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REMARKS IN QUATERNIONIC INTEGRATION: CAUCHY'S THEOREMS AND FORMULAS

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Abstract: The study of quaternions has been developed in the last decades, and some results show like a generalization of the Classical Complex Analysis Theory. The main purpose of this letter is of a presenting new results besides showing new current trends, as for instance, to deal with the integral theorem for Quaternionic Functions. Another proposal to be implemented in this work is the determination of a "closed" formula for the Cauchy Integral. A preliminary formula have been already determined [5].

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1. Introduction

The Quaternionic Analysis has provided along the years many results which may be regarded as extensions of well known theorems, formulas and equations from the Classical Complex Analysis Theory [8]-[15] and [6]-[17]. Moreover, it also

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presents some peculiarities whose interpretations still deserve a consolidation. In this context, the Cauchy theorem (null) will be analysed in this work; through analysis of a certain results it is possible to make an additional assumption on the already consolidated theorem of the Classical Complex Theory.

2. Cauchy's theorem for complex variables and Cauchy like relations for quaternions

The Cauchy theorem for complex variable functions is discussed in detail in [9] and [12]. Furthermore, in this section, we present this theorem and omit its proof. The Cauchy theorem's version that best fits the purpose of this work is the demonstration made by the French mathematician Goursat (1858-1936) in 1883. Thus let us make the following assumptions.

Theorem 1. *Let $f : A \to C$ a continuous function defined in a region $A \subset C$ be the classical complex two dimensional space. Then the following are equivalent:*
(1) f has a primitive in A;
(2) $\int_\gamma f(z)dz = 0$ for any closed path, soft parts for γ in A;
(3) $\int_\gamma f(z)dz$ only depends on the start and end points of any smooth path for parts γ in A.

Theorem 2. *For every pair of points a and b, and any path connecting them in a simply connected dimensional space, the integral $\int f dq$ is independent of given up way, only if there is a function $F = F_1 + iF_2 + jF_3 + kF_4$, with $\int f dq = F(b) - F(a)$ and which satisfies the following relationships:*

$$\frac{\partial F_1}{\partial q_1} = \frac{\partial F_2}{\partial q_2} = \frac{\partial F_3}{\partial q_3} = \frac{\partial F_4}{\partial q_4};$$

$$\frac{\partial F_2}{\partial q_1} = -\frac{\partial F_1}{\partial q_2} = \frac{\partial F_4}{\partial q_3} = -\frac{\partial F_3}{\partial q_4};$$

$$\frac{\partial F_3}{\partial q_1} = -\frac{\partial F_4}{\partial q_2} = -\frac{\partial F_1}{\partial q_3} = \frac{\partial F_2}{\partial q_4};$$

$$\frac{\partial F_4}{\partial q_1} = \frac{\partial F_3}{\partial q_2} = -\frac{\partial F_2}{\partial q_3} = -\frac{\partial F_1}{\partial q_4}. \tag{1}$$

Theorem 3. *For every pair of points a and b, and any path connecting them in a simply connected dimensional space, the integral $\int dqf$ is independent of given up way, only if there is a function $G = G_1 + iG_2 + jG_3 + kG_4$, with $\int dqf = G(b) - G(a)$ and which satisfies the following relationships:*

$$\begin{gathered}\frac{\partial G_1}{\partial q_1} = \frac{\partial G_2}{\partial q_2} = \frac{\partial G_3}{\partial q_3} = \frac{\partial G_4}{\partial q_4};\\ \frac{\partial G_2}{\partial q_1} = -\frac{\partial G_1}{\partial q_2} = -\frac{\partial G_4}{\partial q_3} = G\frac{\partial F_3}{\partial q_4};\\ \frac{\partial G_3}{\partial q_1} = \frac{\partial G_4}{\partial q_2} = -\frac{\partial G_1}{\partial q_3} = -\frac{\partial G_2}{\partial q_4};\\ \frac{\partial G_4}{\partial q_1} = -\frac{\partial G_3}{\partial q_2} = \frac{\partial G_2}{\partial q_3} = \frac{\partial G_1}{\partial q_4}.\end{gathered} \tag{2}$$

3. Cauchy's theorem for quaternionic functions

The next theorem has one more analogue in Classical Complex Analysis, and will be shown that, using some restrictions to $f(q)$, it can be called as Cauchy theorem for quaternionic functions.

Theorem 4. *Let $f(q)$ be a quaternionic function, where $f(q) = f(q_1, q_2, q_1, q_2)$, then*

$$\int_a^b f(q)dq = 0, \tag{3}$$

where a and b are points in a connected simply connected four-dimensional space.

Proof. Using relations, according to [15], the integral (3) to the variables (q_1, q_2, q_1, q_2), is such that we may have that:

$$\begin{aligned}\int_a^b f(q)dq &= \int_a^b (f_1 + if_2 + jf_3 + kf_4)(dq_1 + idq_2 + jdq_3 + kdq_4)\\ &= \int_a^b (f_1dq_1 - f_2dq_2 - f_3dq_3 - f_4dq_4)\\ &+ \int_a^b (f_2dq_1 + f_1dq_2 - f_4dq_3 + f_3dq_4)i\end{aligned}$$

$$+\int_a^b (f_3 dq_1 + f_4 dq_2 + f_1 dq_3 - f_2 dq_4) j$$
$$+\int_a^b (f_4 dq_1 - f_3 dq_2 + f_2 dq_3 + f_1 dq_4) k.$$

Now using the identities obtained in [15], it immediately follows that:

$$\begin{aligned}
&\int_a^b f(q)dq \\
&\quad = \int_a^b (f_1 + if_2 + jf_3 + kf_4)(dq_1 + idq_2 + jdq_3 + kdq_4) \\
&\quad = \int_a^b \Big(\frac{\partial F_3}{\partial q_3} dq_1 + \frac{\partial F_3}{\partial q_4} dq_2 - \frac{\partial F_3}{\partial q_1} dq_3 - \frac{\partial F_3}{\partial q_2} dq_4\Big) \\
&\quad + \int_a^b \Big(\frac{\partial F_4}{\partial q_4} dq_2 + \frac{\partial F_4}{\partial q_3} dq_1 - \frac{\partial F_4}{\partial q_1} dq_3 - \frac{\partial F_4}{\partial q_2} dq_4\Big) i \\
&\quad + \int_a^b \Big(-\frac{\partial F_1}{\partial q_3} dq_1 - \frac{\partial F_1}{\partial q_4} dq_2 + \frac{\partial F_1}{\partial q_1} dq_3 + \frac{\partial F_1}{\partial q_2} dq_4\Big) j \\
&\quad + \int_a^b \Big(-\frac{\partial F_2}{\partial q_3} dq_1 - \frac{\partial F_2}{\partial q_4} dq_2 + \frac{\partial F_2}{\partial q_1} dq_3 + \frac{\partial F_2}{\partial q_2} dq_4\Big) k.
\end{aligned}$$

By hypothesis, $q = (q_1, q_2, q_1, q_2)$. Therefore,

$$\int_a^b f(q)dq = 0.$$

□

Theorem 5. *Let $f(q)$ be a quaternionic function, where $f(q) = f(q_1, q_2, q_1, q_2)$, then*

$$\int_a^b dqf(q) = 0, \tag{4}$$

where a and b are points in a connected simply connected four-dimensional space.

Proof. Using relations, according to [4], the integral (4) to the variables (q_1, q_2, q_1, q_2), is such that we may have that:

$$\int_a^b dqf(q)$$

$$
\begin{aligned}
&= \int_a^b (dq_1 + idq_2 + jdq_3 + kdq_4)(f_1 + if_2 + jf_3 + kf_4) \\
&= \int_a^b (f_1 dq_1 - f_2 dq_2 - f_3 dq_3 - f_4 dq_4) \\
&+ \int_a^b (f_2 dq_1 + f_1 dq_2 + f_4 dq_3 - f_3 dq_4) i \\
&+ \int_a^b (f_3 dq_1 - f_4 dq_2 + f_1 dq_3 + f_2 dq_4) j \\
&+ \int_a^b (f_4 dq_1 + f_3 dq_2 - f_2 dq_3 + f_1 dq_4) k,
\end{aligned}
$$

Now using the identities obtained in [4], it immediately follows that:

$$
\begin{aligned}
&\int_a^b dq f(q) \\
&= \int_a^b (dq_1 + idq_2 + jdq_3 + kdq_4)(f_1 + if_2 + jf_3 + kf_4) \\
&= \int_a^b \Big(\frac{\partial G_3}{\partial q_3} dq_1 - \frac{\partial G_3}{\partial q_4} dq_2 - \frac{\partial G_3}{\partial q_1} dq_3 + \frac{\partial G_3}{\partial q_2} dq_4 \Big) \\
&+ \int_a^b \Big(\frac{\partial G_4}{\partial q_4} dq_2 - \frac{\partial G_4}{\partial q_3} dq_1 + \frac{\partial G_4}{\partial q_1} dq_3 - \frac{\partial G_4}{\partial q_2} dq_4 \Big) i \qquad (5) \\
&+ \int_a^b \Big(- \frac{\partial G_1}{\partial q_3} dq_1 + \frac{\partial G_1}{\partial q_4} dq_2 + \frac{\partial G_1}{\partial q_1} dq_3 - \frac{\partial G_1}{\partial q_2} dq_4 \Big) j \\
&+ \int_a^b \Big(\frac{\partial G_2}{\partial q_3} dq_1 - \frac{\partial G_2}{\partial q_4} dq_2 - \frac{\partial G_2}{\partial q_1} dq_3 + \frac{\partial G_2}{\partial q_2} dq_4 \Big) k.
\end{aligned}
$$

By hypothesis, $q = (q_1, q_2, q_1, q_2)$. Therefore,

$$\int_a^b dq f(q) = 0.$$

□

4. Closed Cauchy integral formula

The statements demonstrated now are crucial in determining a unified formula for the Cauchy integral theorem, shown in detail in [5].

Theorem 6. *Consider Ω a domain simply connected in a four-dimensional space and $f(q)$ a regular feature in Ω. Then,*

$$\int_{\varphi} \frac{f(q)}{q-q_0} dq = f(q_0)\pi(i+2j+k), \tag{6}$$

where φ is a simple closed hypersurface in Ω and q_0 is any point φ.

The next demonstrated theorems follow the same directions as previously, but with the change of orientation of the corresponding coordinates.

Proof. Let us consider φ_0 one hypersphere around the point q_0, $|q-q_0| = r_0$, where r_0 is small enough that φ_0 is within φ. Since the function $\frac{f(q)}{q-q_0}$ is regular in $\Omega/\{q_0\}$, we have:

$$\int_{\varphi} \frac{f(q)}{q-q_0} dq = \int_{\varphi_0} \frac{f(q)}{q-q_0} dq.$$

Therefore,

$$\begin{aligned}
\int_{\varphi} \frac{f(q)}{q-q_0} dq &= \int_{\varphi_0} \frac{f(q)}{q-q_0} dq \\
&= \int_{\varphi_0} [\frac{f(q_0) + f(q) - f(q_0)}{q-q_0}] dq \\
&= f(q_0) \int_{\varphi_0} \frac{dq}{q-q_0} + \int_{\varphi_0} \frac{f(q) - f(q_0)}{q-q_0} dq,
\end{aligned}$$

writing the quaternion $q - q_0$ as follows:

$$q - q_0 = r_0 e^{\theta_1 i + \theta_2 j + \theta_3 k},$$

where $r_0 > 0$. Taking now $-\frac{\pi}{2} \leq \theta_1 \leq \frac{\pi}{2}, 0 \leq \theta_2 \leq 2\pi$ e $-\frac{\pi}{2} \leq \theta_3 \leq \frac{\pi}{2}$ and proceeding analogously to the previous proof of the theorem, we conclude that

$$\int_{\varphi} \frac{f(q)}{q-q_0} dq = f(q_0)\pi(i+2j+k).$$

□

Theorem 7. *Consider Ω as a domain simply connected in a four-dimensional space and $f(q)$ a regular feature in Ω. Then,*

$$\int_{\varphi} \frac{f(q)}{q-q_0} dq = f(q_0)\pi(2i+j+k), \tag{7}$$

where φ is a simple closed hypersurface in Ω e q_0 any point φ.

Proof. Let be φ_0 one hypersphere around the point q_0, $|q-q_0| = r_0$, where r_0 is small enough that φ_0 is within φ. Since the function $\frac{f(q)}{q-q_0}$ is regular in $\Omega/\{q_0\}$, we have:

$$\int_{\varphi} \frac{f(q)}{q-q_0} dq = \int_{\varphi_0} \frac{f(q)}{q-q_0} dq.$$

Therefore,

$$\begin{aligned}\int_{\varphi} \frac{f(q)}{q-q_0} dq &= \int_{\varphi_0} \frac{f(q)}{q-q_0} dq \\ &= \int_{\varphi_0} [\frac{f(q_0)+f(q)-f(q_0)}{q-q_0}] dq \\ &= f(q_0)\int_{\varphi_0} \frac{dq}{q-q_0} + \int_{\varphi_0} \frac{f(q)-f(q_0)}{q-q_0} dq.\end{aligned}$$

Let is write the quaternion $q-q_0$ as follows:

$$q-q_0 = r_0 e^{\theta_1 i+\theta_2 j+\theta_3 k},$$

where $r_0 > 0$. Taking $0 \leq \theta_1 \leq 2\pi, -\frac{\pi}{2} \leq \theta_2 \leq \frac{\pi}{2}$ e $-\frac{\pi}{2} \leq \theta_3 \leq \frac{\pi}{2}$, we conclude:

$$\int_{\varphi} \frac{f(q)}{q-q_0} dq = f(q_0)\pi(2i+j+k).$$

This can be written in the form below, making the sums of the respective formulas found. □

Theorem 8. *Consider Ω as a domain simply connected in a four-dimensional space and $f(q)$ a regular feature in Ω. Then,*

$$\int_{\varphi} \frac{f(q)}{q-q_0} dq = \frac{4}{3}\pi f(q_0)(i+j+k),$$

where φ is a simple closed hypersurface in Ω and q_0 any point φ.

Proof. Let us add the formulas

$$\begin{aligned}\int_{\varphi} \frac{f(q)}{q-q_0} dq &= f(q_0)\pi(i+j+2k) \\ &= f(q_0)\pi(i+2j+k) \\ &= f(q_0)\pi(2i+j+k).\end{aligned}$$

Therefore,

$$\int_{\varphi} \frac{f(q)}{q-q_0} dq = \frac{4}{3}\pi f(q_0)(i+j+k). \tag{8}$$

□

5. Concluding remarks

This study shows that the Cauchy theorem has a "closed" version. This version is more general compared to the version presented in [5]. Another important fact is that there is no similarity of certain formula with its analogous to the case of Classical Complex Analysis [12]. Therefore, the results determined using the formula (9) given in this paper, are actual ones.

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