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OPTIMAL CONTROL PROBLEM WITH COEFFICIENTS FOR THE EQUATION OF VIBRATIONS OF AN ELASTIC PLATE WITH DISCONTINUOUS SOLUTION

H.F. Guliyev[1], Kh.I. Seyfullayeva[2 §]

[1] Baku State University
Z. Khalilov 23, Baku, AZ1148, AZERBAIJAN

[2] Sumgait State University
Sumgait State University, Sumgayit 43th district
Baku street 1, AZ5008, AZERBAIJAN

Abstract: Optimal control problem with coefficients for the equation of vibrations of an elastic plate with discontinuous solution is considered in this work. Existence theorem for optimal pair is proved and necessary condition for optimality in the form of integral inequality is obtained.

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Key Words: elastic plate, equation of vibrations, optimal control, existence theorem, optimality condition

1. Introduction

Fourth order partial differential equations make an important part of mathematical physics. In practice, some real processes are described by fourth order partial differential equations. For example, equations of vibrations of a tuning fork (see [1]), equations of elastic plate (see [6]), equations of thin plate (see [11]), circular plate equations (see [2]), etc. belong to this kind of equations. Therefore, the study of optimal control problems in the processes described by these equations is of great theoretical and practical significance.

Note that different optimal control problems for the vibrations of a elastic plate have been considered in [3], [4], [10].

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§Correspondence author

In case where the control function appears in the right-hand side of the equation, optimal control problem for Petrovski-type correct system with discontinuous solution has been treated in [7].

Of particular interest are the optimal control problems where the control is included in the coefficients of the equation. Some essential difficulties related to their nonlinearity and incorrectness are obtained in investigation of these problems of optimal control (see [5], [8]).

In this work, we consider an optimal control problem using the coefficients of the equation of vibrations of a elastic plate with discontinuous solution.

2. Problem statement

Let the controlled process be described by the equation

$$\rho h \frac{\partial^2 u}{\partial t^2} + \Delta(D\Delta u) + (1-\nu)\left(2\frac{\partial^2 D}{\partial x_1 \partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2}\right.$$

$$\left. - \frac{\partial^2 D}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 D}{\partial x_2^2}\frac{\partial^2 u}{\partial x_1^2}\right) - u^3 = f(x_1, x_2, t), \tag{1}$$

$$(x_1, x_2, t) \in Q,$$

with the initial conditions

$$u(x_1, x_2, 0) = \varphi_0(x_1, x_2), \frac{\partial u(x_1, x_2, 0)}{\partial t}$$

$$= \varphi_1(x_1, x_2), (x_1, x_2) \in \Omega \tag{2}$$

and the boundary conditions

$$u(0, x_2, t) = 0, \frac{\partial u(0, x_2, t)}{\partial x_1} = 0, u(x_1, 0, t) = 0, \ \frac{\partial u(x_1, 0, t)}{\partial x_2} = 0,$$

$$u(a, x_2, t) = 0, \frac{\partial u(a, x_2, t)}{\partial x_1} = 0, u(x_1, b, t) = 0, \frac{\partial u(x_1, b, t)}{\partial x_2} = 0,$$

$$0 \le x_1 \le a, \ \ 0 \le x_2 \le b, \ \ \ 0 \le t \le T, \tag{3}$$

where $(x_1, x_2) \in \Omega = \{(x_1, x_2) : 0 < x_1 < a, \ 0 < x_2 < b\}$, $t \in (0, T)$, $Q = \Omega \times (0, T)$, a, b, T, are the given positive numbers, $\rho(x_1, x_2)$ is a dense of the mass at the point (x_1, x_2), $h(x_1, x_2)$ is the heath thickness of the plate in the point (x_1, x_2), $u(x_1, x_2, t)$- is deflection of the plate in the point (x_1, x_2) at the moment t, Δ is Laplace operator with respect to x_1, x_2, $D = \frac{Eh^3}{12(1-\nu^2)}$ – cylindrical rigidity, ν $\left(0 < \nu < \frac{1}{2}\right)$ – Poisson's coefficient, $E > 0$– Young's modulus, $\rho(x_1, x_2) \in C(\bar{\Omega})$, $f(x_1, x_2, t) \in L_2(Q)$-are given functions, $\varphi_0(x_1, x_2) \in \overset{o}{W}{}_2^2(\Omega)$, $\varphi_1(x_1, x_2) \in L_2(\Omega)$– are given initial functions, $h(x_1, x_2)$– are the control function, belonging to following set

$$U_{ad} = \Big\{ h(x_1, x_2) \in W_2^4(\Omega) \Big| \ |h|,$$

$$\left|\frac{\partial h}{\partial x_1}\right|, \left|\frac{\partial h}{\partial x_2}\right|, \left|\frac{\partial^2 h}{\partial x_1^2}\right|, \left|\frac{\partial^2 h}{\partial x_1 \partial x_2}\right|, \left|\frac{\partial^2 h}{\partial x_2^2}\right|, \left|\frac{\partial^3 h}{\partial x_1^3}\right|, \left|\frac{\partial^3 h}{\partial x_1^2 \partial x_2}\right|,$$

$$\left|\frac{\partial^3 h}{\partial x_1 \partial x_2^2}\right|, \left|\frac{\partial^3 h}{\partial x_2^3}\right|, \left|\frac{\partial^4 h}{\partial x_1^4}\right|, \left|\frac{\partial^4 h}{\partial x_1^3 \partial x_2}\right|, \left|\frac{\partial^4 h}{\partial x_1^2 \partial x_2^2}\right|,$$

$$\left|\frac{\partial^4 h}{\partial x_1 \partial x_2^3}\right|, \left|\frac{\partial^4 h}{\partial x_2^4}\right| \leq M \ \text{ a. e. in } \Omega \Big\},$$

where M is the given positive number.

Define an admissible pair $\{h, u\}$ which satisfies the conditions (1)-(3) and let

$$h(x_1, x_2) \in U_{ad}, u(x_1, x_2, t) \in L_6(Q).$$

Also assume that the set of admissible pairs is nonempty. (4)

Let

$$\varphi_0(x_1, x_2) \in \overset{0}{W}{}_2^2(\Omega), \varphi_1(x_1, x_2) \in L_2(\Omega), \tag{5}$$

here $\overset{0}{W}{}_2^2(\Omega)$ is subspace $W_2^2(\Omega)$, whose elements on the boundary of Ω are equal to zero together with their first derivatives, and $W_2^2(\Omega)$-Hilbert space consisting of all elements $z(x_1, x_2)$ of $L_2(\Omega)$, having generalized derivatives of the first and second orders from $L_2(\Omega)$ with the norm

$$\|z\| = \left\{ \int_\Omega \left[(z)^2 + \left(\frac{\partial z}{\partial x_1}\right)^2 + \left(\frac{\partial z}{\partial x_2}\right)^2 + \left(\frac{\partial^2 z}{\partial x_1^2}\right)^2 \right.\right.$$

$$\left.\left. + \left(\frac{\partial^2 z}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 z}{\partial x_2^2}\right)^2 \right] dx \right\}^{1/2}.$$

Let us denote

$$U = \left\{ u(x_1, x_2, t) : \ u \in u(x_1, x_2, t) \in C\left([0; T]; \overset{0}{W}{}_2^2(\Omega)\right), \right.$$

$$\left.\frac{\partial u(x_1,x_2,t)}{\partial t}\in C\left([0;T]\,;L_2(\Omega)\right)\right\}.$$

For every admissible pair $\{h,u\}$, the satisfaction of conditions (1)-(3) is understood in the sense that $u(x_1,x_2,t)\in U$ and for every function $\eta(x_1,x_2,t)\in U$, $\eta(x_1,x_2,T)=0$ the integral identity

$$\int_Q\left[-\rho h\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t}+D\Delta u\Delta\eta+(1-\nu)\right.$$

$$\left.\times\left(2\frac{\partial^2 D}{\partial x_1\partial x_2}\frac{\partial^2 u}{\partial x_1\partial x_2}-\frac{\partial^2 D}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}-\frac{\partial^2 D}{\partial x_2^2}\frac{\partial^2 u}{\partial x_1^2}\right)\eta-u^3\eta\right]dx_1dx_2dt$$

$$-\int_\Omega \rho h\varphi_1(x_1,x_2)\eta(x_1,x_2,0)dx_1dx_2$$

$$=\int_Q f(x_1,x_2,t)\eta(x_1,x_2,t)dx_1dx_2dt$$

and the condition $u(x_1,x_2,0)=\varphi_0(x_1,x_2)$ holds.

Define the functional

$$J(h,u)=\frac{1}{6}\left\|u-u_d\right\|^6_{L_6(Q)}+\frac{N}{2}\left\|h\right\|^2_{W_2^4(\Omega)}, \tag{6}$$

where $u_d(x_1,x_2,t)\in L_\infty(Q)$-given function, and $N>0$ is a given number.

Consider the following optimal control problem: find the minimum value of the functional (6), where $\{h,u\}$ is varying in the class of admissible pairs.

3. Existence of optimal pair

Theorem 1. *Let the conditions* (4), (5) *hold. Then there exists an optimal pair* $\left\{h^0,u^0\right\}$ *in the problem* (1)-(4), (6), *i.e.*

$$J(h^0,u^0)=\inf_{\{h,u\}}J(h,u).$$

Proof. Let $\left\{h^{(n)},\,u^{(n)}\right\}\in U_{ad}\times U$ be a minimizing sequence, i.e.

$$\lim_{n\to\infty}J(h^{(n)},\,u^{(n)})=\inf_{\{\upsilon,u\}}J(h,u). \tag{7}$$

Hence it follows

$$\left\|h^{(n)}\right\|_{W_2^4(\Omega)}\leq const, \tag{8}$$

$$\left\| u^{(n)} \right\|_{L_6(Q)} \leq const \tag{9}$$

and then by the relation (8) and definition of the class U_{ad} we can derive from $\{h^{(n)}\}$ a subsequence, denoted again by $\{h^{(n)}\}$, such that

$$h^{(n)} \to h^0 \text{ in } W_2^4(\Omega) \text{ weakly as } n \to \infty. \tag{10}$$

From (9) and (1)-(3) we obtain

$$\left\| u^{(n)} \right\|_U \leq const, \left\| u^{(n)} \right\|_{W_2^{2,1}(Q)} \leq const. \tag{11}$$

Hence it follows that the sequence $\{u^{(n)}\}$ belongs to a bounded set in the class U.

Then from embedding theorem of [9, p. 70], we obtain

$$u^{(n)} \to u^0 \text{ in } L_6(Q) \text{ strongly,} \tag{12}$$

$$\frac{\partial^2 D^{(n)}}{\partial x_1^2} \to \frac{\partial^2 D^0}{\partial x_1^2}, \quad \frac{\partial^2 D^{(n)}}{\partial x_1 \partial x_2} \to \frac{\partial^2 D^0}{\partial x_1 \partial x_2},$$
$$\frac{\partial^2 D^{(n)}}{\partial x_2^2} \to \frac{\partial^2 D^0}{\partial x_2^2} \text{ in } W_2^1(\Omega) \text{ strongly,} \tag{13}$$

here $D^{(n)} = \frac{E(h^{(n)})^3}{12(1-\nu^2)}$, $\quad D^{(0)} = \frac{E(h^{(0)})^3}{12(1-\nu^2)}$.

$$\frac{\partial u^{(n)}}{\partial t} \to \frac{\partial u^0}{\partial t}, \ \frac{\partial^2 u^{(n)}}{\partial x_1^2} \to \frac{\partial^2 u^0}{\partial x_1^2}, \ \frac{\partial^2 u^{(n)}}{\partial x_1 \partial x_2} \to \frac{\partial^2 u^0}{\partial x_1 \partial x_2},$$
$$\frac{\partial^2 u^{(n)}}{\partial x_2^2} \to \frac{\partial^2 u^0}{\partial x_2^2} \text{ in } \ L_2(Q) \ \text{ weakly.} \tag{14}$$

Let $h = h^{(n)}$, $u = u^{(n)}$ in the definition of the solution of the problem (1)-(3):

$$\int_Q \left[-\rho h^{(n)} \frac{\partial u^{(n)}}{\partial t} \frac{\partial \eta}{\partial t} + D^{(n)} \Delta u^{(n)} \Delta \eta + (1-\nu) \right.$$
$$\left. \times \left(2 \frac{\partial^2 D^{(n)}}{\partial x_1 \partial x_2} \frac{\partial^2 u^{(n)}}{\partial x_1 \partial x_2} - \frac{\partial^2 D^{(n)}}{\partial x_1^2} \frac{\partial^2 u^{(n)}}{\partial x_2^2} - \frac{\partial^2 D^{(n)}}{\partial x_2^2} \frac{\partial^2 u^{(n)}}{\partial x_1^2} \right) \eta - {u^{(n)}}^3 \eta \right]$$
$$\times dx_1 dx_2 dt - \int_\Omega \rho h^{(n)} \varphi_1(x_1, x_2) \eta(x_1, x_2, 0) dx_1 dx_2$$

$$= \int_Q f(x_1, x_2, t)\eta(x_1, x_2, t)dx_1dx_2dt. \qquad (15)$$

Then, pass to the limit as $n \to \infty$ taking into account (12)-(14), we get

$$\int_Q \left[-\rho h^0 \frac{\partial u^0}{\partial t}\frac{\partial \eta}{\partial t} + D^0 \Delta u^0 \Delta \eta + (1-\nu) \right.$$

$$\times \left(2\frac{\partial^2 D^0}{\partial x_1 \partial x_2}\frac{\partial^2 u^0}{\partial x_1 \partial x_2} - \frac{\partial^2 D^0}{\partial x_1^2}\frac{\partial^2 u^0}{\partial x_2^2} - \frac{\partial^2 D^0}{\partial x_2^2}\frac{\partial^2 u^0}{\partial x_1^2} \right)$$

$$\left. \times\, \eta - u^{0^3}\eta \right] dx_1dx_2dt - \int_\Omega \rho h^0 \varphi_1(x_1, x_2)\eta(x_1, x_2, 0)dx_1dx_2$$

$$= \int_Q f(x_1, x_2, t)\eta(x_1, x_2, t)dx_1dx_2dt.$$

Therefore, $\{h^0, u^0\}$ is an admissible pair. As the functional $J(h,u)$ is continuous in $W_2^4(\Omega) \times L_6(Q)$, we have

$$\lim_{n\to\infty} J(h^{(n)}, u^{(n)}) = J(h^0, u^0). \qquad (16)$$

Then it follows from (7) and (16) that

$$\inf_{\{h,u\}} J(h,u) = J(h^0, u^0).$$

Consequently, the pair $\{h^0, u^0\}$ gives the minimum value of the functional $J(h,u)$, i.e. $\{h^0, u^0\}$ is an optimal pair.

Theorem 1 is proved. □

4. Adaptive penalty method

Introduce adapted functional for the optimal pair $\{h^0, u^0\}$:

$$J_\varepsilon^a(h,u) = \frac{1}{6}\|u - u_d\|_{L_6(Q)}^6 + \frac{N}{2}\|h\|_{W_2^4(\Omega)}^2$$

$$+\frac{1}{2\varepsilon}\left\| \frac{\partial^2 u}{\partial t^2} + \alpha\Delta(h^3\Delta u) + (1-\nu)\alpha\left(2\frac{\partial^2 h^3}{\partial x_1 \partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2} \right.\right.$$

$$\left.\left. -\frac{\partial^2 h^3}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 h^3}{\partial x_2^2}\frac{\partial^2 u}{\partial x_1^2}\right) - u^3 - f \right\|_{L_2(Q)}^2$$

$$+\frac{1}{2}\left\|u-u^{0}\right\|_{L_{2}(Q)}^{2}+\frac{1}{2}\left\|h-h^{0}\right\|_{W_{2}^{4}(\Omega)}^{2}, \tag{17}$$

where $\alpha=\frac{E}{12(1-\nu^{2})}$, $\left\{h^{0},u^{0}\right\}$ is a chosen optimal pair. Let us it minimize this functional for

$$h(x_1,x_2)\in U_{ad}, u(x_1,x_2,t)\in L_6(Q)$$

under the conditions

$$u(x_1,x_2,0)=\varphi_0(x_1,x_2), \frac{\partial u(x_1,x_2,0)}{\partial t}=\varphi_1(x_1,x_2),$$

$$u(0,x_2,t)=0, \frac{\partial u(0,x_2,t)}{\partial x_1}=0, u(x_1,0,t)=0, \frac{\partial u(x_1,0,t)}{\partial x_2}=0,$$

$$u(a,x_2,t)=0, \frac{\partial u(a,x_2,t)}{\partial x_1}=0, u(x_1,b,t)=0, \frac{\partial u(x_1,b,t)}{\partial x_2}=0.$$

Theorem 2. *For every fixed $\varepsilon>0$, there exists a pair $\{h_\varepsilon,u_\varepsilon\}$ that gives the minimum value of the functional $J_\varepsilon^a(h,u)$, i.e.*

$$J_\varepsilon^a(h_\varepsilon,u_\varepsilon)=\inf J_\varepsilon^a(h,u). \tag{18}$$

The proof of Theorem 2 is similar to that of Theorem 1.

5. Convergence of adaptive penalty method

Theorem 3. *Let $\{h_\varepsilon,u_\varepsilon\}$ be some solution of the problem* (18). *Then for $\varepsilon\to0$ we have*

$$h_\varepsilon\to h^0 \text{ in } W_2^4(\Omega) \text{ strongly}, \tag{19}$$

$$u_\varepsilon\to u^0 \text{ in } L_6(Q) \text{ strongly}, \tag{20}$$

where $\left\{h^0,u^0\right\}$ is a chosen optimal pair.

Proof. We have

$$J_\varepsilon^a(h_\varepsilon,u_\varepsilon)=\inf J_\varepsilon^a(h,u)\le J_\varepsilon^a(h^0,u^0)=J(h^0,u^0). \tag{21}$$

By definition of functional, we obtain

$$\|h_\varepsilon\|_{W_2^4(\Omega)}+\|u_\varepsilon\|_{L_6(Q)}\le C \tag{22}$$

and also

$$\rho h_\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} + \Delta(D_\varepsilon \Delta u_\varepsilon) + (1-\nu)\left(2\frac{\partial^2 D_\varepsilon}{\partial x_1 \partial x_2}\frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2}\right.$$

$$\left. -\frac{\partial^2 D_\varepsilon}{\partial x_1^2}\frac{\partial^2 u_\varepsilon}{\partial x_2^2} - \frac{\partial^2 D_\varepsilon}{\partial x_2^2}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\right) - u_\varepsilon^3 = f(x_1, x_2, t), \tag{23}$$

$$u_\varepsilon(x_1, x_2, 0) = \varphi_0(x_1, x_2), \frac{\partial u_\varepsilon(x_1, x_2, 0)}{\partial t} = \varphi_1(x_1, x_2), \tag{24}$$

$$\begin{array}{c} u_\varepsilon(0, x_2, t) = 0, \frac{\partial u_\varepsilon(0, x_2, t)}{\partial x_1} = 0, \\ u_\varepsilon(x_1, 0, t) = 0, \frac{\partial u_\varepsilon(x_1, 0, t)}{\partial x_2} = 0, \\ u_\varepsilon(a, x_2, t) = 0, \frac{\partial u_\varepsilon(a, x_2, t)}{\partial x_1} = 0, \\ u_\varepsilon(x_1, b, t) = 0, \frac{\partial u_\varepsilon(x_1, b, t)}{\partial x_2} = 0. \end{array} \tag{25}$$

From (22), (23)-(25) and the relation $h_\varepsilon \in U_{ad}$ it follows

$$\|u_\varepsilon\|_U \le C, \|u_\varepsilon\|_{W_2^{2,1}(Q)} \le C.$$

Consequently, we can derive from $\{h_\varepsilon, u_\varepsilon\}$ a subsequence, denoted again by $\{h_\varepsilon, u_\varepsilon\}$, such that

$$h_\varepsilon \to \hat{h} \text{ in } W_2^4(\Omega) \text{ weakly as } \varepsilon \to 0 \text{ and } \hat{h} \in U_{ad},$$

$$u_\varepsilon \to \hat{u} \text{ in } W_2^{2,1}(Q) \text{ weakly as } \varepsilon \to 0.$$

Besides, by [9, p. 70],

$$u_\varepsilon \to \hat{u} \text{ in } L_6(Q) \text{ strongly.}$$

Then, in the sense of generalized solution of the problem (1)-(3), the following relations hold:

$$\rho \hat{h} \frac{\partial^2 u}{\partial t^2} + \Delta(\hat{D} \Delta \hat{u}) + (1-\nu)\left(2\frac{\partial^2 \hat{D}}{\partial x_1 \partial x_2}\frac{\partial^2 \hat{u}}{\partial x_1 \partial x_2}\right.$$

$$\left. -\frac{\partial^2 \hat{D}}{\partial x_1^2}\frac{\partial^2 \hat{u}}{\partial x_2^2} - \frac{\partial^2 \hat{D}}{\partial x_2^2}\frac{\partial^2 \hat{u}}{\partial x_1^2}\right) - \hat{u}^3 = f(x_1, x_2, t)\ ,$$

$$\hat{u}(x_1, x_2, 0) = \varphi_0(x_1, x_2), \frac{\partial \hat{u}(x_1, x_2, 0)}{\partial t} = \varphi_1(x_1, x_2),$$

$$\begin{array}{c} \hat{u}(0, x_2, t) = 0, \frac{\partial \hat{u}(0, x_2, t)}{\partial x_1} = 0, \hat{u}(x_1, 0, t) = 0, \frac{\partial \hat{u}(x_1, 0, t)}{\partial x_2} = 0, \\ \hat{u}(a, x_2, t) = 0, \frac{\partial \hat{u}(a, x_2, t)}{\partial x_1} = 0, \hat{u}(x_1, b, t) = 0, \frac{\partial \hat{u}(x_1, b, t)}{\partial x_2} = 0. \end{array}$$

So, the inequality

$$J_\varepsilon^a(h_\varepsilon, u_\varepsilon) \geq J(h_\varepsilon, u_\varepsilon) + \frac{1}{2}\left\|\hat{u} - u^0\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|\hat{h} - h^0\right\|^2_{W_2^4(\Omega)}$$

leads to

$$\underline{\lim}_{\varepsilon \to 0} J_\varepsilon^a(h_\varepsilon, u_\varepsilon) \geq J(\hat{h}, \hat{u}) + \frac{1}{2}\left\|\hat{u} - u^0\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|\hat{h} - h^0\right\|^2_{W_2^4(\Omega)}.$$

And, as by (21) we have

$$\overline{\lim}_{\varepsilon \to 0} J_\varepsilon^a(h_\varepsilon, u_\varepsilon) \leq J(h^0, u^0),$$

it follows that

$$J(\hat{h}, \hat{u}) \leq J(h^0, u^0).$$

Therefore,

$$J(\hat{h}, \hat{u}) = J(h^0, u^0).$$

Then

$$\frac{1}{2}\left\|\hat{u} - u^0\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|\hat{h} - h^0\right\|^2_{W_2^4(\Omega)} = 0,$$

so $\hat{h} = h^0$, $\hat{u} = u^0$. So we get the validity of the relation (20).

By (21),

$$J(h^0, u^0) \geq J_\varepsilon^a(h_\varepsilon, u_\varepsilon) \geq J(h_\varepsilon, u_\varepsilon)$$

and

$$\underline{\lim}_{\varepsilon \to 0} J(h_\varepsilon, u_\varepsilon) \geq J(h^0, u^0).$$

Then

$$J(h^0, u^0) \geq \lim_{\varepsilon \to 0} J_\varepsilon^a(h_\varepsilon, u_\varepsilon) \geq \underline{\lim}_{\varepsilon \to 0} J(h_\varepsilon, u_\varepsilon) \geq J(h^0, u^0).$$

Therefore,

$$J(h_\varepsilon, u_\varepsilon) \to J(h^0, u^0).$$

Hence, by definition of the functional $J(h, u)$ we obtain the relation (19). Theorem 3 is proved. □

6. Optimality system for penalty problem

Now let us find the necessary conditions for $\{h_\varepsilon, u_\varepsilon\}$ to be the solution of the problem (18). For this $\{h_\varepsilon, u_\varepsilon\} \in U_{ad} \times U$

$$\frac{d}{d\lambda} J_\varepsilon^a(h_\varepsilon, u_\varepsilon + \lambda\xi)\Big|_{\lambda=0} = 0, \forall \xi \in C^2(\bar{Q}),$$

$$\xi(x_1, x_2, 0) = 0, \frac{\partial \xi(x_1, x_2, 0)}{\partial t} = 0,$$

$$\xi(0, x_2, t) = 0, \frac{\partial \xi(0, x_2, t)}{\partial x_1} = 0, \xi(x_1, 0, t) = 0, \frac{\partial \xi(x_1, 0, t)}{\partial x_2} = 0,$$

$$\xi(a, x_2, t) = 0, \frac{\partial \xi(a, x_2, t)}{\partial x_1} = 0, \xi(x_1, b, t) = 0, \frac{\partial \xi(x_1, b, t)}{\partial x_2} = 0$$

and

$$\frac{d}{d\lambda} J_\varepsilon^a(h_\varepsilon + \lambda(h - h_\varepsilon), u_\varepsilon)\Big|_{\lambda=0} \geq 0, \forall h \in U_{ad}, h_\varepsilon \in U_{ad}, \tag{26}$$

here $\lambda \in R$.

For this aim, let us calculate the derivative of the functional

$$J_\varepsilon^a(h_\varepsilon, u_\varepsilon + \lambda\xi) = \frac{1}{6}\|u_\varepsilon + \lambda\xi - u_d\|^6_{L_6(Q)} + \frac{N}{2}\|h_\varepsilon\|^2_{W_2^4(\Omega)}$$

$$+\frac{1}{2\varepsilon}\left\|\frac{\partial^2(u_\varepsilon + \lambda\xi)}{\partial t^2} + \alpha\Delta(h_\varepsilon^3\Delta(u_\varepsilon + \lambda\xi)) + (1-\nu)\alpha\right.$$

$$\times\left(2\frac{\partial^2 h_\varepsilon^3}{\partial x_1 \partial x_2}\frac{\partial^2(u_\varepsilon + \lambda\xi)}{\partial x_1 \partial x_2} - \frac{\partial^2 h_\varepsilon^3}{\partial x_1^2}\frac{\partial^2(u_\varepsilon + \lambda\xi)}{\partial x_2^2} - \frac{\partial^2 h_\varepsilon^3}{\partial x_2^2}\frac{\partial^2(u_\varepsilon + \lambda\xi)}{\partial x_1^2}\right)$$

$$\left.-(u_\varepsilon + \lambda\xi)^3 - f\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|u_\varepsilon + \lambda\xi - u^0\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|h_\varepsilon - h^0\right\|^2_{W_2^4(\Omega)}$$

with respect to λ and substitute $\lambda = 0$:

$$\frac{d}{d\lambda} J_\varepsilon^a(h_\varepsilon, u_\varepsilon + \lambda\xi)\Big|_{\lambda=0} = \int_Q (u_\varepsilon - u_d)^5 \xi dx_1 dx_2 dt$$

$$+\frac{1}{\varepsilon}\int_Q \left(\frac{\partial^2 u_\varepsilon}{\partial t^2} + \alpha\Delta(h_\varepsilon^3\Delta u_\varepsilon) + (1-\nu)\alpha\left(2\frac{\partial^2 h_\varepsilon^3}{\partial x_1 \partial x_2}\frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2}\right.\right.$$

$$\left.\left.-\frac{\partial^2 h_\varepsilon^3}{\partial x_1^2}\frac{\partial^2 u_\varepsilon}{\partial x_2^2} - \frac{\partial^2 h_\varepsilon^3}{\partial x_2^2}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\right) - u_\varepsilon^3 - f\right)\left(\frac{\partial^2 \xi}{\partial t^2} + \alpha\Delta(h_\varepsilon^3\Delta\xi)\right.$$

$$+(1-\nu)\alpha\left(2\frac{\partial^2 h_\varepsilon^3}{\partial x_1\partial x_2}\frac{\partial^2\xi}{\partial x_1\partial x_2}-\frac{\partial^2 h_\varepsilon^3}{\partial x_1^2}\frac{\partial^2\xi}{\partial x_2^2}-\frac{\partial^2 h_\varepsilon^3}{\partial x_2^2}\frac{\partial^2\xi}{\partial x_1^2}\right) \tag{27}$$

$$-3u_\varepsilon^2\xi\big)\,dx_1dx_2dt+\int_Q\left(u_\varepsilon-u^0\right)\xi dx_1dx_2dt.$$

Denote

$$\psi_\varepsilon=-\frac{1}{\varepsilon}\left(\frac{\partial^2 u_\varepsilon}{\partial t^2}+\alpha\Delta(h_\varepsilon^3\Delta u_\varepsilon)+(1-\nu)\alpha\right.$$

$$\left.\times\left(2\frac{\partial^2 h_\varepsilon^3}{\partial x_1\partial x_2}\frac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}-\frac{\partial^2 h_\varepsilon^3}{\partial x_1^2}\frac{\partial^2 u_\varepsilon}{\partial x_2^2}-\frac{\partial^2 h_\varepsilon^3}{\partial x_2^2}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\right)-u_\varepsilon^3-f\right). \tag{28}$$

Then from (27) we obtain

$$-\int_Q\psi_\varepsilon\left[\frac{\partial^2\xi}{\partial t^2}+\alpha\Delta(h_\varepsilon^3\Delta\xi)+(1-\nu)\alpha\right.$$

$$\left.\times\left(2\frac{\partial^2 h_\varepsilon^3}{\partial x_1\partial x_2}\frac{\partial^2\xi}{\partial x_1\partial x_2}-\frac{\partial^2 h_\varepsilon^3}{\partial x_1^2}\frac{\partial^2\xi}{\partial x_2^2}-\frac{\partial^2 h_\varepsilon^3}{\partial x_2^2}\frac{\partial^2\xi}{\partial x_1^2}\right)-3u_\varepsilon^2\xi\right]dx_1dx_2dt$$

$$+\int_Q(u_\varepsilon-u_d)^5\xi dx_1dx_2dt+\int_Q(u_\varepsilon-u^0)\xi dx_1dx_2dt=0. \tag{29}$$

The equation (29) means that $\psi_\varepsilon(x_1,x_2,t)$ is solution of the following problem:

$$\rho h_\varepsilon\frac{\partial^2\psi_\varepsilon}{\partial t^2}+\Delta(D_\varepsilon\Delta\psi_\varepsilon)+(1-\nu)\left[2\frac{\partial^2}{\partial x_1\partial x_2}\left(\frac{\partial^2 D_\varepsilon}{\partial x_1\partial x_2}\psi_\varepsilon\right)\right.$$

$$\left.-\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 D_\varepsilon}{\partial x_2^2}\psi_\varepsilon\right)-\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 D_\varepsilon}{\partial x_1^2}\psi_\varepsilon\right)\right]$$

$$-3u_\varepsilon^2\psi_\varepsilon=(u_\varepsilon-u_d)^5+(u_\varepsilon-u^0), \tag{30}$$

$$\psi_\varepsilon(x_1,x_2,T)=0,\frac{\partial\psi_\varepsilon(x_1,x_2,T)}{\partial t}=0, \tag{31}$$

$$\begin{aligned}&\psi_\varepsilon(0,x_2,t)=0,\tfrac{\partial\psi_\varepsilon(0,x_2,t)}{\partial x_1}=0,\psi_\varepsilon(x_1,0,t)=0,\tfrac{\partial\psi_\varepsilon(x_1,0,t)}{\partial x_2}=0,\\&\psi_\varepsilon(a,x_2,t)=0,\tfrac{\partial\psi_\varepsilon(a,x_2,t)}{\partial x_1}=0,\ \psi_\varepsilon(x_1,b,t)=0,\tfrac{\partial\psi_\varepsilon(x_1,b,t)}{\partial x_2}=0.\end{aligned} \tag{32}$$

From (22) follows

$$\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\to\frac{\partial^2 u^0}{\partial x_1^2},\frac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}\to\frac{\partial^2 u^0}{\partial x_1\partial x_2},$$

$$\frac{\partial^2 u_\varepsilon}{\partial x_2^2}\to\frac{\partial^2 u^0}{\partial x_2^2}\text{ in }L_2(Q)\text{ weakly }u^0\in U, \tag{33}$$

and by definition of the class U_{ad} we obtain

$$h_\varepsilon \to h^0 \text{ in } W_2^4(\Omega) \text{ strongly } h^0 \in U_{ad}. \qquad (34)$$

For every admissible pair $\{h_\varepsilon, u_\varepsilon\}$, the satisfaction of conditions (30)-(32) is understood in the sense that $\psi_\varepsilon(x_1,x_2,t) \in U$ and for every function $\eta(x_1,x_2,t) \in U$, $\eta(x_1,x_2,T)=0$ the integral identity

$$\int_Q \left[-\rho h_\varepsilon \frac{\partial \psi_\varepsilon}{\partial t}\frac{\partial \eta}{\partial t} + D_\varepsilon \Delta\psi_\varepsilon \Delta\eta + (1-\nu)\left[2\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial^2 D_\varepsilon}{\partial x_1 \partial x_2}\psi_\varepsilon\right) \right.\right.$$

$$\left.\left. -\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 D_\varepsilon}{\partial x_2^2}\psi_\varepsilon\right) - \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 D_\varepsilon}{\partial x_1^2}\psi_\varepsilon\right)\right]\eta - 3u_\varepsilon^2\psi_\varepsilon\eta \right] dx_1dx_2dt$$

$$-\int_\Omega \rho h_\varepsilon \varphi_1(x_1,x_2)\eta(x_1,x_2,0)dx_1dx_2$$

$$= \int_Q \left[(u_\varepsilon - u_d)^5 + (u_\varepsilon - u^0)\right]\eta(x_1,x_2,t)dx_1dx_2dt$$

and the condition $u(x_1,x_2,0)=\varphi_0(x_1,x_2)$ holds.

Taking into account (33), (34) can pass to the limit in the problem (30)-(32) as $\varepsilon \to 0$, and the limit function $\psi(x_1,x_2,t)$ will be solution of the following adjoint problem:

$$\rho h \frac{\partial^2 \psi}{\partial t^2} + \Delta(D\Delta\psi) + (1-\nu)\left[2\frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial^2 D}{\partial x_1 \partial x_2}\psi\right)\right.$$

$$\left. -\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 D}{\partial x_2^2}\psi\right) - \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 D}{\partial x_1^2}\psi\right)\right] - 3{u^0}^2\psi = (u^0 - u_d)^5,$$

$$\psi(x_1,x_2,T)=0, \frac{\partial \psi(x_1,x_2,T)}{\partial t} = 0,$$

$$\psi(0,x_2,t)=0, \tfrac{\partial\psi(0,x_2,t)}{\partial x_1}=0, \psi(x_1,0,t)=0, \tfrac{\partial\psi(x_1,0,t)}{\partial x_2}=0,$$
$$\psi(a,x_2,t)=0, \tfrac{\partial\psi(a,x_2,t)}{\partial x_1}=0, \ \psi(x_1,b,t)=0, \tfrac{\partial\psi(x_1,b,t)}{\partial x_2}=0.$$

Now let us simplify the condition (26). For this, let us calculate the derivative of the functional

$$J_\varepsilon^a(h_\varepsilon + \lambda(h-h_\varepsilon), u_\varepsilon) = \frac{1}{6}\left\|u_\varepsilon - u_d\right\|_{L_6(Q)}^6 + \frac{N}{2}\left\|h_\varepsilon + \lambda(h-h_\varepsilon)\right\|_{W_2^4(\Omega)}^2$$

$$+\frac{1}{2\varepsilon}\left\|\frac{\partial^2 u_\varepsilon}{\partial t^2} + \alpha\Delta(h_\varepsilon + \lambda(h-h_\varepsilon))^3\Delta u_\varepsilon\right.$$

$$+(1-\nu)\alpha\left(2\frac{\partial^2(h_\varepsilon+\lambda(h-h_\varepsilon))^3}{\partial x_1\partial x_2}\frac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}\right.$$

$$\left.-\frac{\partial^2(h_\varepsilon+\lambda(h-h_\varepsilon))^3}{\partial x_1^2}\frac{\partial^2 u_\varepsilon}{\partial x_2^2}-\frac{\partial^2(h_\varepsilon+\lambda(h-h_\varepsilon))^3}{\partial x_2^2}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\right)$$

$$-u_\varepsilon^3-f\Big\|^2_{L_2(Q)}+\frac{1}{2}\left\|u_\varepsilon-u^0\right\|^2_{L_2(Q)}+\frac{1}{2}\left\|h_\varepsilon+\lambda(h-h_\varepsilon)-h^0\right\|^2_{W_2^4(\Omega)}$$

with respect to λ and substitute $\lambda=0$:

$$\frac{d}{d\lambda}J_\varepsilon^a(h_\varepsilon+\lambda(h-h_\varepsilon),u_\varepsilon)\Big|_{\lambda=0}$$

$$=N\int_\Omega[h_\varepsilon\times(h-h_\varepsilon)]^{(4)}_{2,\Omega}\,dx_1dx_2+\frac{1}{\varepsilon}\int_Q\left[\frac{\partial^2 u_\varepsilon}{\partial t^2}+\alpha\Delta(h_\varepsilon^3\Delta u_\varepsilon)\right.$$

$$\left.+(1-\nu)\alpha\left(2\frac{\partial^2 h_\varepsilon^3}{\partial x_1\partial x_2}\frac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}-\frac{\partial^2 h_\varepsilon^3}{\partial x_1^2}\frac{\partial^2 u_\varepsilon}{\partial x_2^2}-\frac{\partial^2 h_\varepsilon^3}{\partial x_2^2}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\right)-u_\varepsilon^3-f\right]$$

$$\times\left[3\alpha\Delta(h_\varepsilon^2\Delta u_\varepsilon)(h-h_\varepsilon)+(1-\nu)\alpha\left(2\frac{\partial^2(3h_\varepsilon^2(h-h_\varepsilon))}{\partial x_1\partial x_2}\frac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}\right.\right.$$

$$\left.\left.-\frac{\partial^2(3h_\varepsilon^2(h-h_\varepsilon))}{\partial x_1^2}\frac{\partial^2 u_\varepsilon}{\partial x_2^2}-\frac{\partial^2(3h_\varepsilon^2(h-h_\varepsilon))}{\partial x_2^2}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\right)\right]$$

$$\times dx_1dx_2dt+\int_\Omega\left[(h_\varepsilon-h^0)\times(h-h_\varepsilon)\right]^{(4)}_{2,\Omega}dx_1dx_2.$$

Given the designation (28), we obtain the inequality

$$N\int_\Omega[h_\varepsilon\times(h-h_\varepsilon)]^{(4)}_{2,\Omega}\,dx_1dx_2$$

$$-\int_Q\psi_\varepsilon\left[3\alpha\Delta(h_\varepsilon^2\Delta u_\varepsilon)(h-h_\varepsilon)+(1-\nu)\alpha\left(2\frac{\partial^2(3h_\varepsilon^2(h-h_\varepsilon))}{\partial x_1\partial x_2}\frac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}\right.\right.$$

$$\left.\left.-\frac{\partial^2(3h_\varepsilon^2(h-h_\varepsilon))}{\partial x_1^2}\frac{\partial^2 u_\varepsilon}{\partial x_2^2}-\frac{\partial^2(3h_\varepsilon^2(h-h_\varepsilon))}{\partial x_2^2}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\right)\right]dx_1dx_2dt$$

$$+\int_\Omega\left[(h_\varepsilon-h^0)(h-h_\varepsilon)\right]^{(4)}_{2,\Omega}dx_1dx_2\geq 0,\forall h\in U_{ad}. \tag{35}$$

Taking into account the relations (33) and (34), we pass to the limit in (35) and obtain

$$N\int_\Omega\left[h^0\times(h-h^0)\right]^{(4)}_{2,\Omega}dx_1dx_2$$

$$-\int_Q \psi \left[3\alpha\Delta(h^{0^2}\Delta u^0)(h-h^0)+(1-\nu)\alpha\left(2\frac{\partial^2(3h^{0^2}(h-h^0))}{\partial x_1\partial x_2}\frac{\partial^2 u^0}{\partial x_1\partial x_2}\right.\right.$$

$$\left.\left.-\frac{\partial^2(3h^{0^2}(h-h^0))}{\partial x_1^2}\frac{\partial^2 u^0}{\partial x_2^2}-\frac{\partial^2(3h^{0^2}(h-h^0))}{\partial x_2^2}\frac{\partial^2 u^0}{\partial x_1^2}\right)\right]$$

$$\times dx_1dx_2dt \geq 0, \forall h \in U_{ad} \tag{36}$$

Thus, the following theorem is proved.

Theorem 4. *Under the given conditions on the data of the problem* (1)-(6), *for the optimal pair* $\{h^0, u^0\}$ *there exists a triple* $\{h^0, u^0, \psi\} \in U_{ad} \times U$ *such that*

$$\rho h^0\frac{\partial^2 u^0}{\partial t^2}+\Delta(D^0\Delta u^0)+(1-\nu)$$

$$\times\left(2\frac{\partial^2 D^0}{\partial x_1\partial x_2}\frac{\partial^2 u^0}{\partial x_1\partial x_2}-\frac{\partial^2 D^0}{\partial x_1^2}\frac{\partial^2 u^0}{\partial x_2^2}-\frac{\partial^2 D^0}{\partial x_2^2}\frac{\partial^2 u^0}{\partial x_1^2}\right)-u^{0^3}=f(x_1,x_2,t),$$

$$\rho h^0\frac{\partial^2 \psi}{\partial t^2}+\Delta(D^0\Delta\psi)+(1-\nu)\left[2\frac{\partial^2}{\partial x_1\partial x_2}\left(\frac{\partial^2 D^0}{\partial x_1\partial x_2}\psi\right)\right.$$

$$\left.-\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 D^0}{\partial x_2^2}\psi\right)-\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 D^0}{\partial x_1^2}\psi\right)\right]-3u^{0^2}\psi=(u^0-u_d)^5,$$

$$u^0(x_1,x_2,0)=\varphi_0(x_1,x_2), \frac{\partial u^0(x_1,x_2,0)}{\partial t}=\varphi_1(x_1,x_2),$$

$$\psi(x_1,x_2,T)=0, \frac{\partial\psi(x_1,x_2,T)}{\partial t}=0,$$

$$\psi(0,x_2,t)=0, \frac{\partial\psi(0,x_2,t)}{\partial x_1}=0, \psi(x_1,0,t)=0, \frac{\partial\psi(x_1,0,t)}{\partial x_2}=0,$$

$$\psi(a,x_2,t)=0, \frac{\partial\psi(a,x_2,t)}{\partial x_1}=0,\ \psi(x_1,b,t)=0, \frac{\partial\psi(x_1,b,t)}{\partial x_2}=0,$$

$u^0 \in U$, $\psi \in U$ *and the inequality* (36) *holds.*

References

[1] A.N. Tikhonov, A.A. Samarski, *Equations of Mathematical Physics*, Nauka, Moscow, 1972, 736 p.

[2] B.M. Budak, A. A.Samarski, A.N.Tikhonov, *Collection of Problems in Mathematical Physics.* Nauka, Moscow, 1972, 688 p.

[3] H.F. Guliyev, Kh.I. Seyfullayeva, On determination of the right part of linear equation of elastic plate vibration, *International Scientific and Technical Journal "Problems of Management and Informatics"*, No 2 (2017), 76-84.

[4] H.F. Guliyev, Kh.I. Seyfullayeva, Optimal control problem for the equation of vibrations of an elastic plate, *Georgian Mathematical Journal,* **25**, No 3 (2018), 371-380. DOI 10.1515/gmj-2017-0004.

[5] H.F. Guliyev, Kh.I. Seyfullayeva, Optimal control problem with coefficients for the equation of vibrations of a thin plate with discontinuous solution, *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan,* **48**, No 2 (2022), 238-248.

[6] J.L. Arman, *Applications of the Theory of Optimal Control of Distributed-parameter Systems to Structural Optimization.* Mir Moscow, 1977, 144 p.

[7] J.L. Lions, *Optimal Control for the Systems Described by the Partial Differential Equations.* Mir, Moscow, 1972, 416 p.

[8] J.L. Lions, *Control of Distributed Singular Systems.* Nauka, Moscow, 1987, 367 p.

[9] J.L. Lions, *Some Methods of Solving Non-Linear Boundary Value Problems.* Mir, Moscow, 1972, 588 p.

[10] N.V. Banichuk, Optimal design of elastic plates, problems and methods of optimal structural design. *Mathematical Consepts and Methods in Science and Engineering*, **26** (1983), 109-134.

[11] V. Komkov, *Optimal Control Theory for Damping the Vibrations of Simple Elastic Systems.* Mir, Moscow, 1975, 160 p.