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ON ESSENTIAL BI-IDEALS AND ESSENTIAL FUZZY BI-IDEALS IN SEMIGROUPS

Thiti Gaketem[1] §, Aiyared Iampan[2]

[1,2] Fuzzy Algebras and Decision-Making
Problems Research Unit
Department of Mathematics, School of Science
Mae Ka, University of Phayao
Phayao 56000, THAILAND

Abstract: This paper gives the concepts of essential bi-ideals and essential fuzzy bi-ideals in semigroups. We investigate the basic properties of essential bi-ideals and essential fuzzy bi-ideals in semigroups. Finally, we provide the defined essential minimal bi-ideals and essential fuzzy minimal bi-ideals in semigroups and study their properties.

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1. Introduction

The concepts of fuzzy sets were proposed by Zadeh in 1965, [13]. These concepts were applied in many areas such as medical sciences, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, topology, etc. In 1979, Kuroki [6] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them.

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The definition of Essential fuzzy ideals in rings was studied in 1971 by Medhi et al. [7]. Later, Medhi and Saikia learned the concept of T-fuzzy essential ideals and proved the properties of T-fuzzy essential ideals in 2013, [8]. In 2020, Baupradist et al. [1] extended the concept of Essential fuzzy ideals in rings to go essential ideals and essential fuzzy ideals in semigroups. Together with 0-essential ideals and 0-essential fuzzy ideals in semigroups. Moreover, there have been many studies of essential ideals. In 2021, Gaketem and Chimram [2] discussed essential (m, n)-ideals and essential fuzzy (m, n)-ideals in semigroups. In 2021-2022, Gaketem and Iampan [3, 4] used know of essential ideal semigroup to studied in essential UP-subalgebras and essential UP-ideals (filters) and the concepts of t-essential fuzzy UP-subalgebras and t-essential fuzzy UP-ideals (filters) of UP-algebras, and investigated their relationships. In 2023 Panpetch et al. [10] studied essential bi-ideals and fuzzy essential bi-ideals in semigroups. In the same year P. Khamrot and T. Gaketem, [5] studied essential ideals in bipolar fuzzy ideals in semigroups. Recently, R. Rittichuai et al. [12] studied essential ideals and essential fuzzy ideals in ternary semigroups.

Section 2 reviews the definitions and theorem importance of studies in this article. Section 3 gives the definitions of essential subsemigroups and essential fuzzy subsemigroups, and we investigate the properties of this section. Section 4 and Section 6 discuss the properties of essential bi-ideals, essential fuzzy bi-ideals, minimal bi-ideals, essential fuzzy minimal bi-ideal, prime and semiprime essntial ideal in semigroup.

2. Preliminaries

We give here some basic definitions and concepts.

A non-empty subset I of a semigroup S is called a *subsemigroup* of S if $I^2 \subseteq I$. A non-empty subset I of a semigroup S is called a *left (right) ideal* of S if $SI \subseteq I$ $(IS \subseteq I)$. An *ideal* I of S is a non-empty subset which is both a left ideal and a right ideal of S. A subsemigroup I of a semigroup S is called a *bi-ideal* of S if $ISI \subseteq I$. It well-know, every ideal of a semigroup S is a bi-ideal of S.

For any $a, b \in [0, 1]$, we have

$$a \vee b = \max\{a, b\} \quad \text{and} \quad a \wedge b = \min\{a, b\}.$$

A fuzzy set of a non-empty set T is function from T into unit closed interval $[0, 1]$ of real numbers, i.e., $f : T \to [0, 1]$.

For any two fuzzy sets of f and g of a non-empty of T, we defined $f \subseteq g$ if $f(u) \leq g(u)$, $(f \vee g)(u) = f(u) \vee g(u)$ and $(f \wedge g)(u) = f(u) \wedge g(u)$ for all $u \in T$.

A *fuzzy subsemigroup* of a semigroup S if $f(uv) \geq f(u) \wedge f(v)$ for all $u, v \in S$. A *fuzzy left (right) ideal* of a semigroup S if $f(uv) \geq f(v)$ $(f(uv) \geq f(u))$ for all $u, v \in S$. A *fuzzy bi-ideal* of a semigroup S if f is a fuzzy subsemigroup of S and $f(uvw) \geq f(u) \wedge f(w)$ for all $u, v, w \in S$. It well-know, every fuzzy ideal of a semigroup S is a fuzzy bi-ideal of S.

The characteristic fuzzy set χ_I of a non-empty set is defined as follows:

$$\chi_I : T \to [0,1], u \mapsto \begin{cases} 1 & \text{if } u \in I, \\ \emptyset & \text{if } u \notin I. \end{cases}$$

The following theorems hold.

Theorem 1. *[9] Let S be a semigroup. Then I is a subsemigroup (left ideal right ideal, bi-ideal) of S if and only if characteristic function χ_I is a fuzzy subsemigroup (left ideal right ideal, bi-ideal) of S.*

Theorem 2. *[9] Let I and J be subsets of a non-empty set S. Then $\chi_{I\cap J} = \chi_I \wedge \chi_J$. and $\chi_I \circ \chi_J = \chi_{IJ}$.*

For any two fuzzy set of f of a non-empty of T, we defined the *support* of f instead of $\mathrm{supp}(f) = \{u \in T \mid f(u) \neq 0\}$.

Theorem 3. *[9] Let f be a nonzero fuzzy set of a semigroup S. Then f is a fuzzy subsemigroup (ideal, bi-ideal) of S if and only if supp(f) is a subsemigroup (ideal, bi-ideal) of S.*

Next, we review basics of essential ideals and fuzzy essential ideals in a semigroup and properties of those.

Definition 4. An *essential left (right) ideal* I of a semigroup S if I is a left (right) ideal of S and $I \cap J \neq \emptyset$ for every left (right) ideal J of S.

Definition 5. [1] An *essential ideal* I of a semigroup S if I is an ideal of S and $I \cap J \neq \emptyset$ for every ideal J of S.

Theorem 6. *[1] Let I be an essential ideal of a semigroup S. If I_1 is an ideal of S containing I, then I_1 is also an essential ideal of S.*

Theorem 7. *[1] Let I and J be essential ideals of a semigroup S. Then*

$I \cup J$ and $I \cap J$ are essential ideals of S.

Definition 8. [1] An *essential fuzzy ideal* f of a semigroup S if f is a nonzero fuzzy ideal of S and $f \wedge g \neq \emptyset$ for every nonzero fuzzy ideal g of S.

Theorem 9. *[1] Let I be an ideal of a semigroup S. Then I is an essential ideal of S if and only if χ_I is an essential fuzzy ideal of S.*

Theorem 10. *[1] Let f be a nonzero fuzzy ideal of a semigroup S. Then f is an essential fuzzy ideal of S if and only if supp(f) is an essential ideal of S.*

3. Essential subsemigroups and Essential fuzzy subsemigroups

In this section, we study concepts of essential subsemigroups in a semigroup and fuzzy essential subsemigroups in a semigroup and properties of those.

Definition 11. An *essential subsemigroup* I of a semigroup S if I is a subsemigroup of S and $I \cap J \neq \emptyset$ for every subsemigroup J of S.

Example 12. (1) Let E be set of all even integers.

Then $(E, +)$ and $(\mathbb{N}, +)$ are subsemigroups of $(\mathbb{Z}, +)$. Thus $(E, +) \cap (\mathbb{N}, +) \neq \emptyset$. Hence, $(E, +)$ is an essential subsemigroup of $(\mathbb{Z}, +)$.

(2) Let $A = \{2n \mid n \in \mathbb{Z}\}$ and $B = \{3n \mid n \in \mathbb{Z}\}$. Then $(A, \cdot)$ and $(B, \cdot)$ are subsemigroups of $(\mathbb{Z}, \cdot)$. Thus $(A, \cdot) \cap (B, \cdot) \neq \emptyset$. Hence $(A, \cdot)$ is an essential subsemigroup.

Theorem 13. *Let I be an essential subsemigroup of a semigroup S. If I_1 is an ideal of S containing I, then I_1 is also an essential subsemigroup of S.*

Proof. Suppose that I_1 is a subsemigroup of S such that $I_1 \subseteq I$ and let J be any subsemigroup of S. Thus, $I \cap J \neq \emptyset$. Hence, $I_1 \cap J \neq \emptyset$. Therefore I_1 is an essential subsemigroup of S. □

Theorem 14. *Let I and J be essential subsemigroups of a semigroup S. Then $I \cup J$ and $I \cap J$ are essential subsemigroups of S.*

Proof. Since $I \subseteq I \cup J$ and I is an essential subsemigroup we have $I \cup J$ is an essential subsemigroup of S, by Theorem 13.

Since I and J are essential subsemigroups of S we have I and J are subsemigroups of S. Thus $I \cap J$ is a subsemigroups of S. Let K be a subsemigroup of S. Then $I \cap K \neq \emptyset$. Thus there exists $u, v \in I \cap K$. Let $u, v \in J$. Then $uv \in (I \cap J) \cap K$. Thus $(I \cap J) \cap K \neq \emptyset$. Hence $I \cap J$ is an essential subsemigroup of S. □

Definition 15. An *essential fuzzy subsemigroup* f of a semigroup S if f is a nonzero fuzzy subsemigroup of S and $f \wedge g \neq 0$ for every nonzero fuzzy subsemigroup g of S.

Theorem 16. *Let I be a subsemigroup of a semigroup S. Then I is an essential subsemigroup of S if and only if χ_I is an essential fuzzy subsemigroup of S.*

Proof. Suppose that I is an essential subsemigroup of S and let g be a nonzero fuzzy subsemigroup of S. Then $\operatorname{supp}(g)$ is subsemigroup of S. By assumption we have I is a subsemgroup of S. Thus $I \cap \operatorname{supp}(g) \neq 0$. So there exists $u \in I \cap \operatorname{supp}(g)$. It implies that $(\chi_I \wedge g)(u) \neq 0$. Hence, $\chi_I \wedge g \neq 0$. Therefore, χ_I is an essential fuzzy subsemigroup of S.

Conversely, assume that χ_I is an essential fuzzy subsemigroup of S and let J be a subsemigroup of S. Then χ_J is a nonzero fuzzy subsemigroup of S. Sicne χ_I is an essential fuzzy subsemigroup of S we have χ_I is a fuzzy subsemigorup of S. Thus, $\chi_I \wedge \chi_J \neq 0$. So by Theorem 2, $\chi_{I \cap J} \neq \emptyset$. Hence, $I \cap J \neq \emptyset$. Therefore I is an essential subsemigroup of S. □

Theorem 17. *Let f be a nonzero fuzzy subsemigroup of a semigroup S. Then f is an essential fuzzy subsemigroup of S if and only if supp(f) is an essential subsemigroup of S.*

Proof. Assume that f is an essential fuzzy subsemigroup of S. Then $\operatorname{supp}(f)$ is a subsemigroup of S. Let I be a subsemigroup of S. Then by Theorem 1, χ_I is a subsemigroup of S. Since f is an essential fuzzy subsemigroup of S we have f is a fuzzy subsemigroup of S. Thus $f \wedge \chi_I \neq 0$. So there exists $u \in S$ such that $(f \wedge \chi_I)(u) \neq 0$. It implies that $f(u) \neq 0$ and $\chi_I \neq 0$. Hence, $u \in supp(f) \cap I$ so $\operatorname{supp}(f) \cap I \neq \emptyset$. It implies that $\operatorname{supp}(f)$ is an essential subsemigroup of S.

Conversely, assume that $\operatorname{supp}(f)$ is an essential ideal of S and let g be a nonzero fuzzy subsemigroup of S. Then $\operatorname{supp}(g)$ is a subsemigroup of S. Thus supp(f)$\cap$ supp(g) $\neq \emptyset$. So there exists $u \in$ supp(f)$\cap$supp(g), this implies that $f(u) \neq 0$ and $g(u) \neq 0$ for all $u \in S$. Hence, $(f \wedge g)(u) \neq 0$ for all $u \in S$. Therefore, $f \wedge g \neq 0$. We conclude that f is an essential fuzzy subsemigroup of S. □

Theorem 18. *Let f be an essential fuzzy subsemigroup of a semigroup S. If f_1 is a fuzzy subsemigroup of S such that $f \subseteq f_1$, then f_1 is also an essential fuzzy subsemigroup of S.*

Proof. Let f_1 be a fuzzy subsemigroup of S such that $f \subseteq f_1$ and let g be any fuzzy subsemigroup of S. Thus, $f \wedge g \neq 0$. So $f_1 \wedge g \neq 0$. Hence f_1 is an essential fuzzy subsemigroup of S. □

Theorem 19. *Let f_1 and f_2 be essential fuzzy subsemigroups of a semigroup S. Then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are essential fuzzy subsemigroups of S.*

Proof. Let f_1 and f_2 be essential fuzzy subsemigroups of S. Then by Theorem 18, $f_1 \vee f_2$ is an essential fuzzy subsemigroup of S. Since f_1 and f_2 are essential fuzzy subsemigroups of S we have $f_1 \wedge f_2$ is a fuzzy subsemigroup of S. Let g be a nonzero fuzzy subsemigroup of S. Then $f_1 \wedge g \neq 0$. Thus there exists $u \in S$ such that $f_1(u) \neq 0$ and $(g)(u) \neq 0$. Since $f_2 \neq 0$ and let $v \in S$ such that $f_2(v) \neq 0$. Since f_1 and f_2 are fuzzy subsemigroups of S we have $f_1(uv) \geq f_1(u) \wedge f_1(v) > 0$ and $f_2(uv) \geq f_2(u) \wedge f_2(v) > 0$. Thus $(f_1 \wedge f_2)(uv) = f_1(uv) \wedge f_2(uv) \neq 0$. Since g is a fuzzy subsemigroup of S and $g(u) \neq 0$ we have $g(uv) \neq 0$ for all $u, v \in S$. Thus $[(f_1 \wedge f_2) \wedge g](uv) \neq 0$. Hence $[(f_1 \wedge f_2) \wedge g] \neq 0$. Therefore $f_1 \wedge f_2$ is an essential fuzzy subsemigroup of S. □

4. Essential bi-ideals and Essential fuzzy bi-ideals

In this section, we define essential bi-ideals and essential fuzzy bi-ideal in semigroup and integrated properties of its.

Definition 20. An *essential bi-ideal* I of a semigroup S if I is a bi-ideal of S and $I \cap J \neq \emptyset$ for every bi-ideal J of S.

Example 21. Let $S = \{\Psi, \Omega, \Upsilon, \mathit{\Pi}\}$ be semigroup with the following Cayley table:

$\cdot$	Ψ	Ω	Υ	Π
Ψ	Ψ	Ψ	Ψ	Ψ
Ω	Ψ	Ψ	Ψ	Ψ
Υ	Ψ	Ψ	Ω	Ψ
Π	Ψ	Ψ	Ω	Ω

Then $\{\Psi\}, \{\Psi,\Omega\}, \{\Psi,\Omega,\Upsilon\}, \{\Psi,\Omega,\Pi\}$ and $\{\Psi,\Omega,\Upsilon,\Pi\}$ are bi-ideal of S. Thus $\{\Psi\}\cap\{\Psi,\Omega\} \neq \emptyset$ and $\{\Psi,\Omega,\Pi\}\cap\{\Psi,\Omega,\Upsilon,\Pi\} \neq \emptyset$. Hence $\{\Psi\}$ and $\{\Psi,\Omega,\Pi\}$ are essential bi-ideals of S.

Theorem 22. *Let I be an essential bi-ideal of a semigroup S. If I_1 is an ideal of S containing I , then I_1 is also an essential bi-ideal of S.*

Proof. Suppose that I_1 is a bi-ideal of S such that $I_1 \subseteq I$ and let J be any bi-ideal of S. Thus, $I \cap J \neq \emptyset$. Hence, $I_1 \cap J \neq \emptyset$. Therefore I_1 is an essential bi-ideal of S. □

Theorem 23. *Let I and J be essential bi-ideals of a semigroup S. Then $I \cup J$ and $I \cap J$ are essential bi-ideals of S.*

Proof. Since I and J are essential bi-ideals of a semigroup S we have I and J are essential subsemigroups of a semigroup S. Thus by Theorem 14, $I \cup J$ and $I \cap J$ are essential subsemigroups of S. Since $I \subseteq I \cup J$ and I is an essential bi-ideal we have $I \cup J$ is an essential bi-ideal of S.

Let K be a bi-ideal of S. Then $I \cap K \neq \emptyset$. Thus there exists u, v and $w \in I \cap K$.
Let u, v and $w \in J$. Then $uvw \in (I \cap J) \cap K$. Thus $(I \cap J) \cap K \neq \emptyset$. Hence $I \cap J$ is an essential bi-ideal of S. □

Definition 24. An *essential fuzzy bi-ideal* f of a semigroup S if f is a nonzero fuzzy bi-ideal of S and $f \wedge g \neq 0$ for every nonzero fuzzy bi-ideal g of S.

Theorem 25. *Let I be a bi-ideal of a semigroup S. Then I is an essential bi-ideal of S if and only if χ_I is an essential fuzzy bi-ideal of S.*

Proof. Suppose that I is an essential bi-ideal of S and let g be a nonzero fuzzy bi-ideal of S. Then by Theorem 16, supp(g) is subsemigroup of S and χ_I is an essential fuzzy subsemigroup of S. Thus there exists $u, v, w \in I \cap supp(g)$

such that
$(f \wedge \chi_I)(uvw) \neq 0$. It implies that $\chi_I \wedge g \neq 0$. Therefore, χ_I is an essential fuzzy bi-ideal of S.

Conversely, assume that χ_I is an essential fuzzy bi-ideal of S and let J be a bi-ideal of S. Then χ_I is an essential fuzzy subsemigroup of S and J is a subsemigroup of S. Thus by Theorem 16, I is an essential subsemigroup of S. Since J be a bi-ideal of S we have χ_J is a nonzero fuzzy bi-ideal of S. Then, $\chi_I \wedge \chi_J \neq \emptyset$. Thus, $\chi_{I\cap J} \neq \emptyset$. Hence, $I \cap J \neq \emptyset$. Therefore I is an essential bi-ideal of S. □

Theorem 26. *Let f be a nonzero fuzzy bi-ideal of a semigroup S. Then f is an essential fuzzy bi-ideal of S if and only if supp(f) is an essential bi-ideal of S.*

Proof. Assume that f is an essential fuzzy bi-ideal of S. Then f is an essential fuzzy subsemigroup of S. Thus by Theorem 17, supp(f) is an essential subsemigorup of S. Let I be a bi-ideal of S. Then by Thoerem 1, χ_I is a bi-ideal of S. Thus $f \wedge \chi_I \neq 0$. Thus there exists $u \in S$ such that $(f \wedge \chi_I)(u) \neq 0$. It implies that $f(u) \neq 0$ and $\chi_I \neq 0$. Hence, $u \in supp(f) \cap I$ so supp$(f) \cap I \neq \emptyset$ it implies that supp(f) is an essential bi-ideal of S.

Conversely, assume that supp(f) is an essential bi-ideal of S and let g be a nonzero fuzzy bi-ideal of S. Then supp(f) is an essential bi-ideal of S. Since g be a nonzero fuzzy bi-ideal of S we have f is an essential fuzzy subsemigroup of S and supp(g) is a subsemigroup of S, by Theorem 17. This implies that supp$(f)\cap$supp$(g) \neq \emptyset$. So there exists $u \in$ supp$(f)\cap$ supp(g), this implies that $f(u) \neq 0$ and $g(u) \neq 0$. Hence, $(f \wedge g)(u) \neq 0$. Therefore, $f \wedge g \neq 0$. We conclude that f is an essential fuzzy bi-ideal of S. □

Theorem 27. *Let f be an essential fuzzy bi-ideal of a semigroup S.If f_1 is a fuzzy bi-ideal of S such that $f \subseteq f_1$, then f_1 is also an essential fuzzy bi-ideal of S.*

Proof. Let f_1 be a fuzzy bi-ideal of S such that $f \subseteq f_1$ and let g be any fuzzy bi-ideal of S. Thus $f \wedge g \neq 0$. So $f_1 \wedge g \neq 0$. Hence, f_1 is an essential fuzzy bi-ideal of S. □

Theorem 28. *Let f_1 and f_2 be essential fuzzy bi-ideals of a semigroup S. Then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are essential fuzzy bi-ideals of S.*

Proof. Let f_1 and f_2 be essential fuzzy bi-ideal of S. Then by Theorem 27, $f_1 \vee f_2$ is an essential fuzzy bi-ideal of S. Since f_1 and f_2 are essential fuzzy bi-ideals of S we have f_1 and f_2 is an essential fuzzy subsemigroup of S. Thus $f_1 \wedge f_2$ is an essential fuzzy subsemigroup of S. Let g be a nonzero fuzzy bi-ideal of S. Then $f_1 \wedge g \neq 0$. Thus there exists $u, w \in S$ such that $f_1(uw) \neq 0$ and $(g)(uw) \neq 0$. Since $f_2 \neq 0$ and let $v \in S$ such that $f_2(v) \neq 0$. Since f_1 and f_2 are fuzzy subsemigroups of S we have $f_1(uvw) \geq f_1(u) \wedge f_1(w) > 0$ and $f_2(uvw) \geq f_2(u) \wedge f_2(w) > 0$. Thus $(f_1 \wedge f_2)(uvw) = f_1(uvw) \wedge f_2(uvw) \neq 0$. Since g is a fuzzy subsemigroup of S and $g(v) \neq 0$ we have $g(uvw) \neq 0$ for all $u, v \in S$. Thus $[(f_1 \wedge f_2) \wedge g](uvw) \neq 0$. Hence $[(f_1 \wedge f_2) \wedge g] \neq 0$. Therefore $f_1 \wedge f_2$ is an essential fuzzy bi-ideal of S. □

The following theorem we will use as the basic knowledge of ideal and bi-ideal in semigroups to prove essential bi-ideal in semigroup.

Theorem 29. *Every essential ideal of semigroup S is an essential bi-ideal of S.*

Proof. The proof is obvious. □

Theorem 30. *Every essential fuzzy ideal of semigroup S is an essential fuzzy bi-ideal of S.*

Proof. The proof is obvious. □

5. Minimal Essential bi-ideals and Minimal Essential fuzzy bi-ideals in a semigroup.

A bi-ideal I of a semigroup S is called a *minimal bi-ideal* of S if for every bi-ideal M of S such that $M \subseteq I$, we have $M = I$. A nonzero bi-ideal I of a semigroup S with zero is called a *0-minimal bi-ideal* of S if for any nonzero bi-ideal M of S such that $M \subseteq I$, we have $M = I$.

Definition 31. An essential bi-ideal I of a semigroup S is called a *minimal* if for every essential bi-ideal of M of S such that $M \subseteq I$, we have $M = I$.

Definition 32. An essential fuzzy bi-ideal f of a semigroup S is called a *minimal* if for every essential fuzzy bi-ideal of g of S such that $f \subseteq g$, we have $\mathrm{supp}(f) = \mathrm{supp}(g)$.

Definition 33. A 0-essential bi-ideal I of a semigroup with zero S is called a *minimal* if for every 0-essential bi-ideal of M of S such that $M \subseteq I$, we have $M = I$.

Definition 34. A 0-essential fuzzy bi-ideal f of a semigroup with zero S is called a *minimal* if for every 0-essential fuzzy bi-ideal of g of S such that $f \subseteq g$, we have $\mathrm{supp}(f) = \mathrm{supp}(g)$.

Theorem 35. *Let K be a non-empty subset of a semigroup S. Then K is a minimal essential bi-ideal of S if and only if χ_K is a minimal essential fuzzy bi-ideal of S.*

Proof. Suppose that K is a minimal essential bi-ideal of S. Then K is an essential bi-ideal of S. By Theorem 25, χ_K is an essential fuzzy bi-ideal of S. Let f be an essential fuzzy bi-ideal of S such that $f \subseteq \chi_K$. Then $\mathrm{supp}(f) \subseteq \mathrm{supp}(\chi_K)$. By assumption, $\mathrm{supp}(\chi_K) = K$. Thus $\mathrm{supp}(f) \subseteq \mathrm{supp}(\chi_K) = K$. Thus $\mathrm{supp}(f) \subseteq K$. Since f is an essential fuzzy bi-ideal of S we have $\mathrm{supp}(f)$ is an essential bi-ideal of S. By assumption, $\mathrm{supp}(f) = K = \mathrm{supp}(\chi_K)$. Hence, χ_K is a minimal essential fuzzy bi-ideal of S.

Conversely, χ_K is a minimal essential fuzzy bi-ideal of S and let B be an essential bi-ideal of S such that $B \subseteq K$. Then B is an bi-ideal of S. Thus by Theorem 25, χ_B is an essential fuzzy bi-ideal of S such that $\chi_B \subseteq \chi_K$. So $\chi_B = \chi_K$. Hence $B = \chi_B = \chi_K = K$. Therefore K is a minimal essential bi-ideal of S. □

Theorem 36. *Let I be a nonzero bi-ideal of a semigroup with zero S. Then I is a 0-essential bi-ideal of S if and only if χ_I is a 0-essential fuzzy bi-ideal of S.*

Proof. Suppose that I is a 0-essential bi-ideal of S and let g be a nontrivial fuzzy bi-ideal of S. Then by Theorem 3 $\mathrm{supp}(g)$ is a nonzero bi-ideal of S. Since I is a 0-essential bi-ideal of S we have I is a nonzero bi-ideal of S. Thus $I \cap supp(g) \neq \{0\}$. So there exists $x \in I \cap supp(g)$. Since I is a nonzro ideal bi-ideal of S we have χ_I is a nonzero fuzzy bi-ideal of S. Since g is a nonzero fuzzy bi-ideal of S we have $\mathrm{supp}(\chi_I \wedge g)(x) \neq \emptyset$. Thus, $\chi_I \wedge g \neq 0$. Therefore,

χ_I is a 0-essential fuzzy bi-ideal of S.

Conversely, assume that χ_I is a 0-essential fuzzy bi-ideal of S and let J be a nonzero bi-ideal of S. Then χ_J is a nonzero fuzzy bi-ideal of S. Sicne χ_I is a 0-essential fuzzy bi-ideal of S we have χ_I is a nontrivial fuzzy bi-ideal of S. Thus, $\mathrm{supp}(\chi_I \cap \chi_J) \neq \{0\}$. So by Theorem 2, $\chi_{I\cap J} \neq \{0\}$. Hence, $I \cap J \neq \{0\}$. Therefore I is a 0-essential bi-ideal of S. $\square$

Theorem 37. *Let K be a non-empty subset of a semigroup with zero S. Then K is a minimal 0-essential ideal of S if and only if χ_K is a minimal 0-essential fuzzy bi-ideal of S.*

Proof. Suppose that K is a minimal 0-essential bi-ideal of S. Then K is a 0-essential bi-ideal of S. By Theorem 36, χ_K is a 0-essential fuzzy bi-ideal of S. Let f be a 0-essential fuzzy bi-ideal of S such that $f \subseteq \chi_K$. Then $\mathrm{supp}(f) \subseteq \mathrm{supp}(\chi_K)$. By assumption, $\mathrm{supp}(\chi_K) = K$. Thus $\mathrm{supp}(f) \subseteq \mathrm{supp}(\chi_K) = K$. Thus $\mathrm{supp}(f) \subseteq K$. Since f is a 0-essential fuzzy bi-ideal of S we have $\mathrm{supp}(f)$ is a 0-essential bi-ideal of S. By assumption, $\mathrm{supp}(f) = K = \mathrm{supp}(\chi_K)$. Hence, χ_K is a minimal 0-essential fuzzy ideal of S.

Conversely, χ_K is a minimal 0-essential fuzzy ideal of S and let B be a 0-essential bi-ideal of S such that $B \subseteq K$. Then B is a nonzero bi-ideal of S. Thus by Theorem 36, χ_B is a 0-essential fuzzy bi-ideal of S such that $\chi_B \subseteq \chi_K$. So $\chi_B = \chi_K$. Hence $B = \chi_B = \chi_K = K$. Therefore K is a minimal 0-essential bi-ideal of S. $\square$

6. Prime Essential bi-ideals and Prime Essential fuzzy bi-ideals in a semigroup

A bi-ideal B of a semigroup S is called a *prime bi-ideal* if $B_1B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1 and B_2 of S. A bi-ideal B of a semigroup S is called a *semiprime bi-ideal* if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S, [11].

Definition 38. An essential bi-ideal B of a semigroup S is called

(1) a *prime* if $xy \in B$ implies $x \in B$ or $y \in B$,

(2) a *semiprime* if $x^2 \in B$ implies $x \in B$, for all $x, y \in S$.

Definition 39. An essential fuzzy bi-ideal f of a semigroup S is called

(1) a *prime* if $f(xy) \leq f(x) \vee f(y)$,

(2) a *semiprime* if $f(x^2) \leq f(x)$, for all $x, y \in S$.

Theorem 40. *Let A be a non-empty subset of a semigroup S. Then the following statement holds*:

(1) *A is a prime essential bi-ideal of S if and only if χ_A is a prime essential fuzzy bi-ideal of S,*

(2) *A is a semiprime essential ibi-deal of S if and only if χ_A is a semiprime essential fuzzy bi-ideal of S.*

Proof. (1) Suppose that A is a prime essential bi-ideal of S and let $x, y \in S$. By assumption, A is essential ideal of S. Thus by Theorem 25 χ_A is an essential fuzzy ideal of S.

If $xy \in A$, then $x \in A$ or $y \in A$.
Thus $\chi_A(x) \vee \chi_A(y) = 1 \geq \chi_A(xy)$.

If $xy \notin A$, then $\chi_A(xy) = 0 \leq \chi_A(x) \vee \chi_A(y)$.

Thus χ_A is a prime essential fuzzy bi-ideal of S.

Conversely, suppose that χ_A is a prime essential fuzzy bi-ideal of S and let $x, y \in S$. By assumption, χ_A is an essential fuzzy bi-ideal. Thus by Theorem 25, A is an essential bi-ideal
of S. If $xy \in A$, then $\chi_A(xy) = 1$. By assumption,
$\chi_A(xy) \leq \chi_A(x) \vee \chi_A(y)$. Thus $\chi_A(x) \vee \chi_A(y) = 1$ so $x \in A$ or $y \in A$. Hence A is a prime essential bi-ideal of S.

(2) Suppose that A is a semiprime essential bi-ideal of S and let $x \in S$. By assumption, A is essential bi-ideal of S. Thus by Theorem 25 χ_A is an essential fuzzy bi-ideal of S.

If $x^2 \in A$, then $x \in A$ Thus $\chi_A(x) = \chi_A(x^2) = 1$.
Hence $\chi_A(x^2) \leq \chi_A(x)$.

If $x^2 \notin A$, then $\chi_A(x^2) = 0 \leq \chi_A(x)$.

Thus χ_A is a semiprime essential fuzzy bi-ideal of S.

Conversely, suppose that χ_A is a semiprime essential fuzzy bi-ideal of S and let $x \in S$. By assumption, χ_A is an essential fuzzy bi-ideal. Thus by

Theorem 25, A is an essential bi-ideal of S. If $x \in A$, then $\chi_A(x) = 1$. By assumption $\chi_A(x^2) \leq \chi_A(x)$. Thus $\chi_A(x) = 1$ so $x \in A$. Hence A is a semiprime essential bi-ideal of S.

□

7. Conclusion

In this article, we give the concept of essential bi-ideals and fuzzy essential bi-ideals in semigroups. We study properites of essential bi-ideals and fuzzy essential bi-ideals. In the future work, we can study essential kinds ideal and their fuzzifications in other algebraic structures.

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