International Journal of Applied Mathematics

Volume 36 No. 6 2023, 747-756
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v36i6.1

ANTI PRODUCT FUZZY GRAPHS

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Abstract: In this article, we launch the conception of anti product fuzzy graph and two operations on them; namely join and product. We give sufficient conditions for the join and product of two anti product fuzzy graph to be complete. We also provide equivalent conditions for the join of two unbiased anti product fuzzy graphs to be unbiased.

AMS Subject Classification: 05C72
Key Words: anti product fuzzy graph, complete, unbiased, join, product

1. Background

The theory of graph has many applications in mathematics and economics. Since most of the problems on graphs are undetermined, it is necessary to handle these facets via the method of fuzzy logic. Fuzzy relations were introduced by Zadeh [22] in 1965. Rosenfeld [19] in 1975, introduced fuzzy graphs (simply, FG) and some ideas that are generalizations of those of graphs. Nowadays, this theory is having more and more applications in which the information level immanent in the set of things working together as parts of a mechanism differ with various degrees of accuracy. Fuzzy fashion are convenient as they reduce differences between long-established numerical models of expert systems and symbolic models. Peng and Mordeson [14] defined the conceptualization of FGs complement and conscious FGs operations. In [21], it is improved a complement's definition in order to guarantee the original FG is isomorphic to

Received: 2 February 2023

complement of the complement, which concurs with the case of crisp graphs. In addition, self-complementary FGs properties and the complement under FGs join, union and composition (introduced in [14]) were explored. Al-Hawary [1] introduced the concept of balanced in the class of FGs and Al-Hawary and others have deeply explored this ides for many types of FGs. For more on the foregoing concepts and those coming after ones, one can see [2, 4, 5, 6, 7, 8, 9, 10, 14, 16, 17, 18, 21].

For a non-empty finite set Ű, a fuzzy subset of Ű is a mapping ţ:Ű$\to [0,1]$ and a fuzzy subset of Ű×Ű is called a fuzzy relation ς on ţ. We assume that Ű is finite and ς is reflexive and symmetric.

Definition 1. ([19]) A *fuzzy graph (simply, FG),* with Ű as the underlying set, is a pair Ģ: (ţ, ς) where ţ:Ű$\to [0,1]$ is a fuzzy subset and ς :Ű×Ű$\to [0,1]$ is a fuzzy relation on ţ such that $\varsigma(c,s) \leq$ţ$(c)\wedge$ţ(s) for all $c, s \in$Ű, where by $\wedge$, we mean the minimum. Its classical graph is Ģ* : (ţ*, ς*) where ţ$^* = \sup c($ţ$) =$ $\{c \in$Ű:ţ$(c) > 0\}$ and $\varsigma^* = \sup c(\varsigma) = \{(c,s) \in$Ű×Ű: $\varsigma(c,s) > 0\}$.

Definition 2. ([19]) Two FGs Ģ$_1$: (ţ$_1$, ς$_1$) and Ģ$_2$: (ţ$_2$, ς$_2$) are said to be *isomorphic* providing the existence of a bijective τ :Ű$_1$ $\to$Ű$_2$ such that ţ$_1(c) =$ţ$_2(\tau(c))$ for all $c \in$Ű$_1$and $\varsigma_1(c,s) = \varsigma_2(\tau(c), \tau(s))$ for all $(c,s) \in$Ĕ$_1$. We then write Ģ$_1$ $\simeq$Ģ$_2$ and h is called an *isomorphism.*

Using the operation of product instead of minimum, in [20] Ramaswamy and Poornima established PFGs.

Definition 3. ([20]) Let Ģ* : (Ű,Ĕ) be a graph, ţ be a fuzzy subset of Ű and ς be a fuzzy subset of Ű×Ű. We call Ģ: (ţ, ς) a *product fuzzy graph* (simply, PFG), if $\varsigma(c,s) \leq$ţ(c)ţ(s) for all $c, s \in$Ű.

The next result follows immediately.

Theorem 4. *Every PFG is a FG.*

Definition 5. [20] A PFG Ģ: (ţ, ς) is called *complete* if $\varsigma(c,s) =$ţ(c)ţ(s) for all $c, s \in$Ű.

Definition 6. ([20]) The *complement of a PFG* Ģ: (ţ, ς) is Ģ^c : (ţ^c, ς^c)

where ţ^c =ţ and

$$\begin{aligned} \varsigma^c(c,s) &= ţ^c(c)ţ^c(s) - \varsigma(c,s) \\ &= ţ(c)ţ(s) - \varsigma(c,s). \end{aligned}$$

Anti fuzzy graphs were introduced in [13]. The notion of join and product of FGs was introduced and studied in [21] where the complement for these operations was the main idea. In Section 2 of this paper, we launch the conception of anti product fuzzy graph and two operations on them. We give sufficient conditions for the join and product of two anti product fuzzy graphs to be complete. Section 3 is devoted to provide equivalent conditions for the join of two unbiased APFGs to be unbiased.

2. Anti product fuzzy graph

We begin this section by defining the anti product fuzzy graph.

Definition 7. A FG Ģ: (ţ, ς) is said to be ani product fuzzy graph (APFG) if ς(ű, $\ddot{y}$) ≥ţ(ű)ţ(, $\ddot{y}$) for all ű, $\ddot{y}$ ∈ţ*.

Clearly, every complete PFG is an APFG, but the converse is not true in general. In fact, an APFG may not be a PFG. For example, the APFG Ģ: (ţ, ς) where ţ(x)=.1, ţ(y)=.1 and $\varsigma(x,y) = .2$ is not a PFG.

Definition 8. Let Ģ$_1$: (ţ$_1$, ς_1) and Ģ$_2$: (ţ$_2$, ς_2) be APFGs with Ű$_1$∩Ű$_2$ =. The join of Ģ$_1$ and Ģ$_2$ is defined to be Ģ$_1$+Ģ$_2$: (ţ$_1$ + ţ$_2$, $\varsigma_1 + \varsigma_2$), where

$$(ţ_1 + ţ_2)(ű) = \begin{cases} ţ_1(ű) & ű \in Ű_1 \\ ţ_2(ű) & ű \in Ű_2 \end{cases}$$

and

$$(\varsigma_1 + \varsigma_2)(ű, \ddot{y}) = \begin{cases} \varsigma_1(ű, \ddot{y}) & ű\ddot{y} \in \breve{E}_1 \\ \varsigma_2(ű, \ddot{y}) & ű\ddot{y} \in \breve{E}_2 \\ ţ_1(ű)ţ_2(\ddot{y}) & ű \in Ű_1, \ddot{y} \in Ű_2 \end{cases}.$$

Theorem 9. *The join of two APFGs Ģ$_1$: (ţ$_1$, ς_1) and Ģ$_2$: (ţ$_2$, ς_2) is an APFG.*

Proof. To show the join is a APFG, we need only show that $\varsigma(\H{u}, \ddot{y}) \geq (ţ_1{+}ţ_2)(\H{u})(ţ_1{+}ţ_2)(\ddot{y})$ for all ű,$\ddot{y}$.

Case 1: If ű$\ddot{y} \in \breve{E}_1$, then ű,$\ddot{y}$ $\in$Ű$_1$ and as Ģ$_1$ is an APFG,

$$\begin{aligned}(\varsigma_1 + \varsigma_2)(\H{u}, \ddot{y}) &= \varsigma_1(\H{u}, \ddot{y}) \\ &\geq ţ_1(\H{u})ţ_2(\ddot{y}) \\ &= (ţ_1 + ţ_2)(\H{u})(ţ_1 + ţ_2)(\ddot{y}).\end{aligned}$$

The case that ű$\ddot{y} \in \breve{E}_2$ is similar to the case ű$\ddot{y} \in \breve{E}_1$.

Case 2: If ű$\ddot{y} \notin \breve{E}_1$ and ű$\ddot{y} \notin \breve{E}_2$, then

$$\begin{aligned}(\varsigma_1 + \varsigma_2)(\H{u}, \ddot{y}) &= ţ_1(\H{u})ţ_2(\ddot{y}) \\ &= (ţ_1 + ţ_2)(\H{u})(ţ_1 + ţ_2)(\ddot{y})\end{aligned}$$

This completes the proof. □

Theorem 10. *Two APFGs $Ģ_1 : (ţ_1, \varsigma_1)$ and $Ģ_2 : (ţ_2, \varsigma_2) Ģ_1 : (ţ_1, \varsigma_1)$ and $Ģ_2 : (ţ_2, \varsigma_2)$ are complete if and only if their join is complete.*

Proof. If the join of Ģ$_1$ and Ģ$_2$ is complete and ű$\ddot{y} \in \breve{E}_1$, $\varsigma_1(\H{u}, \ddot{y}) = (\varsigma_1 + \varsigma_2)(\H{u}, \ddot{y}) = (ţ_1{+}ţ_2)(\H{u})(ţ_1{+}ţ_2)(\ddot{y}) = ţ_1(\H{u})ţ_2(\ddot{y})$ and hence Ģ$_1$ is complete. Similarly, Ģ$_2$ is complete.

Conversely, let Ģ$_1$ and Ģ$_2$ be complete. If ű$\ddot{y} \in \breve{E}_1$, then ű,$\ddot{y}$ $\in$Ű$_1$ and so $(\varsigma_1 + \varsigma_2)(\H{u}, \ddot{y}) = \varsigma_1(\H{u}, \ddot{y}) = ţ_1(\H{u})ţ_1(\ddot{y})$ because Ģ$_1$ is complete. But $ţ_1(\H{u})ţ_1(\ddot{y}) = (ţ_1{+}ţ_2)(\H{u})(ţ_1{+}ţ_2)(\ddot{y})(ţ_1{+}ţ_2)(\H{u})(ţ_1{+}ţ_2)(\ddot{y})$ and so $(\varsigma_1{+}\varsigma_2)(\H{u}, \ddot{y}) = (ţ_1{+}ţ_2)(\H{u})(ţ_1{+}ţ_2)(\ddot{y})$. Hence the join is complete. □

Theorem 11. *Let $Ģ_1 : (ţ_1, \varsigma_1)$ and $Ģ_2 : (ţ_2, \varsigma_2)$ be APFGs. Then:*
a) $(Ģ_1{+}Ģ_2)^c \simeq (Ģ_1^c{\cup}Ģ_2^c)$,
b) $(Ģ_1{\cup}Ģ_2)^c \simeq (Ģ_1^c{+}Ģ_2^c)$.

Proof. a) For ű$\in$Ű$_1$, $(ţ_1{+}ţ_2)^c(\H{u}) = (ţ_1{+}ţ_2)(\H{u}) = ţ_1(\H{u})$. On the other hand, $ţ_1^c(\H{u}) \vee ţ_2^c(\H{u}) = ţ_1(\H{u}) \vee ţ_2(\H{u}) = ţ_1(\H{u})$. Similar result will occur for ű$\in$Ű$_2$. Now if ű$\ddot{y} \in \breve{E}_1$, then ű,$\ddot{y}$ $\in$Ű$_1$ and so

$$\begin{aligned}(\varsigma_1 + \varsigma_2)^c(\H{u},\ddot{y}) &= (ţ_1 + ţ_2)(\H{u})(ţ_1 + ţ_2)(\ddot{y}) - (\varsigma_1 + \varsigma_2)(\H{u},\ddot{y}) \\ &= ţ_1(\H{u})ţ_1(\ddot{y}) - \varsigma_1(\H{u},\ddot{y}) \\ &= \varsigma_1^c(\H{u},\ddot{y})\end{aligned}$$

$$= \varsigma_1^c(\text{ű},\ddot{y}) \vee \varsigma_2^c(\text{ű},\ddot{y}).$$

Similarly, when ű,$\ddot{y}$ $\in$Ű$_2$, we get $(\varsigma_1+\varsigma_2)^c(\text{ű},\ddot{y}) = \varsigma_1^c(\text{ű},\ddot{y}) \vee \varsigma_2^c(\text{ű},\ddot{y})$. Now if ű$\in$Ű$_1$ and $\ddot{y}$ $\in$Ű$_2$, then

$$\begin{aligned}(\varsigma_1+\varsigma_2)^c(\text{ű},\ddot{y}) &= (\text{ţ}_1+\text{ţ}_2)(\text{ű})(\text{ţ}_1+\text{ţ}_2)(\ddot{y}) - (\varsigma_1+\varsigma_2)(\text{ű},\ddot{y})\\ &= \text{ţ}_1(\text{ű})\text{ţ}_2(\ddot{y}) - \text{ţ}_1(\text{ű})\text{ţ}_2(\ddot{y})\\ &= 0\\ &= \varsigma_1^c(\text{ű},\ddot{y}) \vee \varsigma_2^c(\text{ű},\ddot{y}).\end{aligned}$$

b) For ű$\in$Ű$_1$,

$$(\text{ţ}_1 \cup \text{ţ}_2)^c(\text{ű}) = (\text{ţ}_1 \cup \text{ţ}_2)(\text{ű}) = \text{ţ}_1(\text{ű}) = \text{ţ}_1(\text{ű}) \vee \text{ţ}_2(\text{ű})$$

$$= \text{ţ}_1^c(\text{ű}) \vee \text{ţ}_2^c(\text{ű}) = (\text{ţ}_1^c + \text{ţ}_2^c)(\text{ű}).$$

Similar result will occur for ű$\in$Ű$_2$.

Now if ű$\ddot{y} \in \breve{E}_1$, then ű,$\ddot{y}$ $\in$Ű$_1$ and so,

$$\begin{aligned}(\varsigma_1 \cup \varsigma_2)^c(\text{ű},\ddot{y}) &= (\text{ţ}_1 \cup \text{ţ}_2)(\text{ű})(\text{ţ}_1 \cup \text{ţ}_2)(\ddot{y}) - (\varsigma_1 \cup \varsigma_2)(\text{ű},\ddot{y})\\ &= \text{ţ}_1(\text{ű})\text{ţ}_1(\ddot{y}) - \varsigma_1(\text{ű},\ddot{y})\\ &= \varsigma_1^c(\text{ű},\ddot{y}).\end{aligned}$$

Similarly, when ű,$\ddot{y}$ $\in$Ű$_2$, we get $(\varsigma_1 \cup \varsigma_2)^c(\text{ű},\ddot{y}) = \varsigma_2^c(\text{ű},\ddot{y})$. Now if ű$\in$Ű$_1$ and $\ddot{y}$ $\in$Ű$_2$, then

$$\begin{aligned}(\varsigma_1 \cup \varsigma_2)^c(\text{ű},\ddot{y}) &= (\text{ţ}_1 \cup \text{ţ}_2)(\text{ű})(\text{ţ}_1 \cup \text{ţ}_2)(\ddot{y}) - (\varsigma_1 \cup \varsigma_2)(\text{ű},\ddot{y})\\ &= \text{ţ}_1(\text{ű})\text{ţ}_2(\ddot{y})\\ &= \text{ţ}_1^c(\text{ű})\text{ţ}_1^c(\ddot{y}).\end{aligned}$$

This completes the proof that $(\varsigma_1 \cup \varsigma_2)^c(\text{ű},\ddot{y}) = (\varsigma_1^c \cup \varsigma_2^c)(\text{ű},\ddot{y})$. □

Definition 12. Let Ģ$_1$: (ţ$_1, \varsigma_1$) and Ģ$_2$: (ţ$_2, \varsigma_2$) be APFGs. The product of Ģ$_1$ and Ģ$_2$ is defined to be Ģ$_1\times$Ģ$_2$: (ţ$_1 \times$ ţ$_2, \varsigma_1 \times \varsigma_2$) where (ţ$_1\times$ ţ$_2$)(ű,$\ddot{y}$)=ţ$_1$(ű)ţ$_2$($\ddot{y}$) for all ű$\in$Ű$_1$ and $\ddot{y}$ $\in$Ű$_2$ and $(\varsigma_1 \times \varsigma_2)((\text{ű}_1, \text{ű}_2), (\ddot{y}_1, \ddot{y}_2) = \varsigma_1(\text{ű}_1, \ddot{y}_1)\varsigma_2(\text{ű}_2, \ddot{y}_2)$ for all ű$_1, \ddot{y}_1$ $\in$Ű$_1$ and ű$_2, \ddot{y}_2$ $\in$Ű$_2$.

Next, we show that the above definition is well-defined.

Theorem 13. *Let Ģ$_1$: (ţ$_1, \varsigma_1$) and Ģ$_2$: (ţ$_2, \varsigma_2$) be APFGs. Then Ģ$_1\times$Ģ$_2$ is an APFG.*

Proof. Let $ű_1, \ddot{y}_1 \in Ű_1$ and $ű_2, \ddot{y}_2 \in Ű_2$. Then

$$\begin{aligned}(\varsigma_1 \times \varsigma_2)((ű_1, ű_2), (\ddot{y}_1, \ddot{y}_2) &= \varsigma_1(ű_1, \ddot{y}_1)\varsigma_2(ű_2, \ddot{y}_2)\\ &\geq ţ_1(ű_1)ţ_1(\ddot{y}_1)ţ_2(ű_2)ţ_2(\ddot{y}_2)\\ &= ţ_1(ű_1)ţ_1(ű_1)ţ_1(\ddot{y}_1)ţ_2(\ddot{y}_2)\\ &= (ţ_1 \times ţ_2)(ű_1, ű_2)(ţ_1 \times ţ_2)(\ddot{y}_1, \ddot{y}_2).\end{aligned}$$

□

Theorem 14. *Two APFGs $Ģ_1 : (ţ_1, \varsigma_1)$ and $Ģ_2 : (ţ_2, \varsigma_2)Ģ_1 : (ţ_1, \varsigma_1)$ and $Ģ_2 : (ţ_2, \varsigma_2)$ are complete if and only if $Ģ_1 \times Ģ_2$ is complete.*

Proof. Let $Ģ_1 \times Ģ_2$ be complete. We will first show that at least one APFG is complete by contradiction. So let us assume that both are not complete. Then there exist $ű_1, \ddot{y}_1 \in Ű_1$ and $ű_2, \ddot{y}_2 \in Ű_2$ such that $\varsigma_1(ű_1, \ddot{y}_1) > ţ_1(ű_1)ţ_1(\ddot{y}_1)$ and $\varsigma_2(ű_2, \ddot{y}_2) > ţ_2(ű_2)ţ_2(\ddot{y}_2)$. Now

$$\begin{aligned}(\varsigma_1 \times \varsigma_2)((ű_1, ű_2), (\ddot{y}_1, \ddot{y}_2) &= \varsigma_1(ű_1, \ddot{y}_1)\varsigma_2(ű_2, \ddot{y}_2)\\ &> ţ_1(ű_1)ţ_1(\ddot{y}_1)ţ_2(ű_2)ţ_2(\ddot{y}_2)\\ &= ţ_1(ű_1)ţ_2(ű_1)ţ_1(\ddot{y}_1)ţ_2(\ddot{y}_2)\\ &= (ţ_1 \times ţ_2)(ű_1, ű_2)(ţ_1 \times ţ_2)(\ddot{y}_1, \ddot{y}_2).\end{aligned}$$

This contradicts that $Ģ_1 \times Ģ_2$ is complete. Thus, without loss of generality, assume $Ģ_1$ is complete. To show $Ģ_2$ is complete, as $Ģ_1 \times Ģ_2$ is complete,
$(\varsigma_1 \times \varsigma_2)((ű_1,ű_2), (\ddot{y}_1, \ddot{y}_2) = (ţ_1 \times ţ_2)(ű_1,ű_2)(ţ_1 \times ţ_2)(\ddot{y}_1, \ddot{y}_2)$.
Thus,

$$\begin{aligned}\varsigma_1(ű_1, \ddot{y}_1)\varsigma_2(ű_2, \ddot{y}_2) &= ţ_1(ű_1)ţ_2(ű_1)ţ_1(\ddot{y}_1)ţ_2(\ddot{y}_2)\\ &= ţ_1(ű_1)ţ_1(\ddot{y}_1)ţ_2(ű_2)ţ_2(\ddot{y}_2)\\ &= \varsigma_1(ű_1, \ddot{y}_1)ţ_2(ű_2)ţ_2(\ddot{y}_2),\end{aligned}$$

where the last equality holds because $Ģ_1$ is complete. Now $\varsigma_1(ű_1, \ddot{y}_1) \neq 0$ since otherwise, $\varsigma_1(ű_1, \ddot{y}_1)\varsigma_2(ű_2, \ddot{y}_2) = 0 = ţ_1(ű_1)ţ_1(\ddot{y}_1)ţ_2(ű_2)ţ_2(\ddot{y}_2)$ which means at least one term is zero and that is impossible. Now divide both sides of the above equilaty by $\varsigma_1(ű_1, \ddot{y}_1)$, we get $\varsigma_2(ű_2, \ddot{y}_2) = ţ_2(ű_2)ţ_2(\ddot{y}_2)$, which means $Ģ_2$ is complete.

Conversely, if $Ģ_1$ and $Ģ_2$ are complete,

$$\varsigma_1(ű_1, \ddot{y}_1)\varsigma_2(ű_2, \ddot{y}_2) = ţ_1(ű_1)ţ_2(ű_1)ţ_1(\ddot{y}_1)ţ_2(\ddot{y}_2)$$

$$= ţ_1(ű_1)ţ_1(ÿ_1)ţ_2(ű_2)ţ_2(ÿ_2)$$
$$= ς_1(ű_1, ÿ_1)ţ_2(ű_2)ţ_2(ÿ_2)$$
$$= (ţ_1 \times ţ_2)(ű_1, ű_2)(ţ_1 \times ţ_2)(ÿ_1, ÿ_2).$$

Hence $Ģ_1 \times Ģ_2$ is complete. □

3. Unbiased APFGs

We begin this section by introducing the definition of unbiased APFGs and then proving the following Theorem 16 to make it possible characterize unbiased the join and product of two unbiased APFGs.

Definition 15. ([5]) The *degree of compactness* of an APFG is $c(Ģ) = \frac{2\sum_{űÿ \in \breve{E}} (ς(ű,ÿ))}{\sum_{ű,ÿ \in Ű} (ţ(ű)ţ(ÿ))}$. Ģ is unbiased if $c(Ģ) \le c(H)$ for any non-empty product fuzzy subgraph H of Ģ.

Theorem 16. *Let $Ģ_1$ and $Ģ_2$ be APFGs. Then $c(Ģ_1) \ge c(Ģ_1+Ģ_2)$ and $c(Ģ_2) \ge c(Ģ_1+Ģ_2)$ if and only if $c(Ģ_1) = c(Ģ_2) = c(Ģ_1+Ģ_2)$.*

Proof. If $c(Ģ_1) \ge c(Ģ_1+Ģ_2)$ and $c(Ģ_2) \ge c(Ģ_1+Ģ_2)$, then

$$c(Ģ_1) = 2(\sum_{ű_1,ű_2 \in Ű_1} ς_1(ű_1ű_2))/(\sum_{ű_1,ű_2 \in Ű_1} (ţ_1(ű_1)ţ_1(ű_2)))$$
$$\le 2(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ ÿ_1,ÿ_2 \in Ű_2}} ς_1(ű_1ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2))/$$
$$(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ ÿ_1,ÿ_2 \in Ű_2}} (ţ_1(ű_1)ţ_1(ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)))$$
$$\le 2(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ ÿ_1,ÿ_2 \in Ű_2}} ς_1(ű_1ű_2)ς_2(ÿ_1ÿ_2))/$$
$$(\sum_{\substack{ű_1,ű_2 \in Ű_1 \\ ÿ_1,ÿ_2 \in Ű_2}} (ţ_1(ű_1)ţ_1(ű_2)ţ_2(ÿ_1)ţ_2(ÿ_2)))$$

$$\leq 2\Big(\sum_{\substack{\H{u}_1,\H{u}_2\in\H{U}_1\\ \ddot{y}_1,\ddot{y}_2\in\H{U}_2}} (\varsigma_1+\varsigma_2)((\H{u}_1\ddot{y}_1),(\H{u}_2\ddot{y}_2))/$$
$$\Big(\sum_{\substack{\H{u}_1,\H{u}_2\in\H{U}_1\\ \ddot{y}_1,\ddot{y}_2\in\H{U}_2}} (\c{t}_1+\c{t}_2)((\H{u}_1,\ddot{y}_1))(\c{t}_1+\c{t}_2)((\H{u}_2,\ddot{y}_2))\Big)$$
$$= c(\c{G}_1+\c{G}_2).$$

Hence $c(\c{G}_1) = c(\c{G}_{1\alpha}\c{G}_2)$. Similarly, $c(\c{G}_2) = c(\c{G}_{1\alpha}\c{G}_2)$. Therefore, $c(\c{G}_1) = c(\c{G}_2) = c(\c{G}_{1\alpha}\c{G}_2)$. The converse is trivial. □

Theorem 17. *For two unbiased APFGs $\c{G}_1$ and $\c{G}_2$, $\c{G}_1+\c{G}_2$ is unbiased if and only if $c(\c{G}_1) = c(\c{G}_2) = c(\c{G}_1+\c{G}_2)$.*

Proof. If $\c{G}_1+\c{G}_2$ is unbiased, then $c(\c{G}_1) \geq c(\c{G}_1+\c{G}_2)$ and $c(\c{G}_2) \geq c(\c{G}_1+\c{G}_2)$ and by Theorem 16, $c(\c{G}_1) = c(\c{G}_2) = c(\c{G}_1+\c{G}_2)$.

If $c(\c{G}_1) = c(\c{G}_2) = c(\c{G}_1+\c{G}_2)$ and K is an APFS of $\c{G}_1+\c{G}_2$, then we can find APFSs K_i of $\c{G}_i$ for $i = 1, 2$ with $K \approx K_1 + K_2$. As $\c{G}_1$ and $\c{G}_2$ are unbiased and $c(\c{G}_1) = c(\c{G}_2) = m_1/k_1$, then $c(K_1) = a_1/b_1 \geq m_1/k_1$ and $c(K_2) = a_2/b_2 \geq m_1/k_1$. Thus $a_1k_1 + a_2k_1 \geq b_1m_1 + b_2m_1$ and hence $c(K) \geq (a_1 + a_2)/(b_1 + b_2) \geq m_1/k_1 = c(\c{G}_1+\c{G}_2)$. Therefore, $\c{G}_1+\c{G}_2$ is unbiased. □

We end this section with the following result which states that unbiased notion is preserved under isomorphism:

Theorem 18. *Let $\c{G}_1$ and $\c{G}_2$ be isomorphic APFGs. If one of them is unbiased, then the other is unbiased.*

Proof. Suppose $\c{G}_2$ is unbiased and let $\epsilon : \H{U}_1 \rightarrow \H{U}_2$ be a bijection such that $\c{t}_1(\H{u}) = \c{t}_2(\epsilon(\H{u}))$ and $\varsigma_1(\H{u}\ddot{y}) = \varsigma_2(\epsilon(\H{u})\epsilon(\ddot{y}))$ for all $\H{u}, \ddot{y} \in \H{U}_1$. Now $\sum_{\H{u}\in\H{U}_1} \c{t}_1(\H{u}) = \sum_{\H{u}\in\H{U}_2} \c{t}_2(\H{u})$ and $\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_1(\H{u}\ddot{y}) = \sum_{\H{u}\ddot{y}\in\breve{E}_2} \varsigma_2(\H{u}\ddot{y})$. If $K_1 = (\c{t}_1, \varsigma_1)$ is a APFS of $\c{G}_1$ with underlying set W, then $K_2 = (\c{t}_2, \varsigma_2)$ is a APFS of $\c{G}_2$ with underlying set $\epsilon(W)$ where $\c{t}_2(\epsilon(\H{u})) = \c{t}_1(\H{u})$ and $\varsigma_2(\epsilon(\H{u})\epsilon(\ddot{y})) = \varsigma_1(\H{u}\ddot{y})$ for all $\H{u}, \ddot{y} \in W$. Since $\c{G}_2$ is unbiased, $c(K_1) \geq c(\c{G}_2)$ and so $2\frac{\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_2(\epsilon(\H{u}),\epsilon(\ddot{y}))}{\sum_{\H{u},\ddot{y}\in\H{U}_1}(\c{t}_2(\H{u})\wedge\c{t}_2(\ddot{y}))} \geq 2\frac{\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_2(\H{u},\ddot{y})}{\sum_{\H{u},\ddot{y}\in\H{U}_1}(\c{t}_2(\H{u})\wedge\c{t}_2(\ddot{y}))}$. Hence

$$2\frac{\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_1(\H{u},\ddot{y})}{\sum_{\H{u},\ddot{y}\in\H{U}_1}(\c{t}_2(\H{u}) \wedge \c{t}_2(\ddot{y}))} \geq 2\frac{\sum_{\H{u}\ddot{y}\in\breve{E}_1} \varsigma_1(\H{u},\ddot{y})}{\sum_{\H{u},\ddot{y}\in\H{U}_1}(\c{t}_2(\H{u}) \wedge \c{t}_2(\ddot{y}))}.$$

Therefore, $Ģ_1$ is unbiased. $\square$

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