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A RISK-PARAMETRIC DEFUZZIFICATION FRAMEWORK FOR SYMMETRIC FUZZY TRANSPORTATION PROBLEMS: FROM POSSIBILISTIC QUANTILES TO COHERENT TAIL RISK MEASURES

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Abstract

A class of symmetric Real-world transportation problems involves imprecise parameters arising from volatile fuel prices, labour costs and demand fluctuations. Traditional defuzzification strategies such as the Centroid of Gravity (COG) compute a crisp representative without accommodating the decision-maker's risk attitude toward cost uncertainty. This paper proposes a unified risk-parametric framework grounded in asymmetric loss minimization. By minimizing a pinball loss function with parameter $\lambda \in [0,1]$, we obtain a family of crisp equivalents that spans the spectrum from optimistic ($\lambda \to 0$) to pessimistic ($\lambda \to 1$) cost estimates. For smooth quadratic membership functions constructed via Newton's Forward Interpolation, the λ-quantile (possibilistic VaR) admits an elegant closed-form trigonometric solution. We then derive a closed-form Possibilistic Conditional Value at Risk (PCVaR) — to our knowledge, the first such explicit formula for parabolic fuzzy numbers —that averages the worst $(1-\lambda)$-fraction of the possibility distribution and prove that it satisfies fundamental consistency axioms—monotonicity, translation invariance, positive homogeneity and exact linearity for symmetric parabolic fuzzy numbers—making it suitable for scalable transportation planning. Numerical experiments on two fuzzy transportation problems (3×4 and 4×4) demonstrate that the parameter λ not only shifts total cost but can fundamentally alter the optimal allocation pattern when route cost uncertainties exhibit heterogeneous spreads, while PCVaR provides a conservative, coherent alternative to simple quantile thresholding.

Keywords: Fuzzy transportation problem, defuzzification, asymmetric loss, possibilistic quantile, conditional value at risk, coherent risk measure, Newton forward interpolation.

1 Introduction

The transportation problem is a cornerstone of linear programming, concerned with the optimal allocation of goods from m sources to n destinations at minimum total cost. In practice, however, unit transportation costs are rarely known with certainty. Fluctuating fuel prices, changing labour wages and seasonal demand variations introduce substantial imprecision that deterministic models cannot capture adequately.

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Fuzzy set theory, introduced by Zadeh [1] and Bellman and Zadeh [4], provides a mathematical apparatus for modelling such uncertainty. Triangular Fuzzy Numbers (TFNs) are widely used for their computational simplicity. Nevertheless, classical defuzzification strategies [6]—Centroid of Gravity (COG), mean of maxima and median methods—compute a crisp representative without explicitly encoding the decision-maker's attitude toward uncertainty. A budget-sensitive logistics manager may prefer a *pessimistic* (high) cost estimate to avoid budget overruns, whereas a risk-tolerant planner may accept an *optimistic* (low) estimate. Existing methods offer no systematic mechanism to control this trade-off.

To address this limitation, we develop a **parametric defuzzification paradigm** grounded in statistical decision theory. We model defuzzification as the minimization of an **asymmetric pinball loss function** [9]:

$$\mathcal{L}_\lambda(x, x_0) = (1-\lambda)(x_0 - x)_+ + \lambda(x - x_0)_+ \qquad \lambda \in [0,1] \qquad (1)$$

where $(\cdot)_+ = \max(\cdot, 0)$. The minimizer x_0^* defines a λ-dependent crisp equivalent. For $\lambda < 0.5$, overestimation is penalized more heavily, pulling the crisp value toward the lower bound (optimistic). For $\lambda > 0.5$, underestimation is penalized more, pushing the estimate toward the upper bound (pessimistic). At $\lambda = 0.5$, the loss is symmetric and the crisp value coincides with the modal value for symmetric fuzzy numbers.

While the λ-quantile (possibilistic VaR) provides a threshold guarantee, budget-sensitive planners are often concerned with the **average** cost in adverse scenarios. We therefore derive, for the first time in the fuzzy transportation literature, a **closed-form Possibilistic Conditional Value at Risk (PCVaR)** that averages the worst $(1-\lambda)$-fraction of the possibility distribution. PCVaR satisfies fundamental coherence axioms [2, 3] and exhibits exact **positive linearity** for symmetric parabolic fuzzy numbers, ensuring that scaling shipment volumes or aggregating routes preserves the risk measure proportionally—a critical property for multi-modal logistics planning.

To ensure analytical tractability, we employ a smooth quadratic membership function constructed through Newton's Forward Interpolation over the normalized interval $[0,1]$. This yields closed-form crisp equivalents via elementary calculus.

The principal contributions are:

1. A **parametric family of defuzzification operators** D_λ controlled by a single risk parameter $\lambda \in [0,1]$ with a clear statistical interpretation as a possibilistic quantile.

2. The **first closed-form PCVaR** for parabolic fuzzy numbers, together with a proof of consistency (monotonicity, translation invariance, positive homogeneity and linearity).

3. A **closed-form solution** for possibilistic VaR that is computationally inexpensive and recovers classical strategies as special cases.

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4. **Numerical evidence** that risk-parametric defuzzification can induce **allocation shifts** in the optimal transportation plan when route cost uncertainties exhibit heterogeneous spreads, a phenomenon absent from classical fixed-defuzzification approaches.

2 Preliminaries and Mathematical Foundations

2.1 Triangular Fuzzy Numbers

A Triangular Fuzzy Number (TFN) [8, 15] is denoted by $\tilde{A} = (a, m, b)$ with $a \leq m \leq b$ and membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{m-a}, & a \leq x \leq m \\ \dfrac{b-x}{b-m}, & m \leq x \leq b \\ 0, & \text{otherwise} \end{cases} \qquad (2)$$

While TFNs are computationally convenient, their piecewise-linear membership is non-differentiable at $x = m$. In the present context, non-differentiability prevents the formulation of a smooth optimization objective whose critical point can be located analytically.

2.2 Parabolic Fuzzy Numbers via Newton Interpolation

To obtain a globally differentiable membership curve, we consider a symmetric TFN with equal spread $d = m - a = b - m$ and introduce the normalized variable $u \in [0,1]$ through:

$$x = a + 2du \qquad (3)$$

At $u = 0$, $x = a$; at $u = 0.5$, $x = m = a + d$; and at $u = 1$, $x = b = a + 2d$.

Applying Newton's Forward Interpolation to the equally spaced knots $(0,0)$, $(0.5,1)$, and $(1,0)$ with step $h = 0.5$ yields the quadratic polynomial:

$$\mu(u) = 4u(1-u) = 4u - 4u^2, \qquad 0 \leq u \leq 1 \qquad (4)$$

In terms of the physical variable x:

$$\mu(x) = 1 - \left(\frac{x-m}{d}\right)^2, \qquad a \leq x \leq b \qquad (5)$$

We denote the pair $\tilde{A}_P = (a, m, b)_P$ together with equation (5) as a **Parabolic Fuzzy Number (PFN)**. The following properties are immediate:

Property 2.1 (Normality). The maximum membership grade is $\mu(m) = 1$.

Property 2.2 (Support). The support of $\tilde{A}_P$ is $[a, b]$, since $\mu(a) = \mu(b) = 0$.

Property 2.3 (Quasi-concavity). For any $\alpha \in [0,1]$, the α-cut is the closed interval $[m - d\sqrt{1-\alpha},\ m + d\sqrt{1-\alpha}]$.

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Remark 2.1. PFNs provide smooth symmetric approximations of symmetric TFNs. For asymmetric TFNs, piecewise quadratic or higher-order splines may be employed; this extension is reserved for future work.

3 Proposed Risk-Parametric Defuzzification Methodology

3.1 The Asymmetric Pinball Loss

Let $\tilde{A}_P$ be a PFN with membership $\mu(x)$ and support $[a, b]$. We define the **defuzzification operator** $D_\lambda(\tilde{A}_P)$ as the minimizer of the normalized asymmetric loss:

$$D_\lambda(\tilde{A}_P) = \arg\min_{x_0 \in [a,b]} \frac{\int_a^b \mathcal{L}_\lambda(x, x_0)\, \mu(x)\, dx}{\int_a^b \mu(x)\, dx} \tag{6}$$

Where $$\mathcal{L}_\lambda(x, x_0) = (1-\lambda)(x_0 - x)_+ + \lambda(x - x_0)_+ \tag{7}$$

and $(z)_+ = \max(z, 0)$. The parameter $\lambda \in [0,1]$ controls the relative penalty on overestimation versus underestimation:

- $\lambda < 0.5$: Optimistic stance (overestimation penalized more).
- $\lambda = 0.5$: Neutral stance (symmetric loss).
- $\lambda > 0.5$: Pessimistic stance (underestimation penalized more).

3.2 First-Order Optimality: The Quantile Condition

Theorem 3.1. For a continuous unimodal membership function $\mu(x)$ with support $[a, b]$, the defuzzification operator satisfies:

$$\int_a^{D_\lambda(\tilde{A}_P)} \mu(x)\, dx = \lambda \int_a^b \mu(x)\, dx \tag{8}$$

Proof. Let $J(x_0) = \int_a^b \mathcal{L}_\lambda(x, x_0)\, \mu(x)\, dx$. Expanding:

$$J(x_0) = (1-\lambda) \int_a^{x_0} (x_0 - x)\mu(x)\, dx + \lambda \int_{x_0}^b (x - x_0)\mu(x)\, dx$$

Differentiating under the integral sign (Leibniz rule):

$$J'(x_0) = (1-\lambda) \int_a^{x_0} \mu(x)\, dx - \lambda \int_{x_0}^b \mu(x)\, dx$$

Setting $J'(x_0) = 0$ and noting that $\int_a^b \mu(x)\, dx = \int_a^{x_0} \mu(x)\, dx + \int_{x_0}^b \mu(x)\, dx$, we obtain Equation (8). Since $J''(x_0) = \mu(x_0) > 0$ for $x_0 \in (a, b)$, the minimizer is unique.

Remark 3.2. D_λ is the λ**-quantile** of the normalized possibility distribution. It is the unique point that partitions the total possibility mass in the ratio λ: $(1-\lambda)$.

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3.3 Closed-Form VaR for Parabolic Fuzzy Numbers

For the quadratic membership (5) with $m = a + d$, we evaluate the cumulative possibility mass. Substituting $w = (x - m)/d$ (so $dx = d\,dw$ and $w \in [-1,1]$):

$$\int_a^b \mu(x)\,dx = d\int_{-1}^{1}(1 - w^2)\,dw = \frac{4d}{3} \tag{9}$$

For $x_0 = m + dw_0$ with $w_0 \in [-1,1]$:

$$\int_a^{x_0} \mu(x)\,dx = d\int_{-1}^{w_0}(1 - w^2)\,dw = d\left(w_0 - \frac{w_0^3}{3} + \frac{2}{3}\right) \tag{10}$$

Applying Theorem 3.1:

$$w_0 - \frac{w_0^3}{3} + \frac{2}{3} = \lambda \cdot \frac{4}{3} \tag{11}$$

which simplifies to the depressed cubic:

$$w_0^3 - 3w_0 + (4\lambda - 2) = 0 \tag{12}$$

For $\lambda \in [0,1]$, the discriminant $\Delta = 27[4 - (4\lambda - 2)^2] \geq 0$, yielding three real roots. The unique root in $[-1,1]$ corresponds to the branch $k = 1$ of the trigonometric solution:

$$w_\lambda = 2\cos\left(\frac{1}{3}\arccos(1 - 2\lambda) - \frac{2\pi}{3}\right) \tag{13}$$

Consequently, the crisp equivalent (possibilistic VaR) is:

$$x_{\mathrm{VaR}}(\lambda) = m + d\,w_\lambda = m + 2d\cos\left(\frac{1}{3}\arccos(1 - 2\lambda) - \frac{2\pi}{3}\right) \tag{14}$$

Corollary 3.3 (Boundary Cases).

- $\lambda = 0.5$: $\arccos(0) = \pi/2$, giving $w_{0.5} = 0$ and $x_{\mathrm{VaR}} = m$ (the modal value).
- $\lambda \to 0$: $w_\lambda \to -1$ and $x_{\mathrm{VaR}} \to a$ (lower bound).
- $\lambda \to 1$: $w_\lambda \to +1$ and $x_{\mathrm{VaR}} \to b$ (upper bound).

Remark 3.4. For symmetric PFNs, the neutral case $\lambda = 0.5$ coincides with both the mode and the COG, demonstrating that our method **generalizes** classical strategies rather than replacing them arbitrarily.

3.4 Possibilistic Conditional Value at Risk (PCVaR)

While the λ-quantile (VaR) provides a threshold guarantee, budget-sensitive planners are often concerned with the **average** cost in adverse scenarios. We define the **Possibilistic Conditional Value at Risk** as the average of the worst $(1 - \lambda)$-fraction of the possibility distribution:

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$$\mathrm{PCVaR}_\lambda(\tilde{A}_P) = \frac{1}{1-\lambda}\int_\lambda^1 x_{\mathrm{VaR}}(\alpha)\, d\alpha, \qquad \lambda \in [0,1) \tag{15}$$

Theorem 3.5 (Closed-Form PCVaR). For a PFN $\tilde{A}_P = (a, m, b)_P$ with spread d:

$$\mathrm{PCVaR}_\lambda = m + \frac{3d}{4}\cdot\frac{(1+w_\lambda)^2}{2+w_\lambda} \tag{16}$$

where w_λ is the normalized quantile from Equation (13).

Proof. Substituting $x_{\mathrm{VaR}}(\alpha) = m + d\, w_\alpha$ into (15):

$$\mathrm{PCVaR}_\lambda = m + \frac{d}{1-\lambda}\int_\lambda^1 w_\alpha\, d\alpha$$

From the cubic (12), we have $4\alpha - 2 = 3w_\alpha - w_\alpha^3$, yielding $d\alpha = \frac{3}{4}(1-w_\alpha^2)\, dw_\alpha$. Changing variables from α to w with limits w_λ (at $\alpha = \lambda$) and 1 (at $\alpha = 1$):

$$\int_\lambda^1 w_\alpha\, d\alpha = \int_{w_\lambda}^1 w\cdot\frac{3}{4}(1-w^2)\, dw = \frac{3}{4}\left[\frac{w^2}{2} - \frac{w^4}{4}\right]_{w_\lambda}^1 = \frac{3}{16}(1-w_\lambda^2)^2$$

Moreover, from (12) with the factorization $w^3 - 3w + 2 = (w-1)^2(w+2)$:

$$1-\lambda = \frac{w_\lambda^3 - 3w_\lambda + 2}{4} = \frac{(w_\lambda - 1)^2(w_\lambda + 2)}{4}$$

Therefore:

$$\frac{(1-w_\lambda^2)^2}{1-\lambda} = \frac{(1-w_\lambda)^2(1+w_\lambda)^2}{(w_\lambda-1)^2(w_\lambda+2)/4} = \frac{4(1+w_\lambda)^2}{w_\lambda+2}$$

Substituting back:

$$\frac{1}{1-\lambda}\int_\lambda^1 w_\alpha\, d\alpha = \frac{3}{16}\cdot\frac{4(1+w_\lambda)^2}{w_\lambda+2} = \frac{3}{4}\cdot\frac{(1+w_\lambda)^2}{2+w_\lambda}$$

which yields (16).

Corollary 3.6 (Boundary Cases).

- $\lambda = 0$: $w_0 = -1$, giving $\mathrm{PCVaR}_0 = m$ (the **possibilistic mean**, coinciding with COG for symmetric PFNs).
- $\lambda \to 1$: $w_\lambda \to 1$, giving $\mathrm{PCVaR}_1 \to b$ (the upper bound).
- $\lambda = 0.5$: $w_{0.5} = 0$, giving $\mathrm{PCVaR}_{0.5} = m + \frac{3d}{8}$ (the average of the upper 50% possibility mass).

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3.5 Axiomatic Foundation and Linearity

Theorem 3.7 (Monotonicity). If $\tilde{A} \leq \tilde{B}$ in the fuzzy number ordering, then $\mathrm{VaR}_\lambda(\tilde{A}) \leq \mathrm{VaR}_\lambda(\tilde{B})$ and $\mathrm{PCVaR}_\lambda(\tilde{A}) \leq \mathrm{PCVaR}_\lambda(\tilde{B})$.

Theorem 3.8 (Translation Invariance)$\mathrm{VaR}_\lambda(\tilde{A} \oplus c = \mathrm{VaR}_\lambda(\tilde{A}) + c$ and $\mathrm{PCVaR}_\lambda\ (\tilde{A} \oplus c) = \mathrm{PCVaR}_\lambda(\tilde{A}) + c$ for any crisp constant c.

Theorem 3.9 (Positive Homogeneity). $\mathrm{VaR}_\lambda(k \odot \tilde{A}) = k \cdot \mathrm{VaR}_\lambda(\tilde{A})$ and $\mathrm{PCVaR}_\lambda(k \odot \tilde{A}) = k \cdot \mathrm{PCVaR}_\lambda(\tilde{A})$ for any $k > 0$.

Theorem 3.10 (Linearity for Symmetric PFNs). For two independent symmetric PFNs $\tilde{A}_P$ and $\tilde{B}_P$, the PCVaR operator is exactly additive:

$$\mathrm{PCVaR}_\lambda(\tilde{A}_P \oplus \tilde{B}_P) = \mathrm{PCVaR}_\lambda(\tilde{A}_P) + \mathrm{PCVaR}_\lambda(\tilde{B}_P)$$

Proof. For symmetric PFNs, the sum $\tilde{A}_P \oplus \tilde{B}_P$ has mode $m_A + m_B$ and spread $d_A + d_B$. From (16):

$$\mathrm{PCVaR}_\lambda(\tilde{A}_P \oplus \tilde{B}_P) = (m_A + m_B) + \frac{3(d_A + d_B)}{4} \cdot K_\lambda = \left[m_A + \frac{3d_A}{4} K_\lambda\right] + \left[m_B + \frac{3d_B}{4} K_\lambda\right]$$

where $K_\lambda = (1 + w_\lambda)^2/(2 + w_\lambda)$ depends only on λ. This equals $\mathrm{PCVaR}_\lambda(\tilde{A}_P) + \mathrm{PCVaR}_\lambda(\tilde{B}_P)$.

Remark 3.11. Exact linearity ensures that scaling shipment volumes or aggregating identical route types preserves the risk measure proportionally—a desirable property for logistics networks where contract volumes are frequently adjusted. For general (asymmetric or non-parabolic) fuzzy numbers, the inequality $\mathrm{PCVaR}(\tilde{A} \oplus \tilde{B}) \leq \mathrm{PCVaR}(\tilde{A}) + \mathrm{PCVaR}(\tilde{B})$ recovers the standard subadditivity of coherent risk measures [3, 14].

Theorem 3.12 (Continuity in Risk Parameter). $\mathrm{VaR}_\lambda(\tilde{A})$ and $\mathrm{PCVaR}_\lambda(\tilde{A})$ are continuous in λ.

4 Numerical Experiments on Fuzzy Transportation Problems

4.1 Problem 1: 3 × 4 Balanced Transportation

Consider a balanced fuzzy transportation problem with three sources S_1, S_2, S_3 and four destinations $D_1, \ldots, D_4$. The fuzzy costs (symmetric TFNs approximated as PFNs), supplies, and demands are given in Table 1.

Table 1: Original fuzzy transportation problem (3 × 4)

	D_1	D_2	D_3	D_4	Supply
S_1	(4,7,10)	(8,10,12)	(10,14,18)	(6,8,10)	30
S_2	(4,7,10)	(8,12,16)	(8,12,16)	(5,6,7)	40
S_3	(1,5,9)	(6,9,12)	(10,15,20)	(6,9,12)	50

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	D_1	D_2	D_3	D_4	Supply
Demand	20	30	40	30	120

4.2 Parametric Defuzzification

Using $d = (b - a)/2$ and Equations (14)–(16), crisp equivalents are computed for several risk attitudes. Table 2 reports the defuzzified costs.

Table 2: Defuzzified costs for varying risk parameter λ

Cell	$\text{VaR}_{0.3}$	$\text{VaR}_{0.5}$	$\text{VaR}_{0.7}$	$\text{PCVaR}_{0.5}$	$\text{PCVaR}_{0.7}$
S_1-D_1	6.18	7.00	7.82	8.12	8.61
S_1-D_2	9.45	10.00	10.55	10.75	11.07
S_1-D_3	12.91	14.00	15.09	15.50	16.14
S_1-D_4	7.45	8.00	8.55	8.75	9.07
S_2-D_1	6.18	7.00	7.82	8.12	8.61
S_2-D_2	10.91	12.00	13.09	13.50	14.14
S_2-D_3	10.91	12.00	13.09	13.50	14.14
S_2-D_4	5.73	6.00	6.27	6.38	6.54
S_3-D_1	3.91	5.00	6.09	6.50	7.14
S_3-D_2	8.18	9.00	9.82	10.12	10.61
S_3-D_3	13.63	15.00	16.37	16.88	17.68
S_3-D_4	8.18	9.00	9.82	10.12	10.61

Note: For $\lambda = 0.5$, VaR equals the modal value because all fuzzy costs are symmetric. $\text{PCVaR}_{0.5}$ averages the upper 50% tail, yielding a naturally more conservative estimate.

4.3 Optimal Allocations and Cost–Robustness Trade-off

Each defuzzified tableau was solved to optimality using Vogel's Approximation Method (VAM) followed by the Modified Distribution (MODI) optimality test. Tables 3–5 record the optimal allocations and total costs under three risk configurations.

Table 3: Optimal allocation under $\text{VaR}_{0.5}$ (Classical Modal / COG)

Source	Destination	Units	Unit Cost	Total Cost
S_1	D_4	30	8.00	240.00
S_2	D_3	40	12.00	480.00

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Source	Destination	Units	Unit Cost	Total Cost
S_3	D_1	20	5.00	100.00
S_3	D_2	30	9.00	270.00
Grand Total				**1090.00**

Table 4: Optimal allocation under $\text{VaR}_{0.7}$ (Pessimistic Threshold)

Source	Destination	Units	Unit Cost	Total Cost
S_1	D_3	30	15.09	452.70
S_2	D_3	10	13.09	130.90
S_2	D_4	30	6.27	188.10
S_3	D_1	20	6.09	121.80
S_3	D_2	30	9.82	294.60
Grand Total				**1188.10**

Table 5: Optimal allocation under $\text{PCVaR}_{0.7}$ (Coherent Tail Risk)

Source	Destination	Units	Unit Cost	Total Cost
S_1	D_3	30	16.14	484.20
S_2	D_3	10	14.14	141.40
S_2	D_4	30	6.54	196.20
S_3	D_1	20	7.14	142.80
S_3	D_2	30	10.61	318.30
Grand Total				**1282.90**

Key Observations:

- **VaR** provides a threshold guarantee: the actual fuzzy cost will not exceed the crisp estimate with possibility mass at least λ.
- **PCVaR** provides an average tail guarantee: it represents the expected cost in the worst $(1-\lambda)$-fraction of scenarios, making it suitable for budget planning where occasional overruns are acceptable but average overruns must be controlled.

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- Comparing Table 3 with Tables 4–5 reveals an **allocation shift**: under the neutral $\mathrm{VaR}_{0.5}$, source S_1 serves D_4 exclusively; under $\mathrm{VaR}_{0.7}$ and $\mathrm{PCVaR}_{0.7}$, S_1 shifts to D_3 because the spread penalty on S_2-D_3 makes it less competitive relative to S_2-D_4, freeing D_4 capacity for S_2. This demonstrates that risk attitude can fundamentally restructure the logistics network, not merely inflate costs.

4.4 Problem 2: 4×4 Benchmark

To ensure that the observed behavior generalizes, a second balanced instance with four sources and four destinations was constructed. Table 6 summarizes the data and optimal total costs under $\mathrm{VaR}_{0.5}$ and $\mathrm{PCVaR}_{0.7}$.

Table 6: Second benchmark problem (4×4) and results

	D_1	D_2	D_3	D_4	Supply
S_1	(6,8,10)	(10,12,14)	(8,10,12)	(12,14,16)	25
S_2	(4,6,8)	(6,8,10)	(10,12,14)	(8,10,12)	35
S_3	(7,9,11)	(9,11,13)	(5,7,9)	(11,13,15)	30
S_4	(3,5,7)	(7,9,11)	(9,11,13)	(6,8,10)	40
Demand	20	25	35	50	130

Table 7: Benchmark comparison across problem sizes

Problem	Size	Method	Total Cost
Problem 1	3×4	$\mathrm{VaR}_{0.5}$ (Modal/COG)	1090.00
Problem 1	3×4	$\mathrm{VaR}_{0.7}$	1188.10
Problem 1	3×4	$\mathrm{PCVaR}_{0.7}$	1282.90
Problem 2	4×4	$\mathrm{VaR}_{0.5}$ (Modal/COG)	1040.00
Problem 2	4×4	$\mathrm{PCVaR}_{0.7}$	1179.10

Both instances confirm that PCVaR yields a conservative uplift (17.7% for Problem 1, 13.4% for Problem 2) relative to the modal benchmark, with the magnitude depending on the heterogeneity of cost spreads across the network.

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5 Comparative Analysis and Discussion

5.1 Comparison with Classical Methods

Table 8: Comparative analysis for the fuzzy cost (4,7,10)

Method	Crisp Value	Interpretation
Mode / COG	7.00	Most likely value; no risk adjustment
$\mathbf{VaR}_{0.5}$	**7.00**	**Recovers mode/COG as special case**
$\mathrm{VaR}_{0.3}$	6.18	Optimistic threshold; 30% tail risk accepted
$\mathrm{VaR}_{0.7}$	7.82	Pessimistic threshold; 70% mass covered
$\mathrm{PCVaR}_{0.5}$	8.12	Average of upper 50% tail; coherent risk
$\mathrm{PCVaR}_{0.7}$	8.61	Average of upper 30% tail; highly robust

Unlike ad-hoc utility-based approaches, our method does not claim artificial "cost reduction." Instead, it offers a **controlled spectrum**: the logistics manager who must guarantee a budget selects $\mathrm{PCVaR}_{0.8}$ or $\mathrm{PCVaR}_{0.9}$; the competitive bidder who can tolerate occasional overruns selects $\mathrm{VaR}_{0.2}$ or $\mathrm{VaR}_{0.3}$.

5.2 Allocation Shifts Under Risk Attitude

A distinctive feature of the proposed framework is that defuzzification is not merely a pre-processing step—it can **restructure the optimal transportation plan**. In Problem 1, the transition from $\mathrm{VaR}_{0.5}$ to $\mathrm{VaR}_{0.7}$ shifts S_1 from D_4 to D_3 and splits S_2 between D_3 and D_4. This occurs because the risk premium penalizes routes with larger spreads (S_2-D_3 has spread $d = 4$ versus S_2-D_4 with $d = 1$), altering their relative competitiveness even when their modal costs remain unchanged. Classical fixed defuzzification (COG or mode) cannot capture this effect.

5.3 Sensitivity to Risk Preference

Table 9 reports the VaR and PCVaR for $(4,7,10)$ across the full λ spectrum.

Table 9: Sensitivity of crisp value to λ for $(4,7,10)$

λ	Attitude	VaR	PCVaR	μ(VaR)
0.10	Highly Optimistic	5.17	7.25	0.630
0.25	Optimistic	5.96	7.58	0.879
0.50	Neutral	7.00	8.12	1.000
0.75	Pessimistic	8.04	8.74	0.879
0.90	Highly Pessimistic	8.83	9.23	0.630

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The membership grade $\mu(\text{VaR})$ peaks at $\lambda = 0.5$ and decreases toward the bounds, quantifying the **feasibility sacrifice** incurred for extreme risk attitudes. PCVaR always lies above VaR for $\lambda > 0$, reflecting the additional conservatism of averaging the tail.

5.4 Scalability and Linearity

Theorem 3.10 guarantees that PCVaR scales linearly with shipment size for symmetric PFNs. Table 10 verifies this numerically for the fuzzy cost $(4,7,10)$ at $\lambda = 0.7$.

Table 10: PCVaR scalability under volume scaling

Scale	Fuzzy Cost	$\text{PCVaR}_{0.7}$
0.5	$(2.0, 3.5, 5.0)$	4.3025
1.0	$(4.0, 7.0, 10.0)$	8.6050
2.0	$(8.0, 14.0, 20.0)$	17.2100

The perfect doubling confirms exact positive linearity. For a logistics provider, this means that doubling a contract lane's volume simply doubles the coherent risk estimate—no hidden nonlinearities arise.

6 Conclusion

This paper has presented a **unified risk-parametric defuzzification framework** for fuzzy transportation problems. The principal advances are:

1. **Statistical clarity:** λ is a quantile level with direct possibilistic interpretation, not an abstract utility exponent.

2. **Coherent tail risk:** The first closed-form PCVaR for parabolic fuzzy numbers, satisfying monotonicity, translation invariance, positive homogeneity and exact linearity for symmetric fuzzy costs.

3. **Closed-form tractability:** Both VaR and PCVaR admit explicit formulas involving a single trigonometric root evaluation.

4. **Allocation awareness:** Risk attitude can restructure the optimal transportation plan when route uncertainties exhibit heterogeneous spreads—a phenomenon beyond the reach of classical fixed-defuzzification methods.

Future work may extend the framework to **asymmetric TFNs** via piecewise quadratic interpolation, embed PCVaR directly into the transportation LP via its dual representation [13, 14] and investigate **data-driven calibration** of λ from historical cost realizations.

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