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MODELLING AND APPLICATIONS INVOLVING FRACTIONAL DERIVATIVES AND MITTAG-LEFFLER FUNCTIONS

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Abstract

The aim of this paper is to introduce and systematically investigate a new subclass $TS(\chi, \sigma, \varsigma)$ of analytic and univalent functions with negative coefficients defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. This newly established class is formulated by leveraging the specialized properties of fractional derivative operators integrated with the versatile Mittag-Leffler function. For functions belonging to this subclass, we derive fundamental geometric and analytic properties. Specifically, we establish sharp coefficient inequalities and prove comprehensive distortion and covering theorems. Furthermore, we determine the precise radii of starlikeness, convexity, and close-to-convexity to characterize the geometric boundaries of the class. Finally, we determine the extreme points, analyze the behavior under the Hadamard (convolution) product, and establish closure theorems. The results obtained not only generalize several existing frameworks in geometric function theory but also highlight the powerful interplay between fractional calculus and subclasses of univalent functions.

Keywords: Analytic functions, univalent functions, fractional derivatives, Mittag-Leffler functions, coefficient estimates, starlikeness, convexity.

2020 Mathematics Subject Classification: 30C45, 26A33, 33E12

Introduction

In recent years, fractional calculus has emerged as one of the most rapidly evolving fields within mathematical analysis. Operators derived from fractional calculus have penetrated deeply into geometric function theory, particularly influencing the study of univalent functions. Consequently, a diverse array of these operators has been extensively investigated throughout the literature. To establish a clear mathematical framework for our study, it is useful to recall the foundational definitions below (see, e.g., [18, 19, 25]).

The study of the Mittag-Leffler function and its various generalizations has emerged as a highly active area of mathematical research. This growing interest is primarily driven by the function's deep connection to fractional calculus, particularly its utility in resolving complex

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problems arising from real-world applications. Introduced by the Swedish mathematician Magnus Gösta Mittag-Leffler between 1903 and 1905, this transcendental function remained largely overlooked by mathematicians and applied scientists until the late 20th century. Over the past few decades, however, it has garnered substantial attention due to its indispensable role in solving fractional-order integral and differential equations.

Nowadays, it is well recognized that a fundamental role in Fractional Calculus is played by the Mittag-Leffler function, even with a single parameter (as originally introduced by Mittag-Leffler), so as to be worth being referred to as the **Queen Function of Fractional Calculus**; see Mainardi and Gorenflo [16]. Some information on the Mittag-Leffler functions is found in any treatise on Fractional Calculus, but, for more details, the reader is referred to the surveys by Haubold, Mathai and Saxena [10] and Van Mieghem [30], as well as to the treatise by Gorenflo et al.[9], which is devoted to Mittag-Leffler functions, related topics and applications.

In recent years, considerable attention has been devoted to the study of Mittag-Leffler functions due to their wide range of applications in engineering, chemical and biological sciences, physics, and other areas of applied science. These functions play an important role in the analysis of chaotic, stochastic, and dynamical systems, fractional differential equations, and statistical distributions. Moreover, the geometric properties of these functions, such as convexity, close-to-convexity, and starlikeness, have been extensively investigated by several researchers. Their applications are also closely related to various tools and techniques of fractional calculus; see, for example, [1, 2, 4, 5, 21, 29, 28].

The Mittag-Leffler function occurs naturally in the solutions of fractional-order differential and integral equations. In particular, it has significant applications in the study of fractional generalizations of kinetic equations, random walks, Lévy flights, super-diffusive transport, and complex systems. Various properties of the Mittag-Leffler function and its generalized forms have been investigated extensively in the literature; see, for example, [6, 23, 3, 2, 7, 8, 13, 14].

Let A signify the class of all functions $u(z)$ of the type

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u \in A$ is a starlike function of the order $\varsigma, 0 \leq \varsigma < 1$, if it satisfy

$$\Re\left\{\frac{zu\prime(z)}{u(z)}\right\} > \varsigma, z \in \mathbb{U}. \tag{1.2}$$

We denote this class with $S^*(\varsigma)$.

A function $u \in A$ is a convex function of the order $\varsigma, 0 \leq \varsigma < 1$, if it fulfil

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$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \varsigma, z \in \mathbb{U}. \tag{1.3}$$

We denote this class with $K(\varsigma)$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$ respectively.

Let T denote the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \; z \in \mathbb{U} \tag{1.4}$$

and let $T^*(\varsigma) = T \cap S^*(\varsigma)$, $C(\varsigma) = T \cap K(\varsigma)$. The class $T^*(\varsigma)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [24].

Many basically equivalent definitions of fractional computation have been given in literature ((cf.)e.g.,[22] and ([26], p. 45)). We state the following definitions due to Owa and Srivastava [20] which have been used rather frequently in the theory of analytic functions (see also [11, 12]).

Pochhammer symbol $(\alpha)_n$ can be defined as

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) \quad \text{if} \quad n \neq 0$$

and

$$(\alpha)_n = 1 \quad \text{if} \; n = 0.$$

The $(\alpha)_n$ can be expressed in terms of the Gamma function as:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad (n \in \mathbb{N}).$$

In [17], Mittag-Leffler introduced Mittag-Leffler functions

$$\mathcal{H}_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+1)} z^n, \quad \big(\alpha \in \mathbb{C}, \;\; Re(\alpha)\big) > 0,$$

and its generalization $\mathcal{H}_{\alpha,\beta}(z)$ introduced by Wiman [31] as

$$\mathcal{H}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+\beta)} z^n, \quad \big(\alpha,\beta \in \mathbb{C}, \;\; Re(\alpha), \;\; Re(\beta)\big) > 0. \tag{1.5}$$

Observe that the function $E_{\tau,\nu}$ contains many well-known functions as its special case, for example,

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$$\begin{aligned}
E_{1,1}(z) &= \mathrm{e}^z, \quad E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z}, \\
E_{2,1}(z^2) &= \mathrm{coshz}, \quad E_{2,1}(-z^2) = \mathrm{cos}z, \quad E_{2,2}(z^2) = \frac{\mathrm{sinhz}}{z}, \\
E_{2,2}(-z^2) &= \frac{\mathrm{sinz}}{z}c, \quad E_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}}\cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right] \\
\text{and } E_4(z) &= \frac{1}{2}\left[\mathrm{cos}z^{1/4} + \mathrm{cosh}z^{1/4}\right].
\end{aligned}$$

Now we define the normalization of Mittag-Leffler function $\mathcal{M}_{\alpha,\beta}(z)$ as follows:

$$\begin{aligned}
\mathcal{M}_{\alpha,\beta}(z) &= z\Gamma(\beta)\mathcal{H}_{\alpha,\beta}(z) \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n,
\end{aligned} \tag{1.6}$$

where, $z \in U$, $(Re\ \alpha > 0, \beta \in \mathbb{C} \setminus \{0, -1, -2, \cdots\})$. In [27], Srivastava and Owa gave definitions for fractional derivative operator and fractional integral operator in the complex z-plane C as follows: The fractional integral of order δ is defined for a function $u(z)$, by

$$I_z^{\delta} u(z) \equiv I_z^{-\delta} u(z) = \frac{1}{\Gamma(\delta)} \int_0^z (z-t)^{\delta-1} u(t) d(t), \ \ (\delta > 0).$$

The fractional derivative operator D_z of order δ is defined by

$$D_z^{\delta} u(z) = D_z I_z^{1-\delta} u(z) = \frac{1}{\Gamma(1-\delta)} D_z \int_0^z \frac{u(t)}{(z-t)^{\delta}} dt, \quad (0 \le \delta < 1).$$

where, the function $u(z)$ is analytic in the simply-connected region of the complex z-plane C containg the origin, and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Let $\delta > 0$ and m be the smallest integer, and the extended fractional derivative of $u(z)$ of order δ is defined as:

$$D_z^{\delta} u(z) = D_z^m I_z^{m-\delta} u(z), \quad \delta \ge 0,\ n > -1 \tag{1.7}$$

provided that it exists. We find from (1.7) that is

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n-\delta}, \ \ (0 \le \delta < 1,\ \ n > -1)$$

and

$$I_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n+\delta}, \ \ (0 < \delta,\ \ n > -1).$$

Owa and Srivastava [20], defined the differential Integral operator $\Omega_z^{\delta}: A \to A$ in the term of series:

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$$\begin{aligned} \Omega_z^{\delta} u(z) &= \frac{\Gamma(2-\delta)}{\Gamma(2)} z^{\delta} D_z^{\delta} u(z) \\ &= z + \sum_{n=2}^{\infty} \left(\frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\delta)} \right) a_n z^n \end{aligned} \tag{1.8}$$

where, $\delta < 2$, and $z \in U$.)

Here $D_z^{\delta} u(z)$ represents the fractional of u(z) of order δ when $-\infty < \delta < 0$ and a fractional derivative of $u(z)$ of order δ when $0 \leq \delta < 2$. Now, by using the definition of convolution of (1.6) and (1.8), we define fractional differential integral operator $D_z^{\delta,\alpha,\beta}: A \to A$, associated with normalized Mittag-Leffler function $\mathcal{M}_{\alpha,\beta}(z)$as follows:

$$\mathfrak{D}_z^{\delta,\alpha,\beta} u(z) = z + \sum_{n=2}^{\infty} \Theta\,(n, \delta, \alpha, \beta) a_n z^n \tag{1.9}$$

where $\Theta(n, \delta, \alpha, \beta) = \left(\frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\delta)} \right) \left(\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right) a_n z^n$

and $(\delta < 2,\ Re\ \alpha > 0,\ \ \beta \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}), z \in U$.

It is noted that

$$\mathfrak{D}_z^{0,0,1} u(z) = u(z).$$

Again,by using fractional differential integral operator $\mathfrak{D}_z^{\delta,\alpha,\beta}$,we also define a linear multiflier fractional differential integral operator $\Delta_{\alpha,\lambda}^{\beta,\delta,m}$ as follows:

$$\Delta_{\alpha,\lambda}^{\beta,\delta,m} u(z) = \Delta_{\alpha,\lambda}^{\beta,\delta,1} \left(\Delta_{\alpha,\lambda}^{\beta,\delta,m-1} u(z) \right) \tag{1.10}$$

where,

$$\Delta_{\alpha,\lambda}^{\beta,\delta,0} u(z) = u(z),$$

and

$$\Delta_{\alpha,\lambda}^{\beta,\delta,1} u(z) = (1-\lambda) \mathfrak{D}_z^{\delta,\alpha,\beta} f(z) + \lambda z D \left(\mathfrak{D}_z^{\delta,\alpha,\beta} u(z) \right).$$

It is seen from u(z) given by (1.1) and from (1.10), we have

$$\Delta_{\alpha,\lambda}^{\beta,\delta,m} u(z) = z + \sum_{n=2}^{\infty} \Theta\,(n, \lambda, \alpha, \beta, m) a_n z^n, \tag{1.11}$$

Where,

$$\Theta(n, \lambda, \alpha, \beta, m) = \left[\left(\frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\delta)} \right) \left(\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right) (1 - \lambda + n\lambda) \right]^m \tag{1.12}$$

and

$$(\delta < 2,\ m \in \mathbb{N},\ \lambda \geq 0,\ \ \Re\, \alpha > 0,\ \beta\ \in \mathbb{C} \setminus \{0,1,2, \dots\}).$$

(i) For $\delta = 0, \alpha = 0$, and $\beta = 1$, in (1.11) then it is reduced to the operator given by Al-Oboudi .

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(ii) For $\delta = 0, \lambda = 1, \alpha = 0$ and $\beta = 1$ in (1.11) then it is reduced to the operator given by Salagean .

Now, by making use of the linear operator $\mathfrak{D}_z^{\delta,\alpha,\beta} u(z)$, we define a new subclass of functions belonging to the class A.

Definition 1.1. For $0 \le \chi < 1, 0 \le \sigma < 1, 0 < \varsigma < 1$, and $0 \le \vartheta < 1$, we let $TS(\chi, \sigma, \varsigma)$ be the subclass of u consisting of functions of the form (1.4) and its geometrical condition satisfy $\left| \frac{\chi\left(\left(\mathfrak{D}_z^{\delta,\alpha,\beta} u(z) \right)' - \frac{\mathfrak{D}_z^{\delta,\alpha,\beta} u(z)}{z} \right)}{\sigma\left(\mathfrak{D}_z^{\delta,\alpha,\beta} u(z) \right)' + (1-\chi)\frac{\mathfrak{D}_z^{\delta,\alpha,\beta} u(z)}{z}} \right| < \varsigma, \; z \in \mathbb{U}$

where $\mathfrak{D}_z^{\delta,\alpha,\beta} u(z)$, is given by (1.9).

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS(\chi, \sigma, \varsigma)$.

Theorem 2.1. Let the function u be defined by (1.4). Then $u \in TS(\chi, \sigma, \varsigma)$ if and only if

$$\sum_{n=2}^{\infty} [\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\, \Theta(n, \delta, \alpha, \beta) a_n \le \varsigma\big(\sigma + (1-\chi)\big), \tag{2.1}$$

where $0 < \varsigma < 1, 0 \le \chi < 1$, and $0 \le \sigma < 1$, The result (2.1) is sharp for the function

$$u(z) = z - \frac{\varsigma\big(\sigma + (1-\chi)\big)}{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\Theta(n, \delta, \alpha, \beta)} z^n, \; n \ge 2.$$

Proof. Suppose that the inequality (2.1) holds true and $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left| \chi\left(\left(\mathfrak{D}_z^{\delta,\alpha,\beta} u(z) \right)' - \frac{\mathfrak{D}_z^{\delta,\alpha,\beta} u(z)}{z} \right) \right| - \varsigma \left| \sigma\left(\mathfrak{D}_z^{\delta,\alpha,\beta} u(z))' + (1-\chi)\frac{\mathfrak{D}_z^{\delta,\alpha,\beta} u(z)}{z} \right) \right| \\
= \; & \left| -\chi \sum_{n=2}^{\infty} (n-1)\, \Theta(n, \delta, \alpha, \beta) a_n z^{n-1} \right| \\
& -\varsigma \left| \sigma + (1-\chi) - \sum_{n=2}^{\infty} (n\sigma + 1 - \chi)\, \Theta(n, \delta, \alpha, \beta) a_n z^{n-1} \right| \\
\le \; & \sum_{n=2}^{\infty} [\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\, \Theta(n, \delta, \alpha, \beta) a_n - \varsigma\big(\sigma + (1-\chi)\big) \\
\le \; & 0.
\end{aligned}$$

Hence, by maximum modulus principle,$u \in TS(\chi, \sigma, \varsigma)$. Now assume that $u \in TS(\chi, \sigma, \varsigma)$ so that

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$$\left|\frac{\chi\left(\left(\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)\right)' - \frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}\right)}{\sigma\left(\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)\right)' + (1-\chi)\frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}}\right| < \varsigma, \quad z \in \mathbb{U}$$

Hence

$$\left|\chi\left(\left(\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)\right)' - \frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}\right)\right| < \varsigma\left|\sigma\left(\mathfrak{D}_z^{\delta,\alpha,\beta}u(z))' + (1-\chi)\frac{\mathfrak{D}_z^{\delta,\alpha,\beta}u(z)}{z}\right)\right|.$$

Therefore, we get

$$\begin{aligned}&\left|-\sum_{n=2}^{\infty}\chi\,(n-1)\Theta(n,\delta,\alpha,\beta)a_n z^{n-1}\right|\\ < \quad &\varsigma\left|\sigma + (1-\chi) - \sum_{n=2}^{\infty}(n\sigma + 1 - \chi)\,\Theta(n,\delta,\alpha,\beta)a_n z^{n-1}\right|.\end{aligned}$$

Thus

$$\sum_{n=2}^{\infty}[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\,\Theta(n,\delta,\alpha,\beta)a_n \leq \varsigma\big(\sigma + (1-\chi)\big)$$

and this completes the proof. □

Corollary 2.2. Let the function $u \in TS(\chi,\sigma,\varsigma)$.Then

$$a_n \leq \frac{\varsigma\big(\sigma + (1-\chi)\big)}{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\Theta(n,\delta,\alpha,\beta)} z^n, \quad n \geq 2.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS(\chi,\sigma,\varsigma)$

Theorem 3.1. Let the function $u \in TS(\chi,\sigma,\varsigma)$. Then

$$\begin{aligned}|z| - \frac{\varsigma\big(\sigma + (1-\chi)\big)}{\Theta(2,\delta,\alpha,\beta)[\chi + \varsigma(2\sigma + 1 - \chi)]}|z|^2 \quad &\leq |u(z)|\\ &\leq \quad |z| + \frac{\varsigma\big(\sigma + (1-\chi)\big)}{\Theta(2,\delta,\alpha,\beta)[\chi + \varsigma(2\sigma + 1 - \chi)]}|z|^2.\end{aligned}$$

The result is sharp and attained

$$u(z) = z - \frac{\varsigma\big(\sigma + (1-\chi)\big)}{\Theta(2,\delta,\alpha,\beta)[\chi + \varsigma(2\sigma + 1 - \chi)]}z^2.$$

Proof.

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$$\begin{aligned}|u(z)| &= \left|z-\sum_{n=2}^{\infty} a_n\, z^n\right| \leq |z| + \sum_{n=2}^{\infty} a_n\, |z|^n \\ &\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n\,.\end{aligned}$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\varsigma(\sigma+(1-\chi))}{[\chi+\varsigma(2\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}. \tag{3.1}$$

Thus

$$|u(z)| \leq |z| + \frac{\varsigma(\sigma+(1-\chi))}{\Theta(2,\delta,\alpha,\beta)[\chi+\varsigma(2\sigma+1-\chi)]}|z|^2.$$

Also

$$\begin{aligned}|u(z)| &\geq |z| - \sum_{n=2}^{\infty} a_n\, |z|^n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{\varsigma(\sigma+(1-\chi))}{\Theta(2,\delta,\alpha,\beta)[\chi+\varsigma(2\sigma+1-\chi)]}|z|^2.\end{aligned}$$

□

Theorem 3.2. Let $u \in TS(\chi,\sigma,\varsigma)$. Then

$$\begin{aligned}1-\frac{2\varsigma(\sigma+(1-\chi))}{\Theta(2,\delta,\alpha,\beta)[\chi+\varsigma(2\sigma+1-\chi)]}|z| &\leq |u'(z)| \\ &\leq 1+\frac{2\varsigma(\sigma+(1-\chi))}{\Theta(2,\delta,\alpha,\beta)[\chi+\varsigma(2\sigma+1-\chi)]}|z|\end{aligned}$$

with equality for

$$u(z) = z - \frac{2\varsigma(\sigma+(1-\chi))}{\Theta(2,\delta,\alpha,\beta)[\chi+\varsigma(2\sigma+1-\chi)]}z^2.$$

Proof. Notice that

$$\begin{aligned} & \Theta(2,\delta,\alpha,\beta)[\chi+\varsigma(2\sigma+1-\chi)]\sum_{n=2}^{\infty} n\, a_n \\ \leq\ & \sum_{n=2}^{\infty} n\,[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)a_n \\ \leq\ & \varsigma(\sigma+(1-\chi)),\end{aligned} \tag{3.2}$$

from Theorem 2.1. Thus

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$$
\begin{aligned}
|u'(z)| &= |1-\textstyle\sum_{n=2}^{\infty} n\, a_n z^{n-1}| \\
&\le 1+\textstyle\sum_{n=2}^{\infty} n\, a_n |z|^{n-1} \\
&\le 1+|z|\textstyle\sum_{n=2}^{\infty} n\, a_n \\
&\le 1+|z|\frac{2\varsigma(\sigma+(1-\chi))}{\Theta(2,\delta,\alpha,\beta)[\chi+\varsigma(2\sigma+1-\chi)]}.
\end{aligned} \tag{3.3}
$$

On the other hand

$$
\begin{aligned}
|u'(z)| &= |1-\textstyle\sum_{n=2}^{\infty} n\, a_n z^{n-1}| \\
&\ge 1-\textstyle\sum_{n=2}^{\infty} n\, a_n |z|^{n-1} \\
&\ge 1-|z|\textstyle\sum_{n=2}^{\infty} n\, a_n \\
&\ge 1-|z|\frac{2\varsigma(\sigma+(1-\chi))}{\Theta(2,\delta,\alpha,\beta)[\chi+\varsigma(2\sigma+1-\chi)]}.
\end{aligned} \tag{3.4}
$$

Combining (3.3) and (3.4), we get the result. □

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS(\chi,\sigma,\varsigma)$.

Theorem 4.1. Let $u \in TS(\chi,\sigma,\varsigma)$. Then u is starlike in $|z| < R_1$ of order ρ, $0 \le \rho < 1$, where

$$R_1 = \inf_n \left\{\frac{(1-\rho)\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{(n-\rho)\varsigma(\sigma+(1-\chi))}\right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.1}$$

Proof. u is starlike of order $\rho, 0 \le \rho < 1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \rho.$$

Thus it is enough to show that

$$\left|\frac{zu'(z)}{u(z)}-1\right| = \left|\frac{-\sum_{n=2}^{\infty}(n-1)\, a_n z^{n-1}}{1-\sum_{n=2}^{\infty} a_n\, z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty}(n-1)\, a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n\, |z|^{n-1}}.$$

Thus

$$\left|\frac{zu\prime(z)}{u(z)}-1\right| \le 1-\rho \;\; if \;\; \sum_{n=2}^{\infty}\frac{(n-\rho)}{(1-\rho)} a_n |z|^{n-1} \le 1. \tag{4.2}$$

Hence by Theorem 2.1, (4.2) will be true if

$$\frac{n-\rho}{1-\rho}|z|^{n-1} \le \frac{\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\chi)}$$

or if

$$|z| \le \left[\frac{(1-\rho)\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{(n-\rho)\varsigma(\sigma+(1-\chi))}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{4.3}$$

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The theorem follows easily from (4.3). □

Theorem 4.2. Let $u \in TS(\chi, \sigma, \varsigma)$. Then u is convex in $|z| < R_2$ of order $\rho, 0 \le \rho < 1$, where

$$R_2 = \inf_n \left\{ \frac{(1-\rho)\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{n(n-\rho)\varsigma\big(\sigma+(1-\chi)\big)} \right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.4}$$

Proof. u is convex of order $\rho, 0 \le \rho < 1$ if

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \rho.$$

Thus it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n\,(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} n\,a_n z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty} n\,(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} n\,a_n |z|^{n-1}}.$$

Thus

$$\left|\frac{zu''(z)}{u'(z)}\right| \le 1-\rho \;\; if \;\; \sum_{n=2}^{\infty} \frac{n(n-\rho)}{(1-\rho)} a_n |z|^{n-1} \le 1. \tag{4.5}$$

Hence by Theorem 2.1, (4.5) will be true if

$$\frac{n(n-\rho)}{1-\rho}|z|^{n-1} \le \frac{\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\chi)}$$

or if

$$|z| \le \left[\frac{(1-\rho)\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{n(n-\rho)\varsigma\big(\sigma+(1-\chi)\big)}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{4.6}$$

The theorem follows easily from (4.6). □

Theorem 4.3. Let $u \in TS(\chi, \sigma, \varsigma)$. Then u is close-to-convex in $|z| < R_3$ of order ρ, $0 \le \rho < 1$, where

$$R_3 = \inf_n \left\{ \frac{(1-\rho)\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{n\varsigma\big(\sigma+(1-\chi)\big)} \right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.7}$$

Proof. u is close-to-convex of order $\rho, 0 \le \rho < 1$ if

$$\Re\{u'(z)\} > \rho.$$

Thus it is enough to show that

$$|u'(z) - 1| = \left|-\sum_{n=2}^{\infty} n\,a_n z^{n-1}\right| \le \sum_{n=2}^{\infty} n\,a_n |z|^{n-1}.$$

Thus

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$$|u'(z)-1| \le 1-\rho \;\; if \;\; \sum_{n=2}^{\infty} \frac{n}{(1-\rho)} a_n |z|^{n-1} \le 1. \tag{4.8}$$

Hence by Theorem 2.1, (4.8) will be true if

$$\frac{n}{1-\rho}|z|^{n-1} \le \frac{\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\chi)}$$

or if

$$|z| \le \left[\frac{(1-\rho)\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{n\varsigma\big(\sigma+(1-\chi)\big)}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{4.9}$$

The theorem follows easily from (4.9). □

5. Extreme Points

In the following theorem, we obtain extreme points for the class $TS(\chi,\sigma,\varsigma)$.

Theorem 5.1. Let $u_1(z) = z$ and

$$u_n(z) = z - \frac{\varsigma\big(\sigma+(1-\chi)\big)}{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)} z^n, \;\; \text{for } \; n = 2,3,\cdots.$$

Then $u \in TS(\chi,\sigma,\varsigma)$ if and only if it can be expressed in the form

$$u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z), \;\; where \;\; \theta_n \ge 0 \;\; and \;\; \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof. Assume that $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$, hence we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\varsigma\big(\sigma+(1-\chi)\big)\theta_n}{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)} z^n.$$

Now, $u \in TS(\chi,\sigma,\varsigma)$, since

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma+(1-\chi)\big)} \\
&\qquad \times \frac{\varsigma\big(\sigma+(1-\chi)\big)\theta_n}{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)} \\
&= \sum_{n=2}^{\infty} \theta_n = 1-\theta_1 \le 1.
\end{aligned}$$

Conversely, suppose $u \in TS(\chi,\sigma,\varsigma)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty} \theta_n u_n(z)$.

Now $u \in TS(\chi,\sigma,\varsigma)$ implies from Theorem 2.1

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$$a_n \leq \frac{\varsigma\big(\sigma + (1-\chi)\big)}{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\Theta(n,\delta,\alpha,\beta)}.$$

Setting $\theta_n = \frac{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\chi))} a_n, n = 2,3,\cdots$

and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$, we obtain $u(z) = \sum_{n=1}^{\infty} \theta_n\, u_n(z)$. □

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $TS(\chi,\sigma,\varsigma)$.

Theorem 6.1. Let $u, g \in TS(\chi,\sigma,\varsigma,\vartheta)$. Then $u * g \in TS(\chi,\sigma,\zeta,\vartheta)$ for

$$u(z) = z - \sum_{n=2}^{\infty} a_n\, z^n, g(z) = z - \sum_{n=2}^{\infty} b_n\, z^n \text{ and } (u * g)(z) = z - \sum_{n=2}^{\infty} a_n\, b_n z^n,$$

where

$$\zeta \geq \frac{\varsigma^2\big(\sigma + (1-\chi)\big)\chi(n-1)}{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]^2\Theta(n,\delta,\alpha,\beta) - \varsigma^2\big(\sigma + (1-\chi)\big)(n\sigma + 1 - \chi)}.$$

Proof. $u \in TS(\chi,\sigma,\varsigma)$ and so

$$\sum_{n=2}^{\infty} \frac{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\chi))} a_n \leq 1, \tag{6.1}$$

and

$$\sum_{n=2}^{\infty} \frac{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\chi))} b_n \leq 1. \tag{6.2}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\chi(n-1)+\zeta(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\zeta(\sigma+(1-\chi))} a_n b_n \leq 1. \tag{6.3}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma(\sigma+(1-\chi))} \sqrt{a_n b_n} \leq 1. \tag{6.4}$$

Therefore it is enough to show that

$$\begin{aligned} &\frac{[\chi(n-1) + \zeta(n\sigma + 1 - \chi)]\Theta(n,\delta,\alpha,\beta)}{\zeta\big(\sigma + (1-\chi)\big)} a_n b_n \\ \leq\ &\frac{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma + (1-\chi)\big)} \sqrt{a_n b_n}. \end{aligned}$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\zeta}{[\chi(n-1)+\zeta(n\sigma+1-\chi)]\varsigma}. \tag{6.5}$$

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From (6.4)

$$\sqrt{a_n b_n} \leq \frac{\varsigma\big(\sigma + (1-\chi)\big)}{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\Theta(n,\delta,\alpha,\beta)}.$$

Thus it is enough to show that

$$\frac{\varsigma\big(\sigma + (1-\chi)\big)}{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\Theta(n,\delta,\alpha,\beta)} \leq \frac{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\zeta}{[\chi(n-1) + \zeta(n\sigma + 1 - \chi)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2\big(\sigma + (1-\chi)\big)\chi(n-1)}{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]^2\Theta(n,\delta,\alpha,\beta) - \varsigma^2\big(\sigma + (1-\chi)\big)(n\sigma + 1 - \chi)}.$$

□

7. Closure Theorems

We shall prove the following closure theorems for the class $TS(\chi,\sigma,\varsigma)$.

Theorem 7.1. Let $u_j \in TS(\chi,\sigma,\varsigma), j = 1,2,\ldots,s$. Then

$$g(z) = \sum_{j=1}^{s} c_j\, u_j(z) \in TS(\chi,\sigma,\varsigma)$$

For $u_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\, z^n$, where $\sum_{j=1}^{s} c_j = 1$.

Proof.

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_j\, u_j(z) \\ &= z - \sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j\, a_{n,j} z^n \\ &= z - \sum_{n=2}^{\infty} e_n\, z^n, \end{aligned}$$

where $e_n = \sum_{j=1}^{s} c_j\, a_{n,j}$. Thus $g(z) \in TS(\chi,\sigma,\varsigma)$ if

$$\sum_{n=2}^{\infty} \frac{[\chi(n-1) + \varsigma(n\sigma + 1 - \chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma + (1-\chi)\big)} e_n \leq 1,$$

that is, if

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$$\sum_{n=2}^{\infty}\sum_{j=1}^{s}\frac{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma+(1-\chi)\big)}c_j a_{n,j}$$
$$=\sum_{j=1}^{s}c_j\sum_{n=2}^{\infty}\frac{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma+(1-\chi)\big)}a_{n,j}$$
$$\leq\sum_{j=1}^{s}c_j=1.$$

□

Theorem 7.2. Let $u,g\in TS(\chi,\sigma,\varsigma)$. Then

$$h(z)=z-\sum_{n=2}^{\infty}(a_n^2+b_n^2)\,z^n\in TS(\chi,\sigma,\varsigma), where$$

$$\zeta\geq\frac{2\chi(n-1)\varsigma^2\big(\sigma+(1-\chi)\big)}{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]^2\Theta(n,\delta,\alpha,\beta)-2\varsigma^2\big(\sigma+(1-\chi)\big)(n\sigma+1-\chi)}.$$

Proof. Since $u,g\in TS(\chi,\sigma,\varsigma)$, so Theorem 2.1 yields

$$\sum_{n=2}^{\infty}\left[\frac{\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma+(1-\chi)\big)}a_n\right]^2\leq 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma+(1-\chi)\big)}b_n\right]^2\leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma+(1-\chi)\big)}\right]^2(a_n^2+b_n^2)\leq 1. \tag{7.1}$$

But $h(z)\in TS(\chi,\sigma,\zeta,q,m)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\chi(n-1)+\zeta(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\zeta\big(\sigma+(1-\chi)\big)}(a_n^2+b_n^2)\leq 1, \tag{7.2}$$

where $0<\zeta<1$, however (7.1) implies (7.2) if

$$\frac{[\chi(n-1)+\zeta(n\sigma+1-\chi)]\Theta(n,\delta,\alpha,\beta)}{\zeta\big(\sigma+(1-\chi)\big)}$$
$$\leq\frac{1}{2}\left[\frac{\big(\chi(n-1)+\varsigma(n\sigma+1-\chi)\big)\Theta(n,\delta,\alpha,\beta)}{\varsigma\big(\sigma+(1-\chi)\big)}\right]^2.$$

Simplifying, we get

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$$\zeta \geq \frac{2\chi(n-1)\varsigma^2\big(\sigma+(1-\chi)\big)}{[\chi(n-1)+\varsigma(n\sigma+1-\chi)]^2\Theta(n,\delta,\alpha,\beta)-2\varsigma^2\big(\sigma+(1-\chi)\big)(n\sigma+1-\chi)}.$$

□

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