left(\mu_1 + \frac{\mu_2 m}{mk-1}\right).$$

On the another hand,

$$B^{mk-q} \left(\frac{mk}{mk-1}\right)^m \leq b^{\frac{m(q-1)}{mk-1}} \quad \text{and} \quad q \leq \frac{m\left(k(p+1)+1-p\right)}{m+p+1},$$

it is easy to check that (19) and (20) are valid. It follows from the comparison principle that for the problem (1)-(3) there exists as a solution blowing up in a finite time. □

Theorem 4. $1 \leq p,\ q \leq mk(p+1)$, *then every solution of the problem* (1)-(3) *is blow-up in finite time.*

Proof. Theorem 4 can be proved in the same manner as it was done in [9]. □

3. Global existence

Based on a modification of the energy methods, comparison principle, and regularization methods used in [1, 11], we investigate the secondary critical exponent for the Cauchy problem (1). Before stating the results of the secondary critical exponent, we start with some notations as follows.

Let $C_b(R_+)$ be the space of all bounded continuous functions in R_+. For $a \geq 0$, we define

$$F^a = \left\{ \varphi(x) \in C_b(R_+) \, : \, \varphi(x) \geq 0, \, \lim_{|x| \to \infty} \sup |x|^a \varphi(x) < \infty \right\}. \tag{21}$$

We denote

$$p_c = m(k+1) + 1, \qquad a_c = \frac{m+1-p}{p-km}. \tag{22}$$

Theorem 5. *For $k > 1$, $\frac{1}{k} < m < \frac{1}{k-1}$ and $p > p_c = m(k+1)+1$, suppose that $u_0(x) = \mu\varphi(x)$ for some $\mu > 0$ and $\varphi(x) \in F^a$ for some $a \in (a_c, 1)$, then there is $\mu_0 = \mu_0(\varphi) > 0$ such that the solution $u(x,t)$ of the Cauchy problem* (1) *exists globally for all $t > 0$ and $\mu < \mu_0$ one has*

$$\|u(x,t)\|_\infty \leq Ct^{-a\lambda}, \ \ \forall t > 0, \tag{23}$$

where $\lambda = \dfrac{1}{m+1+a(km-1)}$, $C = const. > 0$.

Proof. We prove Theorem 5 by constructing a global supersolution. To do this, we introduce the self-similar solution $U_{M,a}(x,t)$ to the following Cauchy problem:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left| \frac{\partial u^k}{\partial x} \right|^{m-1} \frac{\partial u^k}{\partial x} \right), \qquad (t,x) \in Q, \tag{24}$$

$$u(x,0) = u_0(x) = M|x|^{-a}, \qquad x \in R. \tag{25}$$

It is well known that the existence and uniqueness of the solution of (24) have been established [4]. By the symmetric properties of (14), the solution $U_{M,a}(x,t)$ is given in the following form

$$U_{M,a}(x,t) = t^{-a\lambda} g_M(r), \qquad r = |x|t^{-\lambda}, \tag{26}$$

where the positive function g_M is the solution of the problem

$$\begin{gathered}\left(\left|\left(g_M^k\right)_r\right|^{m-1}\left(g_M^k\right)_r\right)_r+\lambda r\left(g_M\right)_r+a\lambda g_M\left(r\right)=0, \quad r>0, \\ g_M\left(r\right)\geq 0,\ r\geq 0,\ \left(g_M\right)_r\left(0\right)=0,\ \lim_{r\to+\infty} r^a g_M(r)=M.\end{gathered} \tag{27}$$

We prove the existence of solution $g_M\left(r\right)$ to (27) by the following ordinary differential equation(ODE) and moreover, we obtain the non-increasing property of the solution $g_M\left(r\right)$.

Initially, given a fixed $\delta>0$, we consider the following Cauchy problem:

$$\begin{gathered}\left(\left|\left(z^k\right)_r\right|^{m-1}\left(z^k\right)_r\right)_r+\lambda r z_r+a\lambda z\left(r\right)=0, \qquad r>0, \\ z\left(0\right)=\delta, \qquad z_r\left(0\right)=0.\end{gathered} \tag{28}$$

According to the standard of the Cauchy problem for ODE and the methods used in [4], we can obtain that the solution $z\left(r\right)$ of the Cauchy problem (28) is positive and $z\overset{r\to\infty}{\longrightarrow}0$, moreover

$$\lim_{r\to+\infty} r^a z(r)=M, \tag{29}$$

for some $M=M\left(\delta\right)>0$.

Secondly, we prove that there exists a one-to-one correspondence between $M\in(0,+\infty)$ and $\delta\in(0,+\infty)$. Indeed, this can be seen from the following relation:

$$z_\delta\left(r\right)=\delta z_1\left(\delta^l r\right), \qquad l=\frac{m+1}{1-km}, \tag{30}$$

where $z_1\left(r\right)$ is the solution of (28) for $\delta=1$. Then,

$$M\left(\delta\right)=\delta^{1-al}M\left(1\right) \text{ with } M\left(1\right)=\lim_{r\to\infty} r^a z_1\left(r\right). \tag{31}$$

Therefore, we can deduce that, for each $M>0$, there exists a positive, bounded, and global solution g_M satisfying (27).

Eventually, we prove that the solution $z\left(r\right)$ is non-increasing, that is, g_M is also non-increasing. For this, the following lemmas are necessary.

Lemma 1. Let $z\left(r\right)$ be the solution of (28), then

$$\lim_{r\to 0}\frac{\left|\left(z^k\right)_r\right|^{m-1}\left(z^k\right)_r}{r}=-a\lambda z\left(0\right). \tag{32}$$

Proof. Integrating the (28) over $(0, \epsilon)$ with $\epsilon > 0$, we have

$$\left(\left| \left(z^k \right)_r \right|^{m-1} \left(z^k \right)_r \right) (\epsilon) + \lambda \int_0^\epsilon r z_r dr + a\lambda \int_0^\epsilon z dr = 0. \quad (33)$$

Dividing by ϵ and letting $\epsilon \to 0$ in (33), we obtain

$$\lim_{\epsilon \to 0} \frac{\left(\left| \left(z^k \right)_r \right|^{m-1} \left(z^k \right)_r \right) (\epsilon)}{\epsilon} = -a\lambda \lim_{\epsilon \to 0} z (\epsilon), \quad (34)$$

which implies that (32) holds. The proof of Lemma 1 is complete. □

Lemma 2. If there exists $r_0 \in [0, +\infty)$ such that $z(r_0) = 0$, then $z(r) = 0$ for all $r \geq r_0$.

Proof. We prove this by contradiction. Assuming that Lemma 2 does not hold, it is easy to see that exists $(0, \epsilon)$ such that

$$z(r) > 0, \qquad z'(r) > 0 \quad \text{in } (r_0, r_0 + \epsilon). \quad (35)$$

Integrating (28) over (r_0, r) with $r \in (r_0, r_0 + \epsilon)$, we obtain

$$\left| \left(z^k \right)_r \right|^{m-1} \left(z^k \right)_r + \lambda r z(r) = \lambda (1 - a) \int_{r_0}^r z(r) dr. \quad (36)$$

It follows from (35) and (33) that

$$\lambda r z(r) \leq \lambda (1 - a) \int_{r_0}^r z(r) dr \leq \lambda (1 - a) z(r) (r - r_0), \quad (37)$$

equivalently

$$1 \leq (1 - a)(r - r_0). \quad (38)$$

Letting $r \to r_0$ in (38), we obtain the inequality $1 \leq 0$, which is a contradiction. The proof of Lemma 2 is complete. □

Lemma 3. The solution $z(r)$ of (28) is monotone non-increasing in $[0, +\infty)$.

Proof. We use the method based on the contradiction argument. Suppose that, for some $r_0 > 0$, $z'(r_0) > 0$, by Lemma 1, there exists $r_1 \in (0, r_0)$ such that

$$z'(r_1) = 0, \qquad \left(\left|\left(z^k\right)_r\right|^{m-1}\left(z^k\right)_r\right)_r (r_1) \geq 0. \tag{39}$$

By Lemma 2, we have $z(r_1) > 0$. Using a similar argument in Lemma 1, we obtain

$$\lim_{r \to r_1} \frac{\left(\left|\left(z^k\right)_r\right|^{m-1}\left(z^k\right)_r\right)(r_1)}{r - r_1} = -a\lambda z(r_1) < 0. \tag{40}$$

This is a contradiction with (37). The proof of Lemma 3 is complete. □

Next, we apply the monotone properties to obtain the condition on the global existence of the solution to (1).

Proof of Theorem 5. We demonstrate by taking the steps outlined below.

Since $\varphi(x) \in F^a$, there exists a constant $H > 0$, such that

$$\varphi(x) \leq H(1 + |x|)^{-a}, \qquad \forall\, x \in R_+. \tag{41}$$

Taking $M > H$ and the self-similar solution $U_{M,a}(x,t)$ of (24) defined as (26), since $\lim\limits_{r \to \infty} r^a g_M(r) = M > H$, there exists a positive constant R_0 such that

$$r^a g_M(r) > H \quad \text{for} \quad r \geq R_0. \tag{42}$$

Setting $c = g_{M(R_0)} = \min\limits_{r \in [0,\, R_0]} g_M > 0$, it is easy to verify that $\varphi(x) \leq U_{M,a}(x,t)$ for all $x \in R_+$, where $t_0 \in (0,\, 1)$ and $c t_0^{-a\lambda} > \|\varphi\|_\infty$.

Let $\mu > 0$, then $w(x,t) = \mu U_{M,a}\left(x, \mu^{km-1}t + t_0\right)$ is the solution of the following problem

$$\begin{gathered} \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial w^k}{\partial x}\right|^{m-1}\frac{\partial w^k}{\partial x}\right), \qquad (t,x) \in Q, \\ w(x,0) = \mu U_{M,a}(x,t_0) \geq \mu\varphi(x), \qquad x \in R_+. \end{gathered} \tag{43}$$

Taking $\delta = g_M(0)$ and noting that $g_M(r)$ is non-increasing, we have

$$\|w(x,t)\|_\infty = \delta\mu\left(\mu^{km-1}t + t_0\right)^{-a\lambda}. \tag{44}$$

Set $v(x,t) = I(t)\, w(x, J(t))$, where $I(t)$ and $J(t)$ are solutions of the following problem

$$\begin{aligned} & I'(t) = (\delta\mu)^{p-1}\left(\mu^{km-1}J(t) + t_0\right)^{-a(p-1)\lambda} I^p(t), \quad t \in (0, +\infty), \\ & J'(t) = I^{km-1}(t), \ t \in (0, +\infty), \\ & I(0) = 1, \quad J(0) = 0. \end{aligned} \tag{45}$$

By a direct calculation, we obtain that $v(x,t)$ satisfies

$$\begin{aligned} & \frac{\partial v}{\partial t} \geq \frac{\partial}{\partial x}\left(\left|\frac{\partial v^k}{\partial x}\right|^{m-1}\frac{\partial v^k}{\partial x}\right) + \left|\frac{\partial v}{\partial x}\right|^p, \qquad (t,x) \in Q, \\ & v(x,0) = w(x,0) = \mu U_{M,a}(x,t_0) \geq \mu\varphi(x), \qquad x \in R_+. \end{aligned} \tag{46}$$

We prove that there exists a positive constant $\mu_0 = \mu_0(\varphi)$ such that the problem (45) has a global solution $(I(t), J(t))$ with $I(t)$ bounded in $[0,T)$ if $\mu \in [0, \mu_0)$. According to the standard theory of ODE, the local existence and uniqueness of solution $(I(t), J(t))$ of (45) hold. By (45), we have $I'(t) > 0$, $I(t) > 1$ for $t > 0$, Moreover, the solution is continuous as long as the solution exists and $I(t)$ is finite.

From (45) when $I(t)$ exists in $[0, t]$, then $J(t)$ is uniquely defined by

$$J(t) = \int_0^t I^{km-1}(y)\, dy. \tag{47}$$

Since $m > 1$ and $I(t)$ is increasing, we obtain

$$J(y) = \int_0^y I^{km-1}(\eta)\, d\eta \geq I^{km-1}(0)\, y = y, \qquad \forall\, y \in [0, t]. \tag{48}$$

By (45), (46) and $a > a_c = \dfrac{p-m-1}{km-p}$, it follows that

$$\begin{aligned} & 1 - I^{1-p}(t) = (p-1)(\delta\mu)^{p-1}\int_0^t \left(\mu^{km-1}J(y) + t_0\right)^{-a(p-1)\lambda} dy \\ & \leq (p-1)(\delta\mu)^{p-1}\int_0^t \left(\mu^{km-1}y + t_0\right)^{-a(p-1)\lambda} dy \\ & \leq \frac{(p-1)\,\delta^{p-1}\mu^{p-km}}{a(p-1)\lambda - 1} t_0^{1-a(p-1)\lambda}. \end{aligned} \tag{49}$$

Let $\mu_0 = \mu_0(\varphi)$ be a positive constant defined by

$$\frac{p-1}{\lambda\left(a\left(p-km\right)+m+1-p\right)}\delta^{p-1}\mu_0^{p-km}t_0^{1-a(p-1)\lambda}=\frac{1}{2}. \tag{50}$$

Then from (49), $p>p_c>km>1$ and $a>a_c=\dfrac{m+1-p}{p-km}$, we have $1\leq I(t)\leq 2^{\frac{1}{p-1}}$ for any $\mu\in(0,\,\mu_0]$, as long as $I(t)$ exists globally.

On the other hand, by (45) and (48), we have

$$t\leq J(t)\leq 2^{\frac{m-1}{p-1}}t, \qquad \forall t\geq 0. \tag{51}$$

Consequently, $J(t)$ is also global.

For any $\mu\in(0,\,\mu_0]$, where $\mu_0=\mu_0(\varphi)$ is defined as (50), the solution $u(x,t)$ of (1) with initial value $u_0(x)=\mu\varphi(x)$ exists globally and $u(x,t)\leq v(x,t)$ in Q.

Therefore, there exists a positive constant C, such that

$$\begin{aligned}&\left\|u\left(.,t\right)\right\|_\infty\leq\left\|v\left(.,t\right)\right\|_\infty\\&2^{\frac{1}{p-1}}\delta\mu\left(\mu^{km-1}J\left(y\right)+t_0\right)^{-a\lambda}\leq Ct^{-a\lambda},\ \forall t>0.\end{aligned} \tag{52}$$

The proof of Theorem 5 is complete. The proof of the theorem is similar to the proof of theorems in [6]. □

4. Asymptotics of self-similar solutions

Let us show the asymptotics of self-similar solutions.

The case $\dfrac{m+1}{2-m\left(k-1\right)}<p$ and $q\geq\dfrac{m\left(k\left(p+1\right)+1-p\right)}{m+p+1}$.

Consider the following self-similar solution of problem (1)-(3).

To simplify such auxiliary systems of equations, one can use the following transformations:

$$u_1\left(x,t\right)=t^{\alpha_1}\varphi\left(\xi\right),\ \ \xi=xt^{-\alpha_2}, \tag{53}$$

$$\left(\left|\varphi_\xi^k\right|^{m-1}\varphi_\xi^k\right)_\xi+\alpha_2\xi\varphi_\xi-\alpha_1\varphi+\left|\varphi_\xi\right|^p=0, \tag{54}$$

$$-\left|\varphi_\xi^k\right|^{m-1}\varphi_\xi^k(0)=\varphi^q(0). \tag{55}$$

Consider the function

$$\bar\varphi(\xi)=E\left(a-\xi^{\frac{m+1}{m}}\right)_+^{\frac{m}{mk+1}},\quad E=\left(\frac{mk-1}{m+1}\alpha_2^{\frac{1}{m}}\right)^{\frac{m}{mk-1}}, \tag{56}$$

where $a>0,\ (d)_+=\max\{d,0\}$. We show that the function (56) is the asymptotics of the solutions of problem (54)-(55).

Theorem 6. *The compactly supported solution of problem* (54)-(55) *has the asymptotic*

$$\varphi(\xi)=\bar\varphi(\xi)(1+o(1))$$

when $\xi\to a^{\frac{m}{m+1}}$.

Proof. We are looking for a solution to equation (54) in the format as below:

$$\varphi(\xi)=\bar\varphi(\xi)\,\omega(\tau) \tag{57}$$

with $\tau=-\ln\left(a-\xi^{\frac{m+1}{m}}\right)$, where $\tau\overset{\xi\to a^{\frac{m}{m+1}}}{\longrightarrow}+\infty$.

Substituting (57) into equation (54) in relation to (56) yields the following:

$$\begin{aligned}&\frac{d}{d\tau}(L\omega)^m+(L\omega)^m\left\{a_0(\tau)-\frac{m}{mk-1}\right\}+a_1(\tau)\,\omega^{1-k}L\omega-a_2(\tau)\,\omega\\&+a_3(\tau)\,\omega^{(1-k)p}(L\omega)^p=0,\end{aligned} \tag{58}$$

where

$$L\omega=\frac{d\omega^k}{d\tau}-\frac{mk}{mk-1}\omega^k,$$

$$a_0(\tau)=\frac{e^{-\tau}}{a-e^{-\tau}},\quad a_1(\tau)=\left(1+\frac{1}{m}\right)^{-m}E^{1-mk}\frac{\alpha_2}{k},$$

$$a_2(\tau)=\alpha_1\left(1+\frac{1}{m}\right)^{-1-m}\times E^{1-mk}a_0(\tau),$$

$$a_3(\tau)=\left(1+\frac{1}{m}\right)^{p-1-m}\frac{E^{p-mk}}{k^p}e^{\left(1-\frac{m}{mk-1}\right)(p-1)\tau}\left(a-e^{-\tau}\right)^{\frac{p-m-1}{m+1}}.$$

The solution of the last expression, in a certain vicinity of $+\infty$, satisfies the inequality:

$$\omega > 0, \ \left(\omega^k\right)' - \frac{mk}{mk-1}\omega^k \neq 0. \tag{59}$$

Assuming that $\vartheta(\tau) = (L\omega)^m$, then

$$\vartheta' = -\left(a_0(\tau) - \frac{m}{mk-1}\right)\vartheta - a_1(\tau)\,\omega^{1-k}L\omega + a_2(\tau)\,\omega - a_3(\tau)\,\omega^{(1-k)p}. \tag{60}$$

Furthermore, we consider the function:

$$\theta(\tau,\lambda) = -\left(a_0(\tau) - \frac{m}{mk-1}\right)\lambda - a_1(\tau)\,\omega^{1-k}L\omega + a_2(\tau)\,\omega - a_3(\tau)\,\omega^{(1-k)p}, \tag{61}$$

where $\lambda \in R$.

The function $\theta(\tau,\lambda)$ preserves sign on some interval $[\tau_1;+\infty) \subset [\tau_0;+\infty)$ for every fixed value λ.

Therefore, the functions $\theta(\tau,\lambda)$ satisfies one of the following inequalities, for all $\tau \in [\tau_1;+\infty)$

$$\vartheta' > 0 \quad \text{or} \quad \vartheta' < 0, \tag{62}$$

from what we conclude that $\tau \in [\tau_1;+\infty)$:

$$\lim_{\tau\to+\infty} a_1(\tau) = \left(1+\frac{1}{m}\right)^{-m} E^{1-mk}\frac{\alpha_2}{k},$$

$$\lim_{\tau\to+\infty} a_0(\tau) = \lim_{\tau\to+\infty} a_2(\tau) = \lim_{\tau\to+\infty} a_3(\tau) = 0.$$

Suppose now that for the function $\vartheta(\tau)$ limit at $\tau \to +\infty$ does not exist. Consider the case when one of the inequalities (62) is satisfied. As $\vartheta(\tau)$ is oscillating function around $\bar{\vartheta} = \lambda$ its graph intersects this straight line infinitely many times in $[\tau_1;+\infty)$. But this is impossible, since in the interval $[\tau_1;+\infty)$ just one of the inequalities (62) is valid and therefore, from (61) it follows that graph of the function $\vartheta(\tau)$ intersects the straight line $\bar{\vartheta} = \lambda$ only once in the interval $[\tau_1;+\infty)$. Accordingly, the function $\vartheta(\tau)$ has a limit at $\tau \to +\infty$.

By assumption, the function $\vartheta(\tau)$ has a limit at $\tau \to +\infty$. Then, $w'(\tau)$ has a limit at $\tau \to +\infty$, and this limit is zero. Then

$$\vartheta(\tau) = \left(\frac{mk}{mk-1}\right)^m \left(\overset{0}{\omega}\right)^{km} + o(1)$$

at $\tau \to +\infty$.

And by (60) derivatives of functions $\vartheta(\tau)$ have limits at $\tau \to +\infty$, which is obviously equal to zero.

Consequently, it is necessary

$$\lim_{\eta_i \to \infty} \left(\left(a_0(\tau) - \frac{m}{mk-1} \right) \vartheta + a_1(\tau) \omega^{1-k} L\omega - a_2(\tau) \omega + a_3(\tau) \omega^{(1-k)p} \right) = 0.$$

And we obtain the following algebraic equation

$$\left(\frac{mk}{mk-1} \right)^m \left(\overset{0}{\omega} \right)^{mk} - \alpha_2 \left(1 + \frac{1}{m} \right)^{-m} E^{1-mk} \overset{0}{\omega} = 0.$$

The best case: $\overset{0}{\omega} = 1$. From this equation, we get that $\overset{0}{\omega} \approx 1$, thus we have $\varphi(\xi) = \bar{\varphi}(\xi)(1 + o(1))$. □

5. Conclusion

It is accomplished to acquire the Hamilton–Jacobi equation's solution of the type Zeldovich-Barenblatt. Using the comparison approach, it is possible to study the finite speed properties of the Neumann problem for a parabolic equation with a gradient term. For both slow and fast diffusion cases, the asymptotic behavior of the self-similar solution is examined. Using the modification energy methods we established the secondary critical exponent. Moreover, analyzed the asymptotic behavior of the solution (1) and a nonlinear algebraic equation is demonstrated to be satisfied by the coefficients in the main term of the asymptotic of the solution.

References

[1] M.M. Aripov, A.S. Matyakubov, To the qualitative properties of solution of system equations not in divergence form of polytrophic filtration in variable density, *Nanosystems: Physics, Chemistry, Mathematics*, **8**, No 3 (2017), 317–322.

[2] K. Deng, H.A. Levine, The role of critical exponents in blow-up theorems, *J. Math. Anal. Appl.*, **243** (2000), 85–126.

[3] V.A. Galaktionov, H.A. Levine, On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary, *Israel J. Math.*, **94** (1996), 125–146.

[4] J.I. Diaz and J.E. Saa, Uniqueness of very singular self-similar solution of a quasilinear degenerate parabolic equation with absorption, *Publ. Mat.*, **36**, No 1 (1992), 19–38.

[5] J.N. Zhao, On the Cauchy problem and initial traces for the evolution p-Laplacian equations with strongly nonlinear sources, *J. Differential Equations*, **121**, No 2 (1995), 329–383.

[6] K. Mukai, K. Mochizuki and Q. Huang, Large time behavior and life span for a quasilinear parabolic equation with slowly decaying initial values, *Nonlinear Analysis: Theory, Methods & Applications A*, **39**, No 1, (2000), 33–45.

[7] A.S. Kalashnikov, Some problems of the qualitative theory of nonlinear degenerate parabolic equations of second order, *Uspekhi Mat. Nauk*, **42** (1987), 135–176; *English Transl.: Russian Math. Surveys*, **42** (1987), 169–222.

[8] H.A. Levine, The role of critical exponents in blow up theorems, *SIAM Rev.*, **32** (1990), 262–288.

[9] Z. Rakhmonov, On the properties of solutions of multidimensional nonlinear filtration problem with variable density and nonlocal boundary condition in the case of fast diffusion, *J. of Siberian Federal University. Mathematics & Physics*, **9**, No 2 (2016), 236–245.

[10] Z.R. Rakhmonov, A. Khaydarov, J.E. Urunbaev, Global Existence and nonexistence of solutions to a cross diffusion system with nonlocal boundary conditions, *Mathematics and Statistics*, **8**, No 4 (2020), 404–409; doi:10.13189/ms.2020.080405.

[11] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, A.P. Mikhailov, *Blow-up in Quasilinear Parabolic Equations*, Walter de Gruyter, Berlin (1995).

[12] Z.Q. Wu, J.N. Zhao, J.X. Yin, H.L. Li, *Nonlinear Diffusion Equations*, World Scientific Publ. Co. Inc., River Edge, NJ (2001).

[13] Yongsheng Mi, Chunlai Mu, Rong Zeng, Secondary critical exponent, large time behavior and life span for a quasilinear parabolic equation with slowly decaying initial values, *Z. Angew. Math. Phys.*, (2011).

[14] Z.R. Rakhmonov, A.I. Tillaev, On the behavior of the solution of a nonlinear polytropic filtration problem with a source and multiple nonlinearities, *Nanosystems: Physics, Chemistry, Mathematics*, **9**, No 3 (2018), 1–7.

[15] Z. Rakhmonov, R. Parovik, and A. Alimov, Global existence and nonexistence for a multidimensional system of parabolic equations with nonlinear boundary conditions, *AIP Conference Proc.*, **2365** (2021), 060022.

[16] Z. Rakhmonov, J. Urunbaev, and A. Alimov, Properties of solutions of a system of nonlinear parabolic equations with nonlinear boundary conditions, *AIP Conference Proc.*, **2637** (2022), 040008.

[17] Zhiyong Wang, Jingxue Yin, Gradient blow-up solution with small initial datum for a Hamilton–Jacobi equation with degenerate diffusion, *J. of Mathematical Analysis and Applications*, **472**, No 2 (2019), 1914–1926.

[18] M. Aripov, Asymptotics of the solutions of the non-Newton politropic filtration equation, *ZAMM Zeitschrift fur Angewandte Mathematik und Mechanik*, **80**, No 4, Suppl. 4 (2000).

[19] M. Aripov, The Fujita and secondary type critical exponents in nonlinear parabolic equations and systems, *Springer Proc. in Math. and Stat.*, **268** (2018), 9–23.