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INTEGRAL SOLUTION OF SOME FIXED POINT RESULTS IN ALTERING DISTANCE FUNCTION WITH ORTHOGONAL COMPLETE METRIC SPACE

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Abstract: In this paper, we prove some fixed point theorems for two self-mappings using altering distance function with orthogonal complete metric space. As an application, we derive the existence and uniqueness solution of integral equations.

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1. Introduction and Preliminaries

Banach developed the Banach contraction principle, a fundamental consequence of fixed point theory on metric space, in 1922. This approach has been developed and extended in numerous ways as a result of its applications in diverse domains of nonlinear analysis and applied mathematical analysis. Marr [1] defined a concept of convergence similar to that described for real numbers. It is worth noting that any metric space (shortly, MS) can be embedded in a partially ordered metric space (shortly, POMS), allowing for metric space

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convergence via order convergence. He also demonstrated how to improve the Banach fixed point theorem in complete metric spaces as a prescribed case of a fixed point theorem in partially ordered metric spaces. Later, Khan et al. [2] established some fixed point theorems in complete and compact metric spaces. Vishal Gupta et al. [3] presented some fixed point results involving the generalized altering distance function. Later, Gordji et al. [4],[5], and [6] introduced the concept of orthogonality and its properties. Also, they extended the Banach contraction theorem in the notion of generalized orthogonal MS. In this article, we utilize the concept of orthogonal MS and generalized altering distance function. Here, we also extend the fixed point results in these spaces.

Throughout this paper, we use the notation $[0,\infty) = \mathcal{R}_0^+$.

Definition 1. *Let $\mathcal{Q} \neq \emptyset$, a mapping $\delta : \mathcal{Q} \times \mathcal{Q} \to \mathcal{R}_0^+$ satisfies the following axiom for all $\vartheta, \eta, \mathsf{c} \in \mathcal{Q}$:*

1. $\delta(\vartheta,\eta) \geq 0$; $\delta(\vartheta,\eta) = 0$ *iff* $\vartheta = \eta$;
2. $\delta(\vartheta,\eta) = \delta(\eta,\vartheta)$;
3. $\delta(\vartheta,\eta) \leq \delta(\vartheta,\mathsf{c}) + \delta(\mathsf{c},\eta)$.

Then, $(\mathcal{Q},\delta)$ is said to be a metric space.

Definition 2. *([1]) A POMS $(\mathcal{Q},\leq)$ is a set which satisfy the three axioms:*

1. $\vartheta_1 \leq \vartheta_1$ *for all* $\vartheta_1 \in \mathcal{Q}$,
2. $\vartheta_1 \leq \vartheta_2$ *and* $\vartheta_2 \leq \vartheta_3$ *implies* $\vartheta_1 \leq \vartheta_3$ *for all* $\vartheta_1, \vartheta_2, \vartheta_3 \in \mathcal{Q}$,
3. $\vartheta_1 \leq \vartheta_2$ *and* $\vartheta_2 \leq \vartheta_1$ *implies* $\vartheta_1 = \vartheta_2$ *for all* $\vartheta_1, \vartheta_2 \in \mathcal{Q}$.

Definition 3. *([2]) A function $\Theta : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ is altering distance function satisfies the following axiom:*

1. $\Theta(\hbar)$ *is monotonically increasing and continuous,*
2. $\Theta(\hbar) = 0$ *iff* $\hbar = 0$.

The following are some examples and properties of an orthogonal set as initiated by Gordji et al. [4].

Definition 4. *([4]) Let $\mathcal{Q} \neq \emptyset$. If a binary relation $\perp \subseteq \mathcal{Q} \times \mathcal{Q}$ satisfies the following stipulation:*

$$\exists\ \vartheta_0 \in \mathcal{Q} : (\forall \vartheta \in \mathcal{Q}, \vartheta \perp \vartheta_0) \quad or \quad (\forall \vartheta \in \mathcal{Q}, \vartheta_0 \perp \vartheta).$$

Then, the pair $(\mathcal{Q}, \perp)$ is called an orthogonal set (briefly $\mathcal{O}$-set).

At this point, it is important to remember some basic, separate words like orthogonal sequence, orthogonal continuous, orthogonal complete, orthogonal MS, orthogonal preserving, and weakly orthogonal preserving.

Definition 5. *([4]) A sequence $\{\vartheta_{\mathtt{m}}\}$ of an $\mathcal{O}$-set $(\mathcal{Q}, \perp)$ is called a orthogonal sequence (shortly, $\mathcal{O}$-sequence) if*

$$(\forall \mathtt{m} \in \mathcal{N}, \vartheta_{\mathtt{m}} \perp \vartheta_{\mathtt{m}+1}) \quad or \quad (\forall \mathtt{m} \in \mathcal{N}, \vartheta_{\mathtt{m}+1} \perp \vartheta_{\mathtt{m}}).$$

Definition 6. *([4]) The triplet $(\mathcal{Q}, \perp, \delta)$ is called an orthogonal MS if $(\mathcal{Q}, \perp)$ is an $\mathcal{O}$-set and $(\mathcal{Q}, \delta)$ is a MS.*

Definition 7. *([1]) The $(\mathcal{Q}, \leq, \perp, \delta)$ is called a partially ordered orthogonal MS if $(\mathcal{Q}, \leq)$ is a partially ordered set and $(\mathcal{Q}, \perp, \delta)$ is an orthogonal MS.*

Definition 8. *Let $(\mathcal{Q}, \leq, \perp, \delta)$ be a partially ordered orthogonal MS. Then:*

1. *The $\mathcal{O}$-sequence $\{\vartheta_{\mathtt{m}}\} \subset \mathcal{Q}$ is said to be a convergent if $\exists\ \vartheta^* \in \mathcal{Q}$ such that $\lim_{\mathtt{m} \to \infty} \delta(\vartheta_{\mathtt{m}}, \vartheta^*) = 0$.*
2. *The $\mathcal{O}$-sequence $\{\vartheta_{\mathtt{m}}\}$ in $\mathcal{Q}$ is called an $\mathcal{O}$-Cauchy sequence if for every $\varepsilon > 0$, $\exists\ \mathtt{m}_0 \in \mathcal{N}$ such that $\delta(\vartheta_{\mathtt{m}}, \vartheta_{\ell}) < \varepsilon$ for all $\mathtt{m}, \ell > \mathtt{m}_0$. i.e., $\lim_{\mathtt{m}, \ell \to \infty} \delta(\vartheta_{\mathtt{m}}, \vartheta_{\ell}) = 0$.*
3. *If each $\mathcal{O}$-Cauchy sequence converges in $\mathcal{Q}$, then $(\mathcal{Q}, \leq, \perp, \delta)$ is said to be an orthogonal partial complete (shortly, $\mathcal{O}$-complete).*

Definition 9. *[4] Let $(\mathcal{Q}, \leq, \perp, \delta)$ be a partially ordered orthogonal MS. Then, a function $\mathcal{V} : \mathcal{Q} \to \mathcal{Q}$ is orthogonal partial continuous (or $\leq_{\perp}$-continuous) in $\vartheta \in \mathcal{Q}$ if each $\mathcal{O}$-sequence $\{\vartheta_{\mathtt{m}}\}$ in $\mathcal{Q}$ with $\vartheta_{\mathtt{m}} \to \vartheta$ as $\mathtt{m} \to \infty$, we have $\mathcal{V}(\vartheta_{\mathtt{m}}) \to \mathcal{V}\vartheta$ as $\mathtt{m} \to \infty$. Also, $\mathcal{V}$ is $\leq_{\perp}$-continuous on $\mathcal{Q}$ if $\mathcal{V}$ is $\leq_{\perp}$-continuous in each $\vartheta \in \mathcal{Q}$.*

Definition 10. *[4] Let $(\mathcal{Q}, \perp)$ be an $\mathcal{O}$-set. A function $\mathcal{V} : \mathcal{Q} \to \mathcal{Q}$ is called a $\perp$-preserving if $\mathcal{V}\vartheta \perp \mathcal{V}\eta$ whenever $\vartheta \perp \eta$. Also $\mathcal{V} : \mathcal{Q} \to \mathcal{Q}$ is called a weakly $\perp$-preserving if $\mathcal{V}(\vartheta) \perp \mathcal{V}(\eta)$ or $\mathcal{V}(\eta) \perp \mathcal{V}(\vartheta)$ whenever $\vartheta \perp \eta$.*

2. Main Results

In this section, we inspired the concept of the altering distance function and also to establish some fixed point theorem on orthogonal complete metric space via altering distance function.

Theorem 11. *Let $(\mathcal{Q}, \leq, \perp, \delta)$ be a partially ordered $\mathcal{O}$- complete MS. Let $\mathcal{V}_1, \mathcal{V}_2 : \mathcal{Q} \to \mathcal{Q}$ are monotonically increasing mappings satisfying*

1. *For all $\vartheta, \eta \in \mathcal{Q}$ with $\vartheta \perp \eta$*

$$[\delta(\mathcal{V}_1\vartheta, \mathcal{V}_2\eta) > 0, \xi[\delta(\mathcal{V}_1\vartheta, \mathcal{V}_2\eta)] \leq \Theta_1(\zeta(\vartheta, \eta)) - \Theta_2(\zeta(\vartheta, \eta))], \tag{1}$$

where $\zeta(\vartheta, \eta) = (\frac{\delta(\vartheta, \mathcal{V}_1\vartheta).\delta(\eta, \mathcal{V}_2\eta)}{\delta(\vartheta, \eta)}, \delta(\vartheta, \eta))$ and Θ_1, Θ_2 are altering distance function and $\xi(\vartheta) = \Theta_1(\vartheta, \eta)$;

2. *$\leq_\perp$-continuous;*
3. *$\perp$-preserving.*

Then $\mathcal{V}_1, \mathcal{V}_2$ have a unique common fixed point.

Proof. By the definition of orthogonality, there exists $\vartheta_0 \in \mathcal{Q}$ such that

$$(\forall \eta \in \mathcal{Q}, \vartheta_0 \perp \eta) \text{ or } (\forall \eta \in \mathcal{Q}, \eta \perp \vartheta_0).$$

It implies that

$$\vartheta_0 \perp \mathcal{V}\vartheta_0 \text{ or } \mathcal{V}\vartheta_0 \perp \vartheta_0.$$

Let

$$\vartheta_1 = \mathcal{V}(\vartheta_0); \vartheta_2 = \mathcal{V}(\vartheta_1) = \mathcal{V}^2(\vartheta_0); \ldots ; \vartheta_{2\mathtt{m}+1} = \mathcal{V}(\vartheta_{2\mathtt{m}}) = \mathcal{V}^{2\mathtt{m}+1}(\vartheta_0),$$

for all $\mathtt{m} \in \mathcal{N}$. Since $\mathcal{V}$ is $\perp$-preserving, $\{\vartheta_\mathtt{m}\}_{\mathtt{m} \in \mathcal{Q}}$ is an O-sequence. Let $\vartheta_0 \in \mathcal{Q}$, we define $\vartheta_{2\mathtt{m}+1} = \mathcal{V}_1\vartheta_{2\mathtt{m}}$ and $\vartheta_{2\mathtt{m}+2} = \mathcal{V}_2\vartheta_{2\mathtt{m}+1}$ for all $\mathtt{m} \in \mathcal{N}$. Also assume, $\mathcal{V}_{2\mathtt{m}} = \delta(\vartheta_\mathtt{m}, \vartheta_{\mathtt{m}+1})$. Replacing ϑ by $\vartheta_{2\mathtt{m}}$ and η by $\vartheta_{2\mathtt{m}+1}$ in (1) and we get,

$$\xi[\delta(\mathcal{V}_1\vartheta_{2\mathtt{m}}, \mathcal{V}_2\vartheta_{2\mathtt{m}+1})] = \xi[\mathcal{V}_1\vartheta_{2\mathtt{m}+1}, \mathcal{V}_2\vartheta_{2\mathtt{m}+2}]$$

$$\leq \Theta_1(\zeta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1})) - \Theta_2(\zeta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1})),$$

where

$$\zeta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}) = (\frac{\delta(\vartheta_{2\mathtt{m}}, \mathcal{V}_1\vartheta_{2\mathtt{m}}).\delta(\vartheta_{2\mathtt{m}+1}, \mathcal{V}_2\vartheta_{2\mathtt{m}+1})}{\delta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1})}, \delta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1})),$$

this gives

$$\xi(\mathcal{V}_{2\mathtt{m}+1}) \leq \Theta_1(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}) - \Theta_2(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}). \tag{2}$$

If $\mathcal{V}_{2\mathtt{m}} < \mathcal{V}_{2\mathtt{m}+1}$, then

$$\begin{aligned}\xi(\mathcal{V}_{2\mathtt{m}+1}) &\leq \Theta_1(\mathcal{V}_{2\mathtt{m}}, \mathcal{V}_{2\mathtt{m}+1}) - \Theta_2(\mathcal{V}_{2\mathtt{m}}, \mathcal{V}_{2\mathtt{m}+1}) \\ &< \Theta_1(\mathcal{V}_{2\mathtt{m}}, \mathcal{V}_{2\mathtt{m}+1}) \\ &= \xi(\mathcal{V}_{2\mathtt{m}+1}).\end{aligned}$$

Which is a contradiction. Hence, we have $\mathcal{V}_{2\mathtt{m}} \geq \mathcal{V}_{2\mathtt{m}+1}$. It follows that $\mathcal{V}_{2\mathtt{m}+1} \geq \mathcal{V}_{2\mathtt{m}+2}$. Thus we obtain, $\mathcal{V}_{2\mathtt{m}} \geq \mathcal{V}_{2\mathtt{m}+1}$, it follows that $\{\mathcal{V}_{2\mathtt{m}}\}$ is decreasing $\mathcal{O}$-sequence and converge to ρ. Letting $\mathtt{m} \to \infty$ in equation (2), we have

$$\xi(\mathcal{V}_2) \leq \Theta_1(\mathcal{V}_2, \mathcal{V}_2) - \Theta_2(\mathcal{V}_2, \mathcal{V}_2) = \xi(\mathcal{V}_2) - \Theta_2(\mathcal{V}_2, \mathcal{V}_2)$$

that implies

$$\Theta_2(\mathcal{V}_2, \mathcal{V}_2) = 0,$$

by definition of Θ_2, we have $\mathcal{V}_2 = 0$. Hence,

$$\mathcal{V}_{2\mathtt{m}} = \delta(\vartheta_{\mathtt{m}+1}, \vartheta_{\mathtt{m}}) \to 0 \ as \ \mathtt{m} \to \infty. \tag{3}$$

Next we prove that $\{\vartheta_{\mathtt{m}}\}$ be an $\mathcal{O}$-Cauchy sequence. By (3), it is enough to show that $\{\vartheta_{2\mathtt{m}}\}$ is an $\mathcal{O}$-Cauchy sequence. Suppose that $\{\vartheta_{2\mathtt{m}}\}$ is not an $\mathcal{O}$-Cauchy sequence, if every $\varepsilon > 0$, then there exists two sub $\mathcal{O}$-sequences $\{\vartheta_{2\ell(\flat)}\}$ and $\{\vartheta_{2\mathtt{m}(\flat)}\}$ such that

$$\delta(\vartheta_{2\ell(\flat)}, \vartheta_{2\mathtt{m}(\flat)}) > \varepsilon \tag{4}$$

and

$$\delta(\vartheta_{2\ell(\flat)}, \vartheta_{2\mathtt{m}(\flat)-1}) > \varepsilon \ for \ \mathtt{m}(\flat) > \ell(\flat). \tag{5}$$

Then by equations (4) and (5), we have,

$$\varepsilon < \delta(\vartheta_{2\ell(\flat)}, \vartheta_{2\mathtt{m}(\flat)}) \leq \delta(\vartheta_{2\ell(\flat)}, \vartheta_{2\mathtt{m}(\flat)-1}) + \delta(\vartheta_{2\mathtt{m}(\flat)-1}, \vartheta_{2\mathtt{m}(\flat)})$$

$$< \varepsilon + \delta(\vartheta_{2\mathtt{m}(\flat)-1}, \vartheta_{2\mathtt{m}(\flat)}). \tag{6}$$

Letting $\flat \to \infty$ in (6) and using (3), we get

$$\varepsilon < \lim_{\flat\to\infty} \delta(\vartheta_{2\ell(\flat)}, \vartheta_{2\mathtt{m}(\flat)}) < \varepsilon + 0,$$

implies

$$\lim_{\flat\to\infty} \delta(\vartheta_{2\ell(\flat)}, \vartheta_{2\mathtt{m}(\flat)}) = \varepsilon. \tag{7}$$

Using triangle inequality

$$\delta(\vartheta_{2\mathtt{m}(\flat)+1}, \vartheta_{2\ell(\flat)}) \leq \delta(\vartheta_{2\mathtt{m}(\flat)+1}, \vartheta_{2\mathtt{m}(\flat)}) + \delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)})$$

$$\delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)}) \leq \delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\mathtt{m}(\flat)+1}) + \delta(\vartheta_{2\mathtt{m}(\flat)+1}, \vartheta_{2\ell(\flat)}). \tag{8}$$

Letting $\flat \to \infty$ in (8) and we get

$$\lim_{\flat\to\infty} \delta(\vartheta_{2\mathtt{m}(\flat)+1}, \vartheta_{2\ell(\flat)}) \leq 0 + \varepsilon$$

and

$$\varepsilon \leq 0 + \lim_{\flat\to\infty} \delta(\vartheta_{2\mathtt{m}(\flat)+1}, \vartheta_{2\ell(\flat)}),$$

implies

$$\lim_{\flat\to\infty} \delta(\vartheta_{2\mathtt{m}(\flat)+1}, \vartheta_{2\ell(\flat)}) = \varepsilon. \tag{9}$$

By using triangle inequality we can show that

$$\lim_{\flat\to\infty} \delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1}) = \varepsilon. \tag{10}$$

Replacing ϑ by $\vartheta_{2\mathtt{m}(\flat)}$ and η by $\vartheta_{2\ell(\flat)-1}$ in (1) and we get,

$$\begin{aligned}\xi(\delta(\mathcal{V}_1\vartheta_{2\mathtt{m}(\flat)}, \mathcal{V}_2\vartheta_{2\ell(\flat)-1})) \leq \Theta_1(\zeta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})) \\ - \Theta_2(\zeta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1}))\end{aligned} \tag{11}$$

where

$$\zeta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})$$

$$= (\frac{\delta(\vartheta_{2\mathtt{m}(\flat)}, \mathcal{V}_1\vartheta_{2\mathtt{m}(\flat)}).\delta(\vartheta_{2\ell(\flat)-1}, \mathcal{V}_2\vartheta_{2\ell(\flat)-1})}{\delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})}, \delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1}))$$

this gives

$$\begin{aligned}
&\xi(\delta(\vartheta_{2\mathtt{m}(\flat)+1}, \vartheta_{2\ell(\flat)})) \\
&\leq \Theta_1(\frac{\delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\mathtt{m}(\flat)+1}).\delta(\vartheta_{2\ell(\flat)-1}, \vartheta_{2\ell(\flat)})}{\delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})}, \delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})) \\
&- \Theta_2(\frac{\delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\mathtt{m}(\flat)+1}).\delta(\vartheta_{2\ell(\flat)-1}, \vartheta_{2\ell(\flat)})}{\delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})}, \delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})).
\end{aligned}$$

Letting $\flat \to \infty$ in above inequality, we get $\xi)(\varepsilon) \leq \Theta_1(0, \varepsilon) - \Theta_2)0, \varepsilon < \xi(\varepsilon)$, which is a contradiction. Hence $\{\vartheta_{\mathtt{m}}\}$ is $\mathcal{O}$-Cauchy sequence. Since $\mathcal{O}$-completeness, there exist $\rho \in \mathcal{Q}$ such that $\lim_{\flat\to\infty} \vartheta_{\mathtt{m}} = \rho$. Let $\vartheta = \vartheta_{2\mathtt{m}}$ and $\eta = \rho$ in (1), we get

$$\xi[\delta(\mathcal{V}_1\vartheta_{2\mathtt{m}}, \mathcal{V}_2\rho)] \leq \Theta_1(\zeta(\vartheta_{2\mathtt{m}}, \rho)) - \Theta_2(\zeta(\vartheta_{2\mathtt{m}}, \rho)) \tag{12}$$

where

$$\begin{aligned}
\zeta(\vartheta_{2\mathtt{m}}, \rho) &= (\frac{\delta(\vartheta_{2\mathtt{m}}, \mathcal{V}_1\vartheta_{2\mathtt{m}}).\delta(\rho, \mathcal{V}_2\rho)}{\delta(\vartheta_{2\mathtt{m}}, \rho)}, \delta(\vartheta_{2\mathtt{m}}, \rho)) \\
&= (\frac{\delta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}).\delta(\rho, \mathcal{V}_2\rho)}{\delta(\vartheta_{2\mathtt{m}}, \rho)}, \delta(\vartheta_{2\mathtt{m}}, \rho)).
\end{aligned}$$

Letting $\mathtt{m} \to \infty$ in (12), we get

$$\begin{aligned}
\xi[\delta(\mathcal{V}_1\vartheta_{2\mathtt{m}}, \mathcal{V}_2\rho)] &\leq \Theta_1(\frac{\delta(\rho, \rho).\delta(\rho, \mathcal{V}_2\rho)}{\delta(\rho, \rho)}, \delta(\rho, \rho)) \\
&\quad - \Theta_2(\frac{\delta(\rho, \rho).\delta(\rho, \mathcal{V}_2\rho)}{\delta(\rho, \rho)}, \delta(\rho, \rho)).
\end{aligned}$$

If $\delta(\rho, \mathcal{V}_2\rho) \neq 0$, and since Θ_1 and Θ_2 are non-decreasing, and $\Theta_2(\vartheta, \eta) = 0$ if and only if $\vartheta = \eta = 0$, we get $\xi(\delta(\rho, \mathcal{V}_2\rho)) < \xi(\delta(\rho, \mathcal{V}_2\rho))$, it follows a contradiction. Therefore, we get $\xi(\delta(\rho, \mathcal{V}_2\rho)) = 0$ or $\rho = \mathcal{V}_2\rho$. From this, we obtain $\rho = \mathcal{V}_1\rho$. Hence ρ is a common fixed point of $\mathcal{V}_1$ and $\mathcal{V}_2$.

Now, prove that uniqueness part, suppose that ρ_1 and ρ_2 are two distinct common fixed points of $\mathcal{V}_1$ and $\mathcal{V}_2$. By the choice of ϑ_0 in the proof of the first part, we get

$$[\vartheta_0 \perp \rho_1 \text{ and } \vartheta_0 \perp \rho_2] \text{ or } [\rho_1 \perp \vartheta_0 \text{ and } \rho_2 \perp \vartheta_0]$$

implies $\rho_1 \perp \rho_2$ or $\rho_2 \perp \rho_1$.

Since $\mathcal{V}$ is $\perp$-preserving, we have

$[\mathcal{V}(\vartheta_0) \perp \mathcal{V}(\rho_1)$ or $\mathcal{V}(\rho_1) \perp \mathcal{V}(\vartheta_0)]$ and $[\mathcal{V}(\vartheta_0) \perp \mathcal{V}(\rho_2)$ or $\mathcal{V}(\rho_2) \perp \mathcal{V}(\vartheta_0)]$ implies $\mathcal{V}\rho_1 \perp \mathcal{V}\rho_2$ or $\mathcal{V}\rho_2 \perp \mathcal{V}\rho_1$.

Therefore, by equation (1)

$$\xi[\delta(\mathcal{V}_1\rho_1, \mathcal{V}_2\rho_2)] \leq \Theta_1(\zeta(\rho_1, \rho_2)) - \Theta(\zeta(\rho_1, \rho_2)), \tag{13}$$

where

$$\begin{aligned} \zeta(\rho_1, \rho_2) &= (\frac{\delta(\rho_1, \mathcal{V}_1\rho_1).\delta(\rho_2, \mathcal{V}_2\rho_2)}{\delta(\rho_1, \rho_2)}, \delta(\rho_1, \rho_2)) \\ &= (\frac{\delta(\rho_1, \rho_1).\delta(\rho_2, \rho_2)}{\delta(\rho_1, \rho_2)}, \delta(\rho_1, \rho_2)) \\ &= (0, \delta(\rho_1, \rho_2)). \end{aligned}$$

From (13)

$$\xi[\delta(\rho_1, \rho_2)] \leq \Theta_1(0, \delta(\rho_1, \rho_2)) - \Theta_2(0, \delta(\rho_1, \rho_2)).$$

Clearly, Θ_1 is non-decreasing in all variables, we have $\rho_1 = \rho_2$. □

Theorem 12. *Let $(\mathcal{Q}, \leq, \perp, \delta)$ be a partially ordered $\mathcal{O}$- complete MS. Let $\mathcal{V}, \mathcal{U} : \mathcal{Q} \to \mathcal{Q}$ are monotonically increasing mappings satisfying:*

1. *For all $\vartheta, \eta \in \mathcal{Q}$ with $\vartheta \perp \eta$*

$$\Big[\delta(\mathcal{V}_1\vartheta, \mathcal{V}_2\eta) > 0, \quad \int_0^{\xi[\delta(\mathcal{V}_1\vartheta, \mathcal{V}_2\eta)]} \Omega(\hbar)\delta\hbar \leq \int_0^{\Theta_1(\zeta(\vartheta,\eta))} \Omega(\hbar)\delta\hbar - \int_0^{\Theta_2(\zeta(\vartheta,\eta))} \Omega(\hbar)\delta\hbar\Big], \tag{14}$$

where

$$\zeta(\vartheta, \eta) = (\frac{\delta(\vartheta, \mathcal{V}_1\vartheta).\delta(\eta, \mathcal{V}_2\eta)}{\delta(\vartheta, \eta)}, \delta(\vartheta, \eta))$$

and Θ_1, Θ_2 are altering distance function and $\xi(\vartheta) = \Theta_1(\vartheta, \vartheta)$. Here $\Theta_1 : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ is Lebesgue integrable function as summable for each compact $\mathcal{R}^+$, such that for $\varepsilon > 0$,
$\int \Omega(\hbar)\delta\hbar > 0$;

2. $\leq_\perp$-continuous;

3. $\perp$-preserving.

Then $\mathcal{V}_1$ and $\mathcal{V}_2$ have unique fixed point.

Proof. By the definition of orthogonality, there exists $\vartheta_0 \in \mathcal{Q}$ such that

$$(\forall \eta \in \mathcal{Q}, \vartheta_0 \perp \eta) \text{ or } (\forall \eta \in \mathcal{Q}, \eta \perp \vartheta_0).$$

It implies that

$$\vartheta_0 \perp \mathcal{V}\vartheta_0 \text{ or } \mathcal{V}\vartheta_0 \perp \vartheta_0.$$

Let

$$\vartheta_1 = \mathcal{V}(\vartheta_0); \vartheta_2 = \mathcal{V}(\vartheta_1) = \mathcal{V}^2(\vartheta_0); \ldots; \vartheta_{2\mathtt{m}+1} = \mathcal{V}(\vartheta_{2\mathtt{m}}) = \mathcal{V}^{2\mathtt{m}+1}(\vartheta_0),$$

for all $\mathtt{m} \in \mathcal{N}$. Since $\mathcal{V}$ is $\perp$-preserving, $\{\vartheta_\mathtt{m}\}_{\mathtt{m}\in\mathcal{Q}}$ is an O-sequence. Let $\vartheta_0 \in \mathcal{Q}$ and define $\vartheta_{2\mathtt{m}+1} = \mathcal{V}_1\vartheta_{2\mathtt{m}}$ and $\vartheta_{2\mathtt{m}+2} = \mathcal{V}_2\vartheta_{2\mathtt{m}+1}$. Let $\mathcal{V}_{2\mathtt{m}} = \delta(\vartheta_\mathtt{m}, \vartheta_{\mathtt{m}+1})$. Replacing ϑ by $\vartheta_{2\mathtt{m}}$ and η by $\vartheta_{2\mathtt{m}+1}$ in (1) and we get,

$$\begin{aligned}
&\int_0^{\xi[\delta(\mathcal{V}_1\vartheta_{2\mathtt{m}}, \mathcal{V}_2\vartheta_{2\mathtt{m}+1})]} \Omega(\hbar)\delta\hbar \\
&\leq \int_0^{\Theta_1(\zeta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}))} \Omega(\hbar)\delta\hbar - \int_0^{\Theta_2(\zeta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}))} \Omega(\hbar)\delta\hbar,
\end{aligned}$$

where $\zeta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}) = (\frac{\delta(\vartheta_{2\mathtt{m}}, \mathcal{V}_1\vartheta_{2\mathtt{m}}).\delta(\vartheta_{2\mathtt{m}+1}, \mathcal{V}_2\vartheta_{2\mathtt{m}+1})}{\delta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1})}, \delta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}))$.
This gives

$$\int_0^{\xi(\mathcal{V}_{2\mathtt{m}+1})} \Omega(\hbar)\delta\hbar \leq \int_0^{\Theta_1(\mathcal{V}_{2\mathtt{m}}, \mathcal{V}_{2\mathtt{m}+1})} \Omega(\hbar)\delta\hbar - \int_0^{\Theta_2(\mathcal{V}_{2\mathtt{m}}, \mathcal{V}_{2\mathtt{m}+1})} \Omega(\hbar)\delta\hbar. \tag{15}$$

If $\mathcal{V}_{2\mathtt{m}} < \mathcal{V}_{2\mathtt{m}+1}$, then

$$\begin{aligned}
\int_0^{\xi(\mathcal{V}_{2\mathtt{m}+1})} \Omega(\hbar)\delta\hbar &\leq \int_0^{\Theta_1(\mathcal{V}_{2\mathtt{m}+1}, \mathcal{V}_{2\mathtt{m}+1})} \Omega(\hbar)\delta\hbar - \int_0^{\Theta_2(\mathcal{V}_{2\mathtt{m}}, \mathcal{V}_{2\mathtt{m}+1})} \Omega(\hbar)\delta\hbar \\
&< \int_0^{\Theta_1(\mathcal{V}_{2\mathtt{m}+1}, \mathcal{V}_{2\mathtt{m}+1})} \Omega(\hbar)\delta\hbar \\
&= \int_0^{\xi(\mathcal{V}_{2\mathtt{m}+1})} \Omega(\hbar)\delta\hbar.
\end{aligned}$$

Which is a contradiction, therefore $\mathcal{V}_{2\mathtt{m}} \geq \mathcal{V}_{2\mathtt{m}+1}$. In this way, it follows that $\mathcal{V}_{2\mathtt{m}+1} \geq \mathcal{V}_{2\mathtt{m}+2}$. Thus we have, $\mathcal{V}_{2\mathtt{m}} \geq \mathcal{V}_{2\mathtt{m}+1}$, it follows that $\{\mathcal{V}_{2\mathtt{m}}\}$ is decreasing $\mathcal{O}$-sequence and converge to ρ. Taking $\mathtt{m} \to \infty$ in (15), we have,

$$\begin{aligned}\int_0^{\xi(\mathcal{V}_2)} \Omega(\hbar)\delta\hbar &\leq \int_0^{\Theta_1(\mathcal{V}_2,\mathcal{V}_2)} \Omega(\hbar)\delta\hbar - \int_0^{\Theta_2(\mathcal{V}_2,\mathcal{V}_2)} \Omega(\hbar)\delta\hbar \\ &= \int_0^{\xi(\mathcal{V}_2)} \Omega(\hbar)\delta\hbar - \int_0^{\Theta_2(\mathcal{V}_2,\mathcal{V}_2)} \Omega(\hbar)\delta\hbar.\end{aligned}$$

This implies $\int_0^{\Theta_2(\mathcal{V}_2,\mathcal{V}_2)} \Omega(\hbar)\delta\hbar = 0$, by definition of Θ_2, we have $\mathcal{V}_2 = 0$. Hence,

$$\mathcal{V}_{2\mathtt{m}} = \delta(\vartheta_{\mathtt{m}+1}, \vartheta_{\mathtt{m}}) \to 0 \text{ as } \mathtt{m} \to \infty. \tag{16}$$

Now as in Theorem 11, we have

$$\lim_{\flat \to \infty} \delta(\vartheta_{2\ell(\flat)}, \vartheta_{2\mathtt{m}(\flat)}) = \varepsilon. \tag{17}$$

Replacing ϑ by $\vartheta_{2\mathtt{m}(\flat)}$ and η by $\vartheta_{2\ell(\flat)-1}$ in (14) and we get,

$$\begin{aligned}&\int_0^{\xi[\delta(\mathcal{V}_1\vartheta_{2\mathtt{m}(\flat)},\mathcal{V}_2\vartheta_{2\ell(\flat)-1})]} \\ &\leq \int_0^{\Theta_1(\zeta(\vartheta_{2\mathtt{m}(\flat)},\vartheta_{2\ell(\flat)-1}))} \Omega(\hbar)\delta\hbar - \int_0^{\Theta_2(\zeta(\vartheta_{2\mathtt{m}(\flat)},\vartheta_{2\ell(\flat)-1}))} \Omega(\hbar)\delta\hbar,\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\zeta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1}) \\ &= (\frac{\delta(\vartheta_{2\mathtt{m}(\flat)}, \mathcal{V}_1\vartheta_{2\mathtt{m}(\flat)}).\delta(\vartheta_{2\ell(\flat)-1}, \mathcal{V}_2\vartheta_{2\ell(\flat)-1})}{\delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})}, \delta(\vartheta_{2\mathtt{m}(\flat)}, \vartheta_{2\ell(\flat)-1})).\end{aligned}$$

Letting $\flat \to \infty$ in above inequality, it follows that $\{\vartheta_{\mathtt{m}}\}$ is an $\mathcal{O}$-Cauchy sequence. Since $\mathcal{O}$-completeness, there exist $\rho \in \mathcal{Q}$ such that $\lim_{\flat\to\infty} \vartheta_{\mathtt{m}} = \rho$.

Put $\vartheta = \vartheta_{2\mathtt{m}}$ and $\eta = \rho$ in (14), we get

$$\int_0^{\xi(\delta(\mathcal{V}_1\vartheta_{2\mathtt{m}},\mathcal{V}_2\rho))} \Omega(\hbar)\delta\hbar \leq \int_0^{\Theta_1(\zeta(\vartheta_{2\mathtt{m}},\rho))} \Omega(\hbar)\delta\hbar - \int_0^{\Theta_2(\zeta(\vartheta_{2\mathtt{m}},\rho))} \Omega(\hbar)\delta\hbar, \tag{19}$$

where

$$\zeta(\vartheta_{2\mathtt{m}}, \rho) = (\frac{\delta(\vartheta_{2\mathtt{m}}, \mathcal{V}_1\vartheta_{2\mathtt{m}}).\delta(\rho, \mathcal{V}_2\rho)}{\delta(\vartheta_{2\mathtt{m}}, \rho)}, \delta(\vartheta_{2\mathtt{m}}, \rho))$$

$$= (\frac{\delta(\vartheta_{2\mathtt{m}}, \vartheta_{2\mathtt{m}+1}).\delta(\rho, \mathcal{V}_2\rho)}{\delta(\vartheta_{2\mathtt{m}}, \rho)}, \delta(\vartheta_{2\mathtt{m}}, \rho)).$$

Letting $\mathtt{m} \to \infty$ in (19) and since Θ_1 and Θ_2 are non-decreasing, and $\Theta_2(\vartheta, \eta) = 0$ if and only if $\vartheta = \eta = 0$, we obtain

$$\int_0^{\xi(\delta(\rho,\mathcal{V}_2\rho))} \Omega(\hbar)\delta\hbar < \int_0^{\xi(\delta(\rho,\mathcal{V}_2\rho))} \Omega(\hbar)\delta\hbar.$$

Therefore, we have $\rho = \mathcal{V}_2\rho$. Uniqueness part is the same way as in the proof of Theorem 11. □

3. An application to resolve an integral equations

Consider the integral equations:

$$\begin{aligned} \mathtt{u}(\varsigma) &= \int_0^{\mathcal{V}_1} \mathtt{f}_1(\varsigma, \varrho, \mathtt{u}(\varrho))\mathtt{d}\varrho + \mathtt{h}(\varsigma), \varsigma \in [0, \mathcal{V}_1], \\ \mathtt{u}(\varsigma) &= \int_0^{\mathcal{V}_1} \mathtt{f}_2(\varsigma, \varrho, \mathtt{u}(\varrho))\mathtt{d}\varrho + \mathtt{h}(\varsigma), \varsigma \in [0, \mathcal{V}_1], \end{aligned} \tag{20}$$

where $\mathcal{V}_1 > 0$. The purpose of this section is to give an existence theorem for common solution of (20) using Theorem 11. This application is inspired by [7]. Previously, we consider the space $\mathcal{X} = \mathcal{C}(\mathtt{T})(\mathtt{T} = [0, \mathcal{V}_1])$ of continuous functions defined on $\mathtt{T}$. Obviously, this space with the metric given by

$$\mathtt{d}(\vartheta, \eta) = \sup_{\varsigma \in \mathtt{T}} |\vartheta(\varsigma) - \eta(\varsigma)|, \text{ for all } \vartheta, \eta \in \mathcal{C}(\mathtt{T}), \tag{21}$$

is a complete metric space. Define an orthogonal relation $\perp$ on $\mathcal{C}(\mathtt{T})$ by

$$\vartheta \perp \eta \text{ if and only if } \vartheta(\varsigma)\eta(\varsigma) \geq \vartheta(\varsigma) \vee \eta(\varsigma), \text{ for all } \vartheta, \eta \in \mathcal{C}(\mathtt{T}).$$

Then $(\mathcal{C}(\mathtt{T}), \perp, \mathtt{d})$ is an orthogonal complete metric space. $\mathcal{C}(\mathtt{T})$ can also be equipped with the partial order $\leq$ given by

$$\vartheta, \eta \in \mathcal{C}(\mathtt{T}), \vartheta \leq \eta \Longleftrightarrow \vartheta(\varsigma) \leq \eta(\varsigma), \forall \varsigma \in \mathtt{T}. \tag{22}$$

Now, we will prove the following result.

Theorem 13. *Suppose that the following hypotheses hold:*

(i) $\mathtt{f}_1, \mathtt{f}_2 : \mathtt{T} \times \mathtt{T} \times \mathcal{R} \to \mathcal{R}$ *and* $\mathtt{h} : \mathcal{R} \to \mathcal{R}$ *are continuous,*

(ii) *for all* $\varsigma, \varrho \in \mathtt{T}$,

$$\mathtt{f}_1(\varsigma, \varrho, \mathtt{u}(\varrho)) \leq \mathtt{f}_2(\varsigma, \varrho, \int_0^{\mathcal{V}_1} \mathtt{f}_1(\varrho, \tau, \mathtt{u}(\tau))\mathtt{d}\tau + \mathtt{h}(\varrho)), \tag{23}$$

$$\mathtt{f}_2(\varsigma, \varrho, \mathtt{u}(\varrho)) \leq \mathtt{f}_1(\varsigma, \varrho, \int_0^{\mathcal{V}_1} \mathtt{f}_2(\varrho, \tau, \mathtt{u}(\tau))\mathtt{d}\tau + \mathtt{h}(\varrho)), \tag{24}$$

(iii) *there exist* $\mathtt{f}_1, \mathtt{f}_2 \geq 0$ *such that*

$$\begin{aligned} &|\mathcal{G}_\vartheta(\varsigma) - \mathcal{F}_\eta(\varsigma)| \\ &\leq \Theta_1\left(\frac{|\vartheta(\varsigma) - \mathcal{F}_\eta(\varsigma) - \mathtt{h}(\varsigma)|.|\eta(\varsigma) - \mathcal{G}_\vartheta(\varsigma) - \mathtt{h}(\varsigma)|}{|\vartheta(\varsigma) - \eta(\varsigma)|}, |\vartheta(\varsigma) - \eta(\varsigma)|\right) \\ &- \Theta_2\left(\frac{|\vartheta(\varsigma) - \mathcal{F}_\eta(\varsigma) - \mathtt{h}(\varsigma)|.|\eta(\varsigma) - \mathcal{G}_\vartheta(\varsigma) - \mathtt{h}(\varsigma)|}{|\vartheta(\varsigma) - \eta(\varsigma)|}, |\vartheta(\varsigma) - \eta(\varsigma)|\right), \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathcal{F}_\eta(\varsigma) &= \int_0^{\mathcal{V}_1} \mathtt{f}_1(\varsigma, \varrho, \eta(\varrho))\mathtt{d}\varrho, \varsigma \in \mathtt{T}, \\ \mathcal{G}_\vartheta(\varsigma) &= \int_0^{\mathcal{V}_1} \mathtt{f}_2(\varsigma, \varrho, \vartheta(\varrho))\mathtt{d}\varrho, \varsigma \in \mathtt{T}, \end{aligned} \tag{26}$$

for every $\vartheta, \eta \in \mathcal{X}$ *and* $\vartheta \leq \eta$ *and* $\varsigma \in \mathtt{T}$. *Then, the integral equations (20) have a solution* $\mathtt{u}^* \in \mathcal{C}(\mathtt{T})$.

Proof. Define $\mathcal{V}_1, \mathcal{V}_2 : \mathcal{C}(\mathtt{T}) \to \mathcal{C}(\mathtt{T})$ by

$$\begin{aligned} \mathcal{V}_1\vartheta(\varsigma) &= \mathcal{F}_\vartheta(\varsigma) + \mathtt{h}(\varsigma), \varsigma \in \mathtt{T}, \\ \mathcal{V}_2\vartheta(\varsigma) &= \mathcal{G}_\vartheta(\varsigma) + \mathtt{h}(\varsigma), \varsigma \in \mathtt{T}. \end{aligned} \tag{27}$$

Now, we will prove that $\mathcal{V}_1$ and $\mathcal{V}_2$ are monotonically increasing. From (ii), for all $\varsigma \in \mathtt{T}$, we have

$$\begin{aligned} \mathcal{V}_1\vartheta(\varsigma) &= \int_0^{\mathcal{V}_1} \mathtt{f}_1(\varsigma, \varrho, \vartheta(\varrho))\mathtt{d}\varrho + \mathtt{h}(\varsigma) \\ &\leq \int_0^{\mathcal{V}_1} \mathtt{f}_2\left(\varsigma, \varrho, \int_0^{\mathcal{V}_1} \mathtt{f}_1(\varrho, \tau, \mathtt{u}(\tau))\mathtt{d}\tau + \mathtt{h}(\varrho)\right)\mathtt{d}\varrho + \mathtt{h}(\varsigma) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\mathcal{V}_1} \mathtt{f}_2(\varsigma, \varrho, \mathcal{V}_1\vartheta(\varrho))\mathtt{d}\varrho + \mathtt{h}(\varsigma) \\
&= \mathcal{V}_2\mathcal{V}_1\vartheta(\varsigma). \qquad (28)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{V}_2\vartheta(\varsigma) &= \int_0^{\mathcal{V}_1} \mathtt{f}_2(\varsigma, \varrho, \vartheta(\varrho))\mathtt{d}\varrho + \mathtt{h}(\varsigma) \\
&\leq \int_0^{\mathcal{V}_1} \mathtt{f}_1\Big(\varsigma, \varrho, \int_0^{\mathcal{V}_1} \mathtt{f}_2(\varrho, \tau, \mathtt{u}(\tau))\mathtt{d}\tau + \mathtt{h}(\varrho)\Big)\mathtt{d}\varrho + \mathtt{h}(\varsigma) \\
&= \int_0^{\mathcal{V}_1} \mathtt{f}_1(\varsigma, \varrho, \mathcal{V}_2\vartheta(\varrho))\mathtt{d}\varrho + \mathtt{h}(\varsigma) \\
&= \mathcal{V}_1\mathcal{V}_2\vartheta(\varsigma). \qquad (29)
\end{aligned}
$$

Then, we have $\mathcal{V}_1\vartheta \leq \mathcal{V}_2\mathcal{V}_1\vartheta$ and $\mathcal{V}_2\vartheta \leq \mathcal{V}_1\mathcal{V}_2\vartheta$, for all $\vartheta \in \mathcal{C}(\mathtt{T})$. This implies that $\mathcal{V}_1$ and $\mathcal{V}_2$ are monotonically increasing. Now, for all $\vartheta, \eta \in \mathcal{C}(\mathtt{T})$ such that $\vartheta \leq \eta$, by (iii), we have

$$
\begin{aligned}
&|\mathcal{V}_2\eta(\varsigma) - \mathcal{V}_1\vartheta(\varsigma)| \\
&= |\mathcal{G}_\eta(\varsigma) - \mathcal{F}_\vartheta(\varsigma)| \\
&\leq \Theta_1\Big(\frac{|\vartheta(\varsigma) - \mathcal{F}_\vartheta(\varsigma) - \mathtt{h}(\varsigma)|.|\eta(\varsigma) - \mathcal{G}_\eta(\varsigma) - \mathtt{h}(\varsigma)|}{|\vartheta(\varsigma) - \eta(\varsigma)|}, |\vartheta(\varsigma) - \eta(\varsigma)|\Big) \\
&- \Theta_2\Big(\frac{|\vartheta(\varsigma) - \mathcal{F}_\vartheta(\varsigma) - \mathtt{h}(\varsigma)|.|\eta(\varsigma) - \mathcal{G}_\eta(\varsigma) - \mathtt{h}(\varsigma)|}{|\vartheta(\varsigma) - \eta(\varsigma)|}, |\vartheta(\varsigma) - \eta(\varsigma)|\Big). \qquad (30)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathtt{d}(\mathcal{V}_2\eta, \mathcal{V}_1\vartheta) \\
&= \sup_{\varsigma\in[0,\mathcal{V}_1]} |\mathcal{V}_2\eta(\varsigma) - \mathcal{V}_1\vartheta(\varsigma)| \\
&\leq \sup_{\varsigma\in[0,\mathcal{V}_1]} \Theta_1\Big(\frac{|\vartheta(\varsigma) - \mathcal{F}_\vartheta(\varsigma) - \mathtt{h}(\varsigma)|.|\eta(\varsigma) - \mathcal{G}_\eta(\varsigma) - \mathtt{h}(\varsigma)|}{|\vartheta(\varsigma) - \eta(\varsigma)|}, |\vartheta(\varsigma) - \eta(\varsigma)|\Big) \\
&- \sup_{\varsigma\in[0,\mathcal{V}_1]} \Theta_2\Big(\frac{|\vartheta(\varsigma) - \mathcal{F}_\vartheta(\varsigma) - \mathtt{h}(\varsigma)|.|\eta(\varsigma) - \mathcal{G}_\eta(\varsigma) - \mathtt{h}(\varsigma)|}{|\vartheta(\varsigma) - \eta(\varsigma)|}, |\vartheta(\varsigma) - \eta(\varsigma)|\Big) \\
&= \Theta_1\Big(\frac{\sup_{\varsigma\in[0,\mathcal{V}_1]} |\vartheta(\varsigma) - \mathcal{F}_\vartheta(\varsigma) - \mathtt{h}(\varsigma)|.\sup_{\varsigma\in[0,\mathcal{V}_1]} |\eta(\varsigma) - \mathcal{G}_\eta(\varsigma) - \mathtt{h}(\varsigma)|}{\sup_{\varsigma\in[0,\mathcal{V}_1]} |\vartheta(\varsigma) - \eta(\varsigma)|}, \\
&\qquad \sup_{\varsigma\in[0,\mathcal{V}_1]} |\vartheta(\varsigma) - \eta(\varsigma)|\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \Theta_2\Bigg(\frac{\sup_{\varsigma\in[0,\mathcal{V}_1]}|\vartheta(\varsigma)-\mathcal{F}_\vartheta(\varsigma)-\mathtt{h}(\varsigma)|.\sup_{\varsigma\in[0,\mathcal{V}_1]}|\eta(\varsigma)-\mathcal{G}_\eta(\varsigma)-\mathtt{h}(\varsigma)|}{\sup_{\varsigma\in[0,\mathcal{V}_1]}|\vartheta(\varsigma)-\eta(\varsigma)|},\\
&\quad \sup_{\varsigma\in[0,\mathcal{V}_1]}|\vartheta(\varsigma)-\eta(\varsigma)|\Bigg)\\
&= \Theta_1\Bigg(\frac{\sup_{\varsigma\in[0,\mathcal{V}_1]}|\vartheta(\varsigma)-\mathcal{V}_1\vartheta(\varsigma)|.\sup_{\varsigma\in[0,\mathcal{V}_1]}|\eta(\varsigma)-\mathcal{V}_2\eta(\varsigma)|}{\sup_{\varsigma\in[0,\mathcal{V}_1]}|\vartheta(\varsigma)-\eta(\varsigma)|},\\
&\quad \sup_{\varsigma\in[0,\mathcal{V}_1]}|\vartheta(\varsigma)-\eta(\varsigma)|\Bigg)\\
&- \Theta_2\Bigg(\frac{\sup_{\varsigma\in[0,\mathcal{V}_1]}|\vartheta(\varsigma)-\mathcal{V}_1\vartheta(\varsigma)|.\sup_{\varsigma\in[0,\mathcal{V}_1]}|\eta(\varsigma)-\mathcal{V}_2\eta(\varsigma)|}{\sup_{\varsigma\in[0,\mathcal{V}_1]}|\eta(\varsigma)-\eta(\varsigma)|},\\
&\quad \sup_{\varsigma\in[0,\mathcal{V}_1]}|\vartheta(\varsigma)-\eta(\varsigma)|\Bigg)\\
&= \Theta_1\Bigg(\frac{\mathtt{d}(\vartheta(\varsigma),\mathcal{V}_1\vartheta(\varsigma)).\mathtt{d}(\eta(\varsigma),\mathcal{V}_2\eta(\varsigma))}{\mathtt{d}(\vartheta(\varsigma),\eta(\varsigma))},\mathtt{d}(\vartheta(\varsigma),\eta(\varsigma))\Bigg)\\
&- \Theta_2\Bigg(\frac{\mathtt{d}(\vartheta(\varsigma),\mathcal{V}_1\vartheta(\varsigma)).\mathtt{d}(\eta(\varsigma),\mathcal{V}_2\eta(\varsigma))}{\mathtt{d}(\vartheta(\varsigma),\eta(\varsigma))},\mathtt{d}(\vartheta(\varsigma),\eta(\varsigma))\Bigg).
\end{aligned}
\tag{31}
$$

Then

$$
\mathtt{d}(\mathcal{V}_2\eta,\mathcal{V}_1\vartheta)\le\Theta_1(\mathcal{K}(\vartheta,\eta))-\Theta_2(\mathcal{K}(\vartheta,\eta)),
\tag{32}
$$

where $\mathcal{K}(\vartheta,\eta)=\Bigg(\frac{\mathtt{d}(\vartheta(\varsigma),\mathcal{V}_1\vartheta(\varsigma)).\mathtt{d}(\eta(\varsigma),\mathcal{V}_2\eta(\varsigma))}{\mathtt{d}(\vartheta(\varsigma),\eta(\varsigma))},\mathtt{d}(\vartheta(\varsigma),\eta(\varsigma))\Bigg)$, for all $\vartheta,\eta\in\mathcal{C}(\mathtt{T})$ such that $\vartheta\le\eta$. This implies that for all $\vartheta,\eta\in\mathcal{C}(\mathtt{T})$ such that $\eta\le\vartheta$,

$$
\mathtt{d}(\mathcal{V}_2\eta,\mathcal{V}_1\vartheta)\le\Theta_1(\mathcal{K}(\vartheta,\eta))-\Theta_2(\mathcal{K}(\vartheta,\eta)).
\tag{33}
$$

Hence the contractive condition required by Theorem 11 is satisfied. Now, all the required hypotheses of Theorem 11 are satisfied. Then, there exists $\mathtt{u}^*\in\mathcal{C}(\mathtt{T})$, a common fixed point of $\mathcal{V}_1$ and $\mathcal{V}_2$, that is, $\mathtt{u}^*$ is a solution to (20). □

4. Conclusion

In this paper, we have used altering distance function for proved fixed point theorems in orthogonal complete MS.

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