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A DENSE SUBSET OF THE OPERATOR DOMAIN WITH A POINT INTERACTION

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Abstract: In this paper, we investigate the domain of the operator with a point interaction, specifically the operator $-\frac{d^2}{dx^2} + Z\delta_0$. We demonstrate the existence of a dense subset within this domain consisting of functions that possess compact support and are smooth everywhere except, possibly, at $x = 0$. Our findings shed light on the structural properties of the operator domain, providing valuable insights into its mathematical characterization and potential applications.

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1. Introduction

Point interactions appear in models used in quantum mechanics to represent physical systems where the wavelength of the particle is greater than the range of the potential. An example of this is the work of Kronig and Penney in 1931, where they studied, using a one-dimensional model, an issue involving the reflection of electrons of a given velocity falling from a vacuum onto a lattice, (see [5]). Furthermore, these operators are used to model the interaction of a one-dimensional Bose condensate with an impurity (see [4]).

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In [7], they study existence and stability properties of ground-state standing waves for a two-dimensional nonlinear Schrödinger equation with a point interaction and a focusing power nonlinearity. In [8], they consider one-dimensional Schrödinger operators that involve a finite number of point interactions. The eigenenergies related to the bound states and the complex energies related to the resonance states of the operator are given by the zeros of certain characteristic functions that share the same structure up to an algebraic sign. These functions are obtained explicitly in the form of power series of the spectral parameter, and the computation of the coefficients of the series is given by a recursive integration procedure.

In [9], they construct the norm resolvent approximation for a wide class of point interactions in one dimension.

To better understand the technique used to solve this problem, this paper is divided into two sections. In the second section, we introduce the operator with one-point interaction. In section three, the existence of a dense subspace within the domain of the operator with one-point interaction is demonstrated.

2. Operator with one-point interaction

Formally, the operator with one-point interaction is written as:

$$-\Delta_Z = -\frac{d^2}{dx^2} + Z\delta_0, \tag{1}$$

where $Z \in \mathbb{R}$ and δ_0 is the Dirac's distribution centered at the origin $x = 0$. Berezin provided a rigorous mathematical definition of this in three dimensions. Faddeev [2] used the theory of self-adjoint extensions of symmetric operators (see [3]), but in the one-dimensional case, it was extensively studied in [1], using that same theory and explicitly describing its domain and its spectrum. Another perspective to view this operator is as a Laplacian perturbed by a potential supported on a single point set $\{0\}$.

To provide a mathematical description of (1) we consider the Laplacian operator $-\Delta$ in $L^2(\mathbb{R})$,

$$\begin{cases} -\Delta = -\frac{d^2}{dx^2} \\ D(-\Delta) = H^2(\mathbb{R}), \end{cases}$$

and take the restriction operator A of $-\Delta$ to space $D(A) = \{f \in H^2(\mathbb{R}) : f(0) = 0\}$, which is closed, densely defined, and symmetric. In [1], it was shown that the adjoint of the operator A is given by

$$\begin{cases} A^* = -\frac{d^2}{dx^2} \\ D(A^*) = H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}). \end{cases}$$

The deficiency indices of A are equal to one because the equation $A^*\phi = k^2\phi$, where $\phi \in D(A^*)$, $k^2 \in \mathbb{C} \setminus \mathbb{R}$ and $Im(k) > 0$, has one unique solution $\phi_k(x) = e^{ik|x|}$. Then all self-adjoint extensions A_θ of A are given by the one-parameter family

$$\begin{cases} A_\theta(g + c\psi_{+i} + ce^{i\theta}\psi_{-i}) &= Ag + ic\psi_{+i} - ice^{i\theta}\psi_{-i}, \\ D(A_\theta) = \{g + c\psi_{+i} + ce^{i\theta}\psi_{-i} : & g \in D(A),\ c \in \mathbb{C}\}, \end{cases} \tag{2}$$

where $\theta \in [0, 2\pi)$ is such that $\lambda = e^{i\theta}$ and $\psi_{\pm i}(x) = \frac{i}{2\sqrt{\pm i}} e^{i\sqrt{\pm i}|x|}$ $\mathrm{Im}\sqrt{\pm i} > 0$ (see [1]). If we choose

$$Z = Z(\theta) = \frac{-2\cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)}, \quad \text{for } \theta \in [0, 2\pi) \setminus \{\pi/2\} \text{ and setting}$$
$$Z((\pi/2)^-) = +\infty, \tag{3}$$

in (2) to obtain the following.

Theorem 1. *All self-adjoint extensions of A are given by*

$$\begin{cases} -\Delta_Z &= -\frac{d^2}{dx^2}, \quad x \neq 0, \\ D(-\Delta_Z) &= \{g \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : g'(0^+) - g'(0^-) \\ &= 2Zg(0)\}, \end{cases}$$

where $-\infty < Z \leq \infty$. If $Z = 0$, the self-adjoint extension $-\Delta_0$ matches the Laplacian operator in space $L^2(\mathbb{R})$,

$$\begin{cases} -\Delta &= -\frac{d^2}{dx^2}, \\ D(-\Delta) &= H^2(\mathbb{R}). \end{cases}$$

In the case $Z = \infty$, the real line is divided into two intervals $(-\infty, 0)$ and $(0, \infty)$. It happens due to the appearance of the boundary condition of the Dirichlet type at point 0, that is,

$$\begin{cases} -\Delta_\infty &= (-\Delta_{D_-}) \oplus (-\Delta_{D_+}), \\ D(-\Delta_\infty) &= \{g \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : g(0) = 0\} \\ &= H_0^2((-\infty, 0)) \cap H_0^2((0, \infty)), \end{cases}$$

where $(-\Delta_{D_\pm})$ *denotes the Dirichlet Laplacian over* $(-\infty, 0)$ *and* $(0, \infty)$*, respectively (see [6]), with* $D(-\Delta_{D_-}) = H_0^2((-\infty, 0))$ *and* $D(-\Delta_{D_+}) = H_0^2((0, \infty))$*. So, for every* Z *we have*

$$-\Delta_Z f(x) = -\frac{d^2 f(x)}{dx^2}, \quad x \neq 0.$$

3. Proof of the existence of the subspace dense into the operator domain with one-point interaction

Now, we consider the subspace $\mathbb{Y}$ of $D(-\Delta_Z)$, defined by

$$\mathbb{Y} = \{g \in D(-\Delta_Z) : \quad g \in C_0(\mathbb{R}) \cup C_0^\infty(\mathbb{R} \setminus \{0\})\}.$$

In the present paper, we prove that $\mathbb{Y}$ is dense in $D(-\Delta_Z)$, and besides also the following.

Theorem 2. *Given a function* $\vartheta \in W^{m,p}(\mathbb{R})$*, with* $\vartheta(0) = 0$*, there exists a sequence of functions* $(\psi_n)_{n\geq 1}$ *in the space of smooth and compact support functions,* $C_0^\infty(\mathbb{R})$*, satisfying* $\psi_n(0) = 0$ *for all* $n \in \mathbb{N}$ *and converging to* ϑ *in* $W^{m,p}(\mathbb{R})$ *norm.*

Proof. Suppose that $\vartheta \in W^{m,p}(\mathbb{R})$ and we consider for every $n \in \mathbb{Z}^+$, the function $\varphi_n \in C_0^\infty(\mathbb{R})$ such that $0 \leq \varphi_n(x) \leq 1$ for all $x \in \mathbb{R}$. Furthermore,

$$\varphi_n(x) = \begin{cases} 1, & \text{if } \frac{2}{n^2} \leq |x| \leq 2 \\ 0, & \text{if } |x| < \frac{1}{n^2} \quad \text{or} \quad |x| > 3. \end{cases}$$

We define the functions $\xi_n(x) = \varphi_n(\frac{x}{n})$, for every $n \in \mathbb{Z}^+$. Thus, $(\xi_n)_{n\geq 1}$ is a sequence of test functions satisfying:

a) $\xi_n = 1$ on $\{x \in \mathbb{R} : \frac{2}{n} \leq |x| \leq 2n\}$, $\text{supp}(\xi_n) \subset \{x \in \mathbb{R} : \frac{1}{n} \leq |x| \leq 3n\}$ and $0 \leq \xi_n(x) \leq 1$,

b) For every $r \in \mathbb{N}$, we have $D^r \xi_n(x) = \frac{1}{n^r} D^r \varphi_n(\frac{x}{n})$.

It is easy using the Dominated Convergence Theorem and the considerations (a) and (b) above that the sequence $(\xi_n \vartheta)_{n\geq 1}$ converge to ϑ in the $W^{m,p}(\mathbb{R})-$norm.

We can observe that every function $\vartheta_n = \xi_n \vartheta$ has compact support contained in set $\mathbb{R} \setminus (-\frac{1}{n}, \frac{1}{n})$.

We now take a sequence $(\rho_k)_{k\geq 1}$ of regularizing functions on $\mathbb{R}$ such that

$$supp(\rho_k) \subset (-\frac{1}{k}, \frac{1}{k}).$$

It follows that for every $n \in \mathbb{N}$, the terms of the sequence $(\rho_k * \vartheta_n)_{k\geq 1}$ are test functions on $\mathbb{R}$. Furthermore, for every $r \leq m$, we have

$$D^r(\rho_k * \vartheta_n) = \rho_k * D^r \vartheta_n,$$

and since $\rho_k * v \to v$ in $L^p(\mathbb{R})$ for all $v \in L^p(\mathbb{R})$ we obtain

$$D^r(\rho_k * \vartheta_n) = \rho_k * D^r \vartheta_n \to D^r \vartheta_n$$

in $L^p(\mathbb{R})$, as $k \longrightarrow \infty$. Therefore

$$\rho_k * \vartheta_n \to \vartheta_n$$

in $W^{m,p}(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$, as $k \longrightarrow \infty$. Now, sub-sequence $(\psi_n = \rho_n * \vartheta_n)_{n\geq 1}$ satisfies $\psi_n \to \vartheta$ in $W^{m,p}(\mathbb{R})$-norm and

$$\psi_n(0) = \int_{\mathbb{R}} \rho_n(y)\vartheta_n(-y)\ dy = \int_{supp(\rho_n)} \rho_n(y)\vartheta_n(-y)\ dy = 0,$$

because $\vartheta_n \equiv 0$ on the support of ρ_n. □

Theorem 3. *Given a function $\vartheta \in D(-\Delta_Z)$, there exists a sequence of functions $(\vartheta_n)_{n\geq 1}$ in $D(-\Delta_Z)$ of differentiable infinitely for all $x \in \mathbb{R}$, except possibly at $x = 0$, with compact support such that*

$$\|\vartheta - \vartheta_n\|_{H^2(\mathbb{R}\setminus\{0\})} \to 0,$$

as $n \to \infty$.

Proof. We consider $\vartheta \in D(-\Delta_Z)$. Then, there are a function $g \in H^2(\mathbb{R})$, a constant $C \in \mathbb{C}$, and a real number $\theta \in [0, 2\pi)$, such that, $g(0) = 0$ and

$$\vartheta = g + C\psi_i + Ce^{i\theta}\psi_{-i},$$

where $\psi_{\pm i}(x) = \frac{i}{2\sqrt{\pm i}} e^{i\sqrt{\pm i}|x|}$, with $Im\sqrt{\pm i} > 0$, a relationship between Z and θ is given by the relation (3). By Proposition 2, there is a sequence $(\kappa_n)_{n\geq 1}$ in space $C_0^\infty(\mathbb{R})$ satisfying $\kappa_n(0) = 0$ and

$$\|\kappa - \kappa_n\|_{H^2} \longrightarrow \infty, \quad \text{when} \quad n \longrightarrow \infty.$$

On the other hand, for each $n \in \mathbb{Z}^+$, we consider function $\tau_n(x) = \varrho(\frac{x}{n})$ where $\varrho \in C_0^\infty(\mathbb{R})$ has an image contained in $[0,1]$, is the same 1 on set $\{x \in \mathbb{R} : |x| \leq 1\}$ and 0 on $\{x \in \mathbb{R} : |x| \geq 2\}$. Then, a sequence of functions $(w_n)_{n\geq 1}$, where $w_n = (c\psi_i + ce^{i\theta}\psi_{-i})\tau_n$ is continuous, has compact support in $\mathbb{R}$, and is of class C^∞ for all $x \neq 0$ and its derivative satisfies the same condition of jump at $x = 0$ as the function $c\psi_i + ce^{i\theta}\psi_{-i}$. Since the sequences $(w_n)_{n\geq 1}$, $(\partial_x w_n)_{n\geq 1}$ and $(-\Delta_Z w_n)_{n\geq 1}$ converge the functions $c\psi_i + ce^{i\theta}\psi_{-i}$, $\partial_x(c\psi_i + ce^{i\theta}\psi_{-i})$ and $-\Delta_Z(c\psi_i + ce^{i\theta}\psi_{-i})$, respectively, the Dominated Convergence Theorem shows that

$$\|c\psi_i + ce^{i\theta}\psi_{-i} - w_n\|_{H^2(\mathbb{R}\backslash\{0\})} \longrightarrow 0, \quad \text{when} \quad n \longrightarrow \infty.$$

Now, for each $n \in \mathbb{Z}^+$, we define $\vartheta_n \equiv \kappa_n + w_n = \kappa_n + (c\psi_i + ce^{i\theta}\psi_{-i})\tau_n$. We observe that every function ϑ_n has compact support, is infinitely differentiable for all $x \in \mathbb{R}$ except at $x \neq 0$ and $(\vartheta_n)_{n\geq 1} \subset D(-\Delta_Z)$. Furthermore,

$$\|\vartheta - \vartheta_n\|_{H^2(\mathbb{R}\backslash\{0\})} \leq \|\kappa - \kappa_n\|_{H^2} + \|c\psi_i + ce^{i\theta}\psi_{-i} - w_n\|_{H^2(\mathbb{R}\backslash\{0\})} \to 0,$$

as $n \to \infty$. It shows the proposition. □

Corollary 4. *Let $a, b \in \mathbb{R}$ and $u \in C([a,b] : D(-\Delta_Z)) \cap C^1([a,b] : L^2(\mathbb{R}))$. Then for $n > 0$, there exists a sequence of functions $u_n \in C_0^\infty([a,b] : \mathbf{Y})$, such that*

$$\|u(t) - u_n(t)\|_{H^2(\mathbb{R}\backslash\{0\})} \to 0,$$

$$\int_a^b \|u(t) - u_n(t)\|_{H^2(\mathbb{R}\backslash\{0\})}\, dt \to 0,$$

$$\int_a^b \|\partial_t u(t) - \partial_t u_n(t)\|_{L^2}\, dt \to 0,$$

as $n \to \infty$.

Proof. By uniform continuity of function

$$u \in C([A,B] : D(-\Delta_Z)) \cap C^1([A,B] : L^2(\mathbb{R})),$$

for each $n \in \mathbb{Z}^+$, exists $\delta > 0$ such that

$$\|u(t) - u(s)\|_{H^2(\mathbb{R}\backslash\{0\})} < \frac{1}{n^2}, \quad \text{whenever} \quad |t - s| < \delta.$$

Now, choose $m_n \in \mathbb{Z}^+$ such that $m_n > n$, $m_n < m_{n+1}$ and $(B - A)/m_n < \delta$ and we consider the partition of interval $[A, B]$ with points:

$$t_j = A + (j-1)\frac{B-A}{m_n}, \quad j = 1, \ldots, m_n + 1. \tag{4}$$

On the other hand, as $\mathbb{Y}$ is dense in $D(-\Delta_Z)$, exist functions $\omega_{m_n}(t)$ defined on $[A, B]$ with the image in $\mathbb{Y}$ satisfying

$$\|\omega_{m_n}(t) - u(t)\|_{H^2(\mathbb{R}\setminus\{0\})} < \frac{1}{m_n^2}.$$

Now, we defined the functions

$$\Theta_{m_n}(x,t) = \sum_{j=1}^{m_n+1} \gamma_{j-1}(x,t)\chi_{[t_{j-1},t_j)}(t),$$

where the t_j's are as in (4) and

$$\gamma_{j-1}(x,t) = \frac{\omega_{m_n}(x,t_j) - \omega_{m_n}(x,t_{j-1})}{t_j - t_{j-1}}(t - t_{j-1}) + \omega_{m_n}(x,t_{j-1}).$$

We have constructed the sequence of functions $(\Theta_{m_n})_{n\geq 1}$, $\Theta_{m_n}(t) \in \mathbb{Y}$, verifying the following:

(i) For each $t \in [A, B]$, $\|u(t) - \Theta_{m_n}(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \to 0$, as $n \to \infty$. In fact, for $t \in [t_{j-1}, t_j)$

$$\begin{aligned}
&\|u(t) - \Theta_{m_n}(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \\
&= \|\frac{\omega_{m_n}(t_j) - \omega_{m_n}(t_{j-1})}{t_j - t_{j-1}}(t - t_{j-1}) \\
&+ \omega_{m_n}(t_{j-1}) - u(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \\
&\leq \|\omega_{m_n}(t_j) - u(t)\|_{H^2(\mathbb{R}\setminus\{0\})} + 2\|\omega_{m_n}(t_{j-1}) - u(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \\
&\leq \|\omega_{m_n}(t_j) - u(t_j)\|_{H^2(\mathbb{R}\setminus\{0\})} + \|u(t_j) - u(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \\
&+ 2\|\omega_{m_n}(t_{j-1}) - u(t_{j-1})\|_{H^2(\mathbb{R}\setminus\{0\})} \\
&+ 2\|u(t_{j-1}) - u(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \\
&< \frac{6}{n^2}.
\end{aligned}$$

(ii) The dominated convergence theorem implies,

$$\int_A^B \|u(t) - \Theta_{m_n}(t)\|_{H^2(\mathbb{R}\setminus\{0\})}\, dt \to 0$$

when $n \to \infty$.

Now, we consider an extension of function Γ_n defined on $\mathbb{R} \times \mathbb{R}$, by

$$\Gamma_{m_n}(x,t) = \begin{cases} \Theta_{m_n}(x,t), & \text{if} \quad (x,t) \in \mathbb{R} \times [A,B], \\ 0, & \text{in another case.} \end{cases}$$

Here, we continue with the regularization of each of the functions Γ_{m_n} with respect to the temporal variable t, choosing a test function $\tau \in C_0^\infty(\mathbb{R})$, which has the property $\int \tau(t)dt = 1$, and defined $\tau_r(t) = r\tau(rt)$. Then, by Proposition 3, the sequence $(\Gamma_{m_n} * \tau_r)_{r\geq 1}$ content in the space of the functions $C_0^\infty(\mathbb{R} : \mathbf{Y})$ that satisfy

(iii) For each $t \in [A,B]$, we have $\|\Gamma_{m_n} * \tau_r(t) - \Gamma_{m_n}(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \to 0$, when $r \to \infty$,

(iv) $\int_A^B \|\Gamma_{m_n} * \tau_r(t) - \Gamma_{m_n}(t)\|_{H^2(\mathbb{R}\setminus\{0\})} dt \to 0$, as $r \to \infty$.

Hence of (i)-(iv), we obtain the sequence, $(\Gamma_{m_n} * \tau_n)_{n\geq 1}$

(v) For each $t \in [A,B]$, we have $\|\Gamma_{m_n} * \tau_n(t) - u(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \to 0$, as $n \to \infty$,

(vi) $\int_A^B \|\Gamma_{m_n} * \tau_n(t) - u(t)\|_{H^2(\mathbb{R}\setminus\{0\})} dt \to 0$, when $n \to \infty$.

Besides, since the support $\Gamma_{m_n}(t)$ and $u(t)$ is the interval $[A,B]$, and the support of $u(t + \frac{B-A}{m_n})$ is $[A - \frac{B-A}{m_n}, B - \frac{B-A}{m_n}]$ then, the support of $\partial_t \Gamma_{m_n} * \tau_n(t)$ and $u(\cdot + \frac{B-A}{m_n}) * \tau_{m_n}(t)$ is contained in the interval $[A - \frac{B-A+1}{m_n}, B + \frac{1}{m_n}]$, so

$$\begin{aligned}
&\int \| \int \partial_t \Gamma_{m_n}(t-s)\tau_{m_n}(s)ds \\
&- \int \frac{u(t-s+\frac{(B-A)}{m_n}) - u(t-s)}{\frac{(B-A)}{m_n}} \tau_{m_n}(s)ds\|dt \\
&= \int \| \int \Big(\sum_{j-2}^{n+1} \frac{\omega_{m_n}(t_j) - \omega_{m_n}(t_{j-1})}{t_j - t_{j-1}} \chi_{[t_{j-1},t_j)}(t-s) \\
&- \frac{u(t-s+\frac{(B-A)}{m_n}) - u(t-s)}{\frac{(B-A)}{m_n}} \Big) \tau_{m_n}(s)ds\|dt \\
&\leq \int_{A-\frac{B-A+1}{m_n}}^{B+\frac{1}{m_n}} \sum_{j-2}^{m_n+1} \int_{t-t_j}^{t-t_{j-1}} \tau_{m_n}(s) \Big(\frac{\|\omega_{m_n}(t_j) - u(t-s+\frac{(B-A)}{m_n})\|}{t_j - t_{j-1}} \\
&\qquad + \frac{\|\omega_{m_n}(t_{j-1}) - u(t-s)\|}{\frac{(B-A)}{m_n}} \Big) dsdt
\end{aligned}$$

$$< \frac{m_n}{(B-A)}\frac{2}{m_n^2}\Big(B-A+\frac{(B-A)+2}{m_n}\Big) \leq \frac{(B-A)+2}{(B-A)m_n}\,.$$

Therefore,

$$\begin{aligned}&\int \| \int \partial_t \Gamma_{m_n}(t-s)\tau_{m_n}(s)ds \\ &- \int \frac{u(t-s+\frac{(B-A)}{m_n})-u(t-s)}{\frac{(B-A)}{m_n}}\tau_{m_n}(s)ds\|dt \to 0,\end{aligned} \tag{5}$$

as $n \to \infty$.

On the other hand, for $\epsilon > 0$, we obtain

$$\begin{aligned}&\int \| \int \frac{u(t-s+\frac{(B-A)}{m_n})-u(t-s)}{\frac{(B-A)}{m_n}}\tau_{m_n}(s)ds - \partial_t(u*\tau_{m_n})(t)\|dsdt \\ &\leq \int \tau_{m_n}(s)\int \|\frac{u(t-s+\frac{(B-A)}{m_n})-u(t-s)}{\frac{(B-A)}{m_n}} - \partial_t u(t-s)\|dtds \\ &< \epsilon \int \tau_{m_n}(s)ds = \epsilon,\end{aligned} \tag{6}$$

from $n \in \mathbb{Z}^+$ big enough.

Since

$$\begin{aligned}&\int \|\partial_t(\Gamma_{m_n}*\tau_{m_n})(t) - \partial_t u(t)\|dt \leq \int \|\partial_t(\Gamma_{m_n}*\tau_{m_n})(t) \\ &- \int \frac{u(t-s+\frac{(B-A)}{m_n})-u(t-s)}{\frac{(B-A)}{m_n}}\tau_{m_n}(s)ds\|dt \\ &+ \int \| \int \frac{u(t-s+\frac{(B-A)}{m_n})-u(t-s)}{\frac{(B-A)}{m_n}}\tau_{m_n}(s)ds - \partial_t(u*\tau_{m_n})(t)\|dt \\ &+ \int \|\partial_t(u*\tau_{m_n})(t) - \partial_t u(t)\|dt\end{aligned}$$

and from (5) and (6) we obtain

$$\int \|\partial_t(\Gamma_{m_n}*\tau_{m_n})(t) - \partial_t u(t)\|dt \to \ 0, \quad \text{when} \quad n\to\infty.$$

□

4. Conclusions

In this paper we show that in the domain of operator with point interaction $-\frac{d^2}{dx^2}+Z\delta_0$ there exists a dense subset of compact support and smooth functions except, possibly at $x=0$.

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