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BLOCK INVERSION ON ℘WORDS WITH ZERO PALINDROMIC DEFECT

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Abstract: The role of ℘words and inversions in molecular biology led to a unified study of inversions on ℘words. In this paper, we use a generalization of the concept of inversion termed as block inversion on finite rich ℘words. A comparison of block inversion on finite total words and on finite ℘words is made. We conclude that the total number of ℘words in the block inversion set of a finite rich ℘word of length n is strictly less than 2^{n-1}.

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1. Introduction

Combinatorics on words and the study of formal languages are related since

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both fields investigate different aspects of words [10, 16]. Recent molecular genetic research have suggested that inversion (or reversal) operation is one of the vital operations on DNA sequences [12]. Inversion is a concept of rearrangement of a word. If a word x exists as the mirror image of another word y then x is the inversion of y. From the formal language theory point of view, many biological operations such as string matching and alignment problems consider hairpin inversion, pseudo-inversion and non-overlapping inversion [4, 11, 19]. The authors in [7] proved that context-free as well as regular languages are closed under the inversion but not closed under the iterated inversion. The decidability and closure properties of some language classes with respect to hairpin inversion was scrutinized in [5, 6]. In the literature of combinatorics of words, block inversion operation is used for many combinatorial manipulations [9, 17]. Block inversion of a word is a process of partitioning the word into blocks instead of letters and writing it in the reverse order. Sorting by block reversal problem make use of this operation.

Partial words are words with holes and are considered in gene comparisons [8, 14]. For instance, orientation of two DNA sequences can be seen as construction of two compatible $\wp$words. In DNA computation, DNA strands are considered as finite words and are utilized for encoding information. While encoding, some parts of information may be unseen or missing. These parts are revealed by using $\wp$words that represent the positions of the missing symbols in a word. The holes present in a partial word over an alphabet does not belong to that alphabet but remains as a standby symbol for the unknown letter. The study of $\wp$words was initiated in [1] and the study was later extended by Blanchet Sadri [2, 3]. Here, we use a generalisation of the concept of inversion termed as block inversion on finite $\wp$words. We study the significant differences between block inversion on finite total words and block inversion on finite $\wp$words. We show that the total count of words in the block inversion set of a finite total word of length n is less than or equal to 2^{n-1} but the total count of words in the block inversion set of a finite rich $\wp$word of length n is strictly less than 2^{n-1}. In Section 2 the fundamental definitions pertaining to $\wp$words and inversions are recalled. The block inversion operation on rich $\wp$words explained in Section 3 and finally in Section 4, conclusion and future work are discussed.

2. Preliminaries

Here, we briefly recall the standard notations with respect to $\wp$words.

Let the set $\mathbb{A}$ termed as *alphabet* represent a non-empty finite set of symbols (or letters). A *total word* or *string* is a sequential arrangement of letters over $\mathbb{A}$. The set of all total words from $\mathbb{A}$ is denoted by $\mathbb{A}^*$. $\mathbb{A}^+ = \mathbb{A}^*$excluding the empty word $\{\lambda\}$. A language L is a subset of $\mathbb{A}^*$. Let $w \in \mathbb{A}^*$, by $|w|$ we denote the length of the word w. $Alph(w)$ denotes the set of all letters in w. A finite word w is called a *palindrome* if $w = w^R$ where w^R is the *reversal (mirror image)* of w. A *partial word* is a word made up of a number of holes or wild card letters (denoted as $\Diamond$) present anywhere in the sequence of letters of the word. The symbol $\Diamond \notin \mathbb{A}$, but remains as a standby symbol for the unknown letter, for instance, $u = aa\Diamond b$ is a partial word with $|u| = 4$. Formally, a partial word u with $|u| = n$ over $\mathbb{A}$ is a partial function $u : \{0, 1, 2 \cdots, n-1\} \rightarrow \mathbb{A}$. For $0 \leq j < n$, if $u(j)$ is defined, then we say j belongs to the domain of u (defined as $d(u)$), otherwise j belongs to the set of holes (defined as $h(u)$). The following definition is used in order to represent the locations of the holes of the ℘words. $u_\Diamond$ representing the companion of u is the total function $u_\Diamond$: $\{0, 1, 2 \cdots, n-1\} \rightarrow \mathbb{A}_\Diamond = \mathbb{A} \cup \{\Diamond\}$ defined by

$$u_\Diamond(j) = \begin{cases} u(j) & \text{if } j \in d(u), \\ \diamond & \text{if } j \in h(u). \end{cases}$$

A finite partial word p is primitive (non-periodic) if a finite partial word q exists such that $p = q^m$, for all $m \geq 2$.

3. Block Inversion of a Rich ℘word

The inversion of a non-empty finite total word of length k say $u_1u_2...u_k$ such that $u_m \in \mathbb{A}_\Diamond$ for all m is the word derived by partitioning u into k non-empty segments and listing them in the reverse order as $u_k...u_2u_1$. For instance the inversion of the total word aab is baa. The inversion operation on a finite rich ℘word is not similar to the inversion of a finite word since the presence of $\Diamond$ in a rich ℘word cannot exist as a non-empty segment without a companion while partitioning. In order to fulfill this gap we use block inversion operation on finite rich ℘word which shows variation from the classical definition of block inversion of finite total word. Here we study the palindromic properties along with block inversion operation on rich ℘words. The empty word λ is regarded as a palindrome.

Definition 1. A factor $p_\Diamond$ of a partial word $u_\Diamond$ over $\mathbb{A}_\Diamond$ is called a partial palindromic proper factor if $p_\Diamond$ is compatible with its reversal (denoted by

$p_\Diamond \uparrow p_\Diamond^R$). The set of all non-empty partial palindromic proper factors of $u_\Diamond$ is denoted by $PPPF(u_\Diamond)$.

Example 2. Consider a partial word $u_\Diamond = abba\Diamond a$ over $\mathbb{A}_\Diamond = \{a,b\} \cup \{\Diamond\}$. The palindromic factors of $u_\Diamond$ are

$$\{\lambda, a, b, bb, a\Diamond, \Diamond a, ba\Diamond, a\Diamond a, abba\}.$$

Here the factors $\{a\Diamond, \Diamond a, ba\Diamond, a\Diamond a\}$ are termed as partial palindromic factors.

Definition 3. Any partial word over $\mathbb{A}_\Diamond$ with length n is a rich partial word if it has at least n distinct partial palindromic proper factors.

Example 4. Consider a partial word $u_\Diamond = aa\Diamond ba$ over $\mathbb{A}_\Diamond = \{a,b\} \cup \{\Diamond\}$ with $|u_\Diamond| = 5$.
The set of all distinct palindromic proper factors of $u_\Diamond$ are

$$\{\lambda, a, b, aa, a\Diamond, \Diamond b, aa\Diamond, \Diamond ba, bab, a\Diamond ba\}.$$

Among the above set, the set of all distinct partial palindromic factors of $u_\Diamond$ are

$$\{a\Diamond, \Diamond b, aa\Diamond, \Diamond ba, a\Diamond ba\}.$$

Here the number of distinct partial palindromic proper factors is equal to $|u_\Diamond|$. Hence $u_\Diamond$ is a rich partial word.

Example 5. Consider a partial word $v_\Diamond = \Diamond ababb$ with length $|v_\Diamond| = 6$ over $\mathbb{A}_\Diamond = \{a,b\} \cup \{\Diamond\}$. Then the partial palindromic proper factors of $v_\Diamond$ are

$$v_\Diamond = \{\Diamond a, \Diamond ab, \Diamond abab\}.$$

Here the number of distinct partial palindromic proper factors is less than $|v_\Diamond|$. Hence $v_\Diamond$ is not a rich partial word.

Definition 6. Consider a finite alphabet $\mathbb{A}_\Diamond = \mathbb{A} \cup \{\Diamond\}$. For any integer m, let $u_\Diamond, I_m \in \mathbb{A}_\Diamond^+$ where $u_\Diamond$ represent the rich $\wp$word over $\mathbb{A}_\Diamond$; I_m denotes non-empty blocks of a rich $\wp$word in $\mathbb{A}_\Diamond^+$ such that

$$|I_m| \geq \begin{cases} 1 & \text{if } \Diamond \notin I_m, \\ 2 & \text{if } \Diamond \in I_m. \end{cases}$$

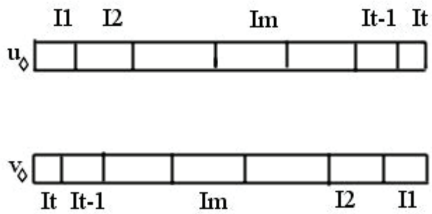

Figure 1: For $u_\Diamond \in \mathbb{A}_\Diamond^+, v_\Diamond \in I_B(u_\Diamond)$

The block inversion of $u_\Diamond$ represented by $I_B(u_\Diamond)$ is the set

$$I_B(u_\Diamond) = \{I_p I_{p-1}...I_2 I_1 : \ u_\Diamond = I_1 I_2 ... I_{p-1} I_p, \ p \geq 1\}.$$

A rich ℘word $u_\Diamond$ can be partitioned at most of $|u_\Diamond|$ blocks. The block inversion in $L_\Diamond \subseteq \mathbb{A}_\Diamond^+$ which is denoted by $I_B(L_\Diamond)$ is

$$I_B(L_\Diamond) = \bigcup_{u \in L_\Diamond} I_B(u_\Diamond).$$

Example 7. Assume $u_\Diamond = ab\Diamond b$ over $\mathbb{A}_\Diamond = \{a, b\} \in \{\Diamond\}$. Then

$$I_B(u_\Diamond) \quad = \quad \{ab\Diamond b, \Diamond bba, bb\Diamond a, b\Diamond ba, bab\Diamond, \Diamond bab\}.$$

Theorem 8. *[17] Assume a total word $x \in \mathbb{A}^+$ with $|x| = n$. Then $|I_B(x)| = 2^{n-1}$ iff $|Alph(x)| = n$.*

Theorem 9. *Assume a rich ℘word $u_\Diamond \in \mathbb{A}_\Diamond^+$ and $|u_\Diamond| = n$. Then $|I_B(u_\Diamond)| < 2^{n-1}$ iff $|Alph(u_\Diamond)| \leq n$.*

Proof. Consider $u_\Diamond \in \mathbb{A}_\Diamond^+$ and $|u_\Diamond| = n$ such that $|I_B(u_\Diamond)| = 2^{n-1}$. If $a \in \mathbb{A}_\Diamond^+$ is a letter, then the number of occurences of a in the rich ℘word $u_\Diamond$ is denoted by $|u_\Diamond|_a$. Let $|u_\Diamond|_a \geq 1$ and $|u_\Diamond|_\Diamond \geq 1$. Also let $a \in Alph(u_\Diamond)$ and $\Diamond \in Alph(u_\Diamond), \Diamond \notin \mathbb{A}$. There are four cases to consider:

1. Assume that $|u_\Diamond|_a > 1$ and $|u_\Diamond|_\Diamond > 1$. Then for some $x, y, z \in \mathbb{A}^*$, $u_\Diamond = axay\Diamond z\Diamond$. For $I_1 = axa, I_2 = y, I_3 = \Diamond z\Diamond$, we get $I_3 I_2 I_1 = \Diamond z\Diamond yaxa \in I_B(u_\Diamond)$. But we do not get $J_7 J_6 J_5 J_4 J_3 J_2 J_1 = \Diamond z\Diamond yaxa \in I_B(u_\Diamond)$ for $J_1 = a, J_2 = x, J_3 = a, J_4 = y, J_5 = \Diamond, J_6 = z, J_7 = \Diamond$ since $\Diamond$ without a companion does not exist as a block. Since $J_7 J_6 J_5 J_4 J_3 J_2 J_1$ cannot exist, then $|I_B(u_\Diamond)| = 2^{|u_\Diamond|-1}$ is a contradiction.

2. Assume that $|u_\Diamond|_a = 1$ and $|u_\Diamond|_\Diamond > 1$. Then for some $x, y, z \in \mathbb{A}^*$, $u_\Diamond = xay\Diamond z\Diamond$. For $I_1 = x, I_2 = a, I_3 = y, I_4 = \Diamond z\Diamond$, we get $I_4I_3I_2I_1 = \Diamond z\Diamond yax \in I_B(u_\Diamond)$. But we do not get $J_6J_5J_4J_3J_2J_1 = \Diamond z\Diamond yax \in I_B(u_\Diamond)$ for $J_1 = x, J_2 = a, J_3 = y, J_4 = \Diamond, J_5 = z, J_6 = \Diamond$. Thus $I_4I_3I_2I_1 \neq J_6J_5J_4J_3J_2J_1$, then $|I_B(u_\Diamond)| < 2^{|u_\Diamond|-1}$.

3. Assume that $|u_\Diamond|_a > 1$ and $|u_\Diamond|_\Diamond = 1$. Then $u_\Diamond = axay\Diamond z$ for some $x, y, z \in \mathbb{A}^*$. For $I_1 = axa, I_2 = y, I_3 = \Diamond z$, we get $I_3I_2I_1 = \Diamond zyaxa \in I_B(u_\Diamond)$. But we do not get $J_6J_5J_4J_3J_2J_1 = \Diamond zyaxa \in I_B(u_\Diamond)$ for $J_1 = a, J_2 = x, J_3 = a, J_4 = y, J_5 = z, J_6 = \Diamond$ since $\Diamond$ without a companion does not exist as a block. Since $J_6J_5J_4J_3J_2J_1$ cannot exist, then $|I_B(u_\Diamond)| = 2^{|u_\Diamond|-1}$ is a contradiction.

4. Assume that $|u_\Diamond|_a = 1$ and $|u_\Diamond|_\Diamond = 1$. Then for some $x, y, z \in \mathbb{A}^*$, $u_\Diamond = xay\Diamond z$. For $I_1 = x, I_2 = a, I_3 = y, I_4 = \Diamond z$, we get $I_4I_3I_2I_1 = \Diamond zyax \in I_B(u_\Diamond)$. But we do not get $J_5J_4J_3J_2J_1 = \Diamond zyax \in I_B(u_\Diamond)$ for $J_1 = x, J_2 = a, J_3 = y, J_4 = z, J_5 = \Diamond$. Since $J_5J_4J_3J_2J_1$ cannot exist, then $|I_B(u_\Diamond)| = 2^{|u_\Diamond|-1}$ is a contradiction.

   Thus, for each $a \in Alph(u_\Diamond)$ and $\Diamond \in Alph(u_\Diamond), \Diamond \notin \mathbb{A}$, $|u_\Diamond|_a \geq 1$ and $|u_\Diamond|_\Diamond \geq 1$, i.e, $|Alph(u_\Diamond)| \leq n$ if $|I_B(u_\Diamond)| < 2^{n-1}$.

Conversely, consider $u_\Diamond$ such that for each $a \in Alph(u_\Diamond)$ and $\Diamond \in Alph(u_\Diamond), \Diamond \notin \mathbb{A}, |u_\Diamond|_a \geq 1$ and $|u_\Diamond|_\Diamond \geq 1$. For $1 \leq k < |u_\Diamond|$, the rich $\wp$word $u_\Diamond$ can be divided into k non-empty blocks in $\binom{|u_\Diamond|-1}{k-1}$ distinct ways. Since all the letters in $u_\Diamond$ are distinct, the block inversion of such a rich $\wp$word contains $2^{|u_\Diamond|-1}$ elements. Hence, the proof. □

Example 10. Consider the rich $\wp$words $u^1_\Diamond = aa\Diamond b\Diamond, u^2_\Diamond = ca\Diamond b\Diamond, u^3_\Diamond = aba\Diamond, u^4_\Diamond = ab\Diamond c$ over $\mathbb{A}_\Diamond = \{a, b, c\} \cup \{\Diamond\}$ with $|u^1_\Diamond| = 5, |u^2_\Diamond| = 5, |u^3_\Diamond| = 4$ and $|u^4_\Diamond| = 4$. Here $|Alph(u^1_\Diamond)| = 3, |Alph(u^2_\Diamond)| = 4, |Alph(u^3_\Diamond)| = 3$ and $|Alph(u^4_\Diamond)| = 4$.

$$\begin{aligned} I_B(u^1_\Diamond) &= \{aa\Diamond b\Diamond, \Diamond b\Diamond aa, b\Diamond a\Diamond a, b\Diamond aa\Diamond\} \\ I_B(u^2_\Diamond) &= \{ca\Diamond b\Diamond, \Diamond b\Diamond ac, \Diamond b\Diamond ca, b\Diamond a\Diamond c, b\Diamond ca\Diamond\} \\ I_B(u^3_\Diamond) &= \{a\Diamond ba, aba\Diamond, a\Diamond ab, ba\Diamond a\} \\ I_B(u^4_\Diamond) &= \{ab\Diamond c, \Diamond cba, \Diamond cab, b\Diamond ca\}. \end{aligned}$$

Here

$$|I_B(u^1_\Diamond)| = 4 < 2^{|u_\Diamond|-1}$$

$$|I_B(u_\Diamond^2)| = 5 < 2^{|u_\Diamond|-1}$$
$$|I_B(u_\Diamond^3)| = 4 < 2^{|u_\Diamond|-1}$$
$$|I_B(u_\Diamond^4)| = 4 < 2^{|u_\Diamond|-1}.$$

A subset of the set $I_B(u_\Diamond)$ that considers rich ℘words with the inversion of minimum two non-empty blocks is termed as a proper block inversion.

Definition 11. The proper block inversion of $u_\Diamond$ denoted by $PI_B(u_\Diamond)$ is

$$PI_B(u_\Diamond) = \{I_p I_{p-1}...I_2 I_1 : \ u_\Diamond = I_1 I_2 ... I_{p-1} I_p, \ p \geq 2\}.$$

The proper block inversion of $(u_\Diamond)_\circ$ denoted by $PI_B((u_\Diamond)_\circ)$ is

$$PI_B((u_\Diamond)_\circ) = \{I_B(PI_B(u_\Diamond)) : \ u_\Diamond = I_1 I_2 ... I_{p-1} I_p, \ p \geq 2\}.$$

Example 12. Assume $u_\Diamond = ac\Diamond ca$ over $\mathbb{A}_\Diamond$. Then

$$PI_B(u_\Diamond) = \{c\Diamond caa, \Diamond caac, caac\Diamond, aac\Diamond c, cac\Diamond a, a\Diamond cca, a\Diamond cac, acac\Diamond, ac\Diamond ca, acc\Diamond a, \Diamond caca\},$$
$$PI_B((u_\Diamond)_\circ) = \{c\Diamond caa, \Diamond caac, caac\Diamond, aac\Diamond c, cac\Diamond a, a\Diamond cca, a\Diamond cac, acac\Diamond, ac\Diamond ca, acc\Diamond a, \Diamond caca, aacc\Diamond, aa\Diamond cc, acac\Diamond, aca\Diamond c, ca\Diamond ca, c\Diamond aca, ac\Diamond ac, \Diamond ccaa\}.$$

Definition 13. A rich ℘word $u_\Diamond$ over $\mathbb{A}_\Diamond$ is called a block palindrome (denoted as b_{pal}) if $u_\Diamond$ exists as

$$u_\Diamond = I_{-p} I_{-(p-1)} ... I_{-1} I_0 I_1 ... I_{(p-1)} I_p,$$

where $I_m, I_{-m} \in \mathbb{A}_\Diamond^+$ and $I_m = I_{-m}$ for $1 \leq m \leq p$. Here $I_0 \in \mathbb{A}^*$ is the mid-block and it may or may not be a palindrome.

Remark 14. *A palindromic rich ℘word is a b_{pal} formed by palindromic partitioning but the converse does not holds. For instance, the partial word $u_\Diamond = bab\Diamond b\Diamond babb$ is not a palindrome but $u_\Diamond$ is a b_{pal} with palindromic partitioning $b \,|\, ab \,|\, \Diamond b \,|\, \Diamond b \,|\, ab \,|\, b$.*

Remark 15. *A unique representation of a rich ℘word as a b_{pal} is not a necessary condition.*

Example 16. Consider $u_\Diamond = b\Diamond aab\Diamond$ over $\mathbb{A}_\Diamond = \{a, b\} \cup \{\Diamond\}$. Here $u_\Diamond$ is a b_{pal} with palindromic partitioning $b\Diamond \,|\, a \,|\, a \,|\, b\Diamond$ as well as $b\Diamond \,|\, aa \,|\, b\Diamond$.

Theorem 17. *For any rich ℘word $u_\Diamond \in \mathbb{A}_\Diamond^+$, $u_\Diamond \in PI_B(u_\Diamond)$ iff $u_\Diamond$ is a b_{pal}.*

Proof. Let the rich ℘word $u_\Diamond \in PI_B(u_\Diamond)$. Then by Definition 13, there exists $I_m, I_{-m} \in \mathbb{A}_\Diamond^+$ for $2 \le m \le p$ such that $u_\Diamond = I_{-p}I_{-(p-1)}...I_{-1} = I_1...I_{(p-1)}I_p$. Let $|I_1| \le |I_p|$, then we have the following cases:

1. If $|I_1| = |I_p|$, it is trivial that $u_\Diamond \in PI_B(u)$ is a b_{pal}.
2. Let $|I_1| < |I_p|$. Then $I_p = I_1x$ for some $x \in \mathbb{A}^*$. We have

$$\begin{aligned} u_\Diamond &= I_pI_{p-1}...I_2I_1 \\ &= I_1xI_{p-1}...I_2I_1 \\ &= I_1I_0I_1 \text{ where } I_0 = xI_{p-1}...I_2. \end{aligned}$$

Thus $u_\Diamond$ is a b_{pal}.

Conversely, let $u_\Diamond$ be a b_{pal}. Then $u_\Diamond \in PI_B(u_\Diamond)$ since $I_m = I_{-m}$ for $1 \le m \le p$. □

Lemma 18. *[3] For any non-empty rich ℘words $u_\Diamond$ and $v_\Diamond$, the equality $u_\Diamond v_\Diamond = v_\Diamond u_\Diamond$ holds iff $u_\Diamond = w_\Diamond^i$ and $v_\Diamond = w_\Diamond^j$ for some positive integers i, j and rich ℘word $w_\Diamond$.*

Remark 19. *For all integers $i, j \ge 1$, the rich ℘word $u_\Diamond^i$ is a palindrome iff $u_\Diamond^j$ is a palindrome.*

Theorem 20. *For any non-empty partial words $u_\Diamond$ and $v_\Diamond$, the concatenation $u_\Diamond v_\Diamond$ is a b_{pal} iff both $u_\Diamond, v_\Diamond$ are powers of some b_{pal} $w_\Diamond$.*

Proof. Suppose $u_\Diamond v_\Diamond$ is a b_{pal}, by Definition 13 we have

$$\begin{aligned} u_\Diamond &= \{I_{-p}I_{-(p-1)}...I_{-1}I_0I_1...I_{(p-1)}I_p\}, \\ v_\Diamond &= \{I_{-q}I_{-(q-1)}...I_{-1}I_0I_1...I_{(q-1)}I_q\}, \end{aligned}$$

where $I_m, I_{-m}, I_n, I_{-n}, \in \mathbb{A}_\Diamond^+$ and $I_m = I_{-m}, I_n = I_{-n}$ for $1 \le m \le p$ and $1 \le n \le q$. Then we get

$$u_\Diamond v_\Diamond = \{I_{-p}I_{-(p-1)}...I_{(p-1)}I_pI_{-q}I_{-(q-1)}...I_{(q-1)}I_q\}$$

$$
\begin{aligned}
&= \{I_q I_{(q-1)} ... I_{-(q-1)} I_{-q} I_p I_{(p-1)} ... I_{-(p-1)} I_{-p}\} \\
&= (u_\Diamond v_\Diamond)^R \\
&= (v_\Diamond)^R (u_\Diamond)^R \\
&= v_\Diamond u_\Diamond .
\end{aligned}
$$

By Lemma 18 and Remark 19, we have that both $u_\Diamond, v_\Diamond$ are powers of some block palindrome $w_\Diamond$. Conversely, if $u_\Diamond = w_\Diamond^i$ and $v_\Diamond = w_\Diamond^j$, we have $u_\Diamond v_\Diamond = w_\Diamond^{i+j}$, which is a b_{pal} by Remark 19. □

The total count of total words in the block inversion set of a total word of length n is equal to 2^{n-1}. But it is not so, in the case of rich ℘words. The following theorem proves that the total count of rich ℘words in the block inversion set of a rich ℘word of length n is strictly less than 2^{n-1}.

Lemma 21. *[10] The count of palindromic segments of an integer n is $2^{\lfloor \frac{n}{2} \rfloor}$. Also,*

Count of palindromic segments of n with

$$
\begin{cases}
2k+1 \ parts \ \ equals & \binom{\lceil \frac{n}{2} \rceil - 1}{k}, \\
2k \ parts \ \ equals & \binom{\lceil \frac{n}{2} \rceil - 1}{k-1}.
\end{cases}
$$

Theorem 22. *Assume $u_\Diamond$ to be a palindromic partial word with $|u_\Diamond| = n$. Let $Pal(I_B(u_\Diamond))$ denote the total number of palindromic words in $I_B(u_\Diamond)$. Then*

$$
Pal(I_B(u_\Diamond)) = \begin{cases}
< 2^{\frac{n}{2}-1} & if \ \ n \ \ is \ \ even, \ \ |Alph(u_\Diamond)| = \frac{n}{2}, \\
< 2^{\frac{n-1}{2}-1} & if \ \ n \ \ is \ \ odd, \ \ |Alph(u_\Diamond)| = \frac{n+1}{2}.
\end{cases}
$$

Proof. Consider a palindromic rich ℘word $u_\Diamond$ with $|u_\Diamond| = n$. The following two cases arises with respect to n.

Case 1 : Let n be even and $|Alph(u_\Diamond)| = \frac{n}{2}$. By Lemma 23, all elements of $I_B(u_\Diamond)$ with palindromic blocks are palindromes. Let $v_\Diamond \in I_B(u_\Diamond)$ be a palindrome with palindromic blocks $n_1 + n_2 + ... + n_m + ... + n_2 + n_1$. Since n is even, n_m is also even. This implies that palindromic blocks with an even number of parts forms $v_\Diamond$. Then followed by the Theorem 17 and the Lemma 21, we get the total number of palindromic elements in $I_B(u_\Diamond)$ less than $2^{\frac{n}{2}-1}$.

Case 2 : Let n be odd and $|Alph(u_\Diamond)| = \frac{n+1}{2}$. This shows that a letter occurs once in the rich ℘word and also in the mid position of $u_\Diamond$. Let $v \in I_B(u_\Diamond)$ be a palindrome. The palindromic blocks of $v_\Diamond$ are of the form $n_1 + n_2 + ... +$

$n_m + n_m + ... + n_2 + n_1$. Then by Theorem 17, the total number of palindromic elements in $I_B(u_\Diamond)$ is less than $2^{\frac{n+1}{2}-1}$. □

Remark 23. *If $u_\Diamond$ over $\mathbb{A}_\Diamond^+$ is a palindromic rich ℘word then all elements of $I_B(u_\Diamond)$ with palindromic partitions are palindromes.*

4. Conclusion and Future Work

We extended block inversion operation on finite rich ℘words and also discussed the classification of palindromic words in the block inversion of a finite rich ℘word. In future, it would be interesting to study the combinatorial properties of finite ℘words using non-overlapping inversion and pseudo inversion operations.

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