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# ANALYSIS STABILITY FOR A FRACTIONAL VOLTERRA MODEL WITH TWO CONTROLS

Mihai Ivan

West University of Timişoara
Geometry and Topology Seminar, Teacher Training Dept.
Timişoara – 300223, ROMANIA

**Abstract:** The main purpose of this paper is to study the fractional-order system with Caputo derivative associated to 3-dimensional Volterra model with two controls. For this fractional system we investigate the existence and uniqueness of solution of initial value problem, asymptotic stability of its equilibrium states, stabilization problem using appropriate controls and numerical integration via the fractional Euler method.

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## 1. Introduction

Fractional calculus is a field of mathematics based on a generalization of integer derivatives to fractional order. The studies in this area are very important and attractive because several phenomena have been described better by fractional derivatives that take into account not only the local properties, but also global correlations of dynamical systems ([25], [13], [23]). The fractional calculus has deep and natural connections with many fields of science and engineering ([20], [14],[19]).

In the last three decades, increasing attention has been paid to the study of the dynamic behaviors (in particular, the chaotic behavior) of some classical differential systems, as well as some fractional-order differential systems. For example, the fractional models played an important role in applied mathematics ([16], [7], [9]), applied physics ([2], [24], [11]), study of biological systems ([1],

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[21], [15]), chaos synchronization and secure communications ([17], [8] and so on.

## 2. Some theoretical aspects on fractional dynamical systems

In this section we introduce definitions and preliminaries facts which are used in this paper.

There are several definitions of fractional derivatives. One of the more commonly used is the Caputo definition, suitable for concrete applications. Let $f$ be a real-valued function which has infinitely many derivatives on $\mathbf{R}$ and $q \in \mathbf{R}, q > 0$. The $q$-order Caputo differential operator ([4]) is defined by

$$D_t^q f(t) = J^{m-q} f^{(m)}(t),\ q > 0, \tag{1}$$

where $f^{(m)}(t)$ represents the $m$-order derivative of the function $f$, $m \in \mathbf{N}^*$ is an integer such that $m = [q] + 1$ (i.e. $[q]$ denotes the integer part of $q$) and $J^\alpha$ is the $\alpha$- order Riemann-Liouville integral operator ([20]) as follows

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, \tag{2}$$

where $\Gamma(.)$ is the Euler's gamma function. If $q = 1$, then $D_t^q f(t) = \frac{df}{dt}$.

Related to (1) and (2), one has the following definition.

**Definition 1.** The Caputo fractional derivative of order $q > 0$ for a function $f \in C^\infty(\mathbf{R})$ is described by:

$$D_t^q f(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} f^{(m)}(s) ds, \quad q > 0. \tag{3}$$

In particular, the Caputo derivative of order $q \in (0,1]$ for a function $f \in C^\infty(\mathbf{R})$ is described by

$$D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'(s) ds, \quad q \in (0,1]. \tag{4}$$

The Caputo derivative defined by (4) is often used in concrete applications. In this paper we suppose that $q \in (0,1]$.

In the Euclidean space $\mathbf{R}^n$ with local coordinates $\{x^1, x^2, \ldots, x^n\}$, we consider the following system of fractional differential equations:

$$D_t^q x^i(t) = f_i(x^1(t), x^2(t), \ldots, x^n(t)), \quad i = \overline{1,n}, \tag{5}$$

where $q \in (0,1), f_i \in C^\infty(\mathbf{R}^n, \mathbf{R})$, $D_t^q x^i(t)$ is the Caputo fractional derivative of order $q$ for $i = \overline{1,n}$ and $t \in [0, \tau)$ is the time.

The fractional dynamical system (5) can be written as follows:

$$D_t^q x(t) = f(x(t)), \tag{6}$$

where $f(x(t)) = (f_1(x^1(t), \ldots, x^n(t)), \ldots, f_n(x^1(t), \ldots, x^n(t)))^T$ and $D_t^q x(t) = (D_t^q x^1(t), \ldots, D_t^q x^n(t))^T$.

Consider the initial value problem with Caputo derivative for the system (6):

$$D_t^q x(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I = [0, T], \ T > 0 \tag{7}$$

where $x : I \to \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function and $q \in (0,1)$.

**Theorem 1.** ([4]) *Let by the initial value problem for the fractional system* (7). *If the function $f$ satisfies the following two assumptions:*
($i$) *$f$ is differentiable and bounded on $D = \{x \in \mathbf{R}^n | |x^i - x_0^i| < \delta, i = \overline{1,n}\}$ for any $\delta > 0$.*
($ii$) *$f(x(t))$ satisfies the Lipschitz condition, i.e.* $(\exists) L > 0$, *such that*

$$|f(x(t)) - f(y(t))| < L \cdot |x(t) - y(t)|, \quad (\forall) x(t), y(t) \in D.$$

A point $x_e = (x_e^1, x_e^2, \ldots, x_e^n) \in \mathbf{R}^n$ is said to be *equilibriun point* or *equilibrium state* of the system (5), if $D_t^q x^i(t) = 0$ for $i = \overline{1,n}$.

The equilibrium states of the fractional dynamical system (5) are determined by solving the following set of equations:

$$f_i(x^1(t), x^2(t), \ldots, x^n(t)) = 0, \quad i = \overline{1,n}. \tag{8}$$

**Definition 2.** ([20]) The equilibrium point $x_e$ of the system (5) is said to be:
($i$) *(locally) stable,* if for each $\varepsilon > 0$, $\exists \delta > 0$ such that

$$\|x_e\| < \delta \quad \Rightarrow \quad \|x(t)\| < \varepsilon, \quad (\forall) t > 0. \tag{9}$$

($ii$) *(locally) asymptotically stable,* if it is stable and $\lim_{t\to\infty} \|x(t) - x_e\| = 0$, where $\| \cdot \|$ is the Euclidean norm.

The Jacobian matrix associated to system (5) is:

$$J(x) = \left(\frac{\partial f_i}{\partial x^j}\right), \quad i, j = \overline{1,n}.$$

**Proposition 1.** ([17]) *Let $x_e$ be an equilibrium state of fractional system* (5) *and $J(x_e)$ be the Jacobian matrix $J(x)$ evaluated at $x_e$.*

($i$) *$x_e$ is locally asymptotically stable, if and only if all eigenvalues $\lambda(J(x_e))$ of $J(x_e)$ satisfy:*

$$|arg(\lambda(J(x_e)))| > \frac{q\pi}{2}. \qquad (10)$$

*(ii)* $x_e$ *is locally stable, if and only if either it is asymptotically stable, or the critical eigenvalues satisfying* $|arg(\lambda(J(x_e)))| = \dfrac{q\pi}{2}$ *have geometric multiplicity one.*

*(iii)* $x_e$ *is locally unstable, if and only if there exists one eigenvalue* $\lambda(J(x_e))$ *of* $J(x_e)$ *such that* $|arg(\lambda(J(x_e)))| < \dfrac{q\pi}{2}$.

According to Proposition 1, it is easy to prove the following lemma.

**Lemma 1.** ([6]) *Let* $x_e$ *be an equilibrium state of the fractional model* (5), $\lambda_i$, $i = \overline{1,n}$ *the eigenvalues of* $J(x_e)$ *and* $q \in (0,1)$.

*(i) If one of the eigenvalues* $\lambda_i$, $i = \overline{1,n}$ *is equal to zero or it is positive, then* $x_e$ *is unstable.*

*(ii) If* $\lambda_i < 0$, *for all* $i = \overline{1,n}$, *then* $x_e$ *is asymptotically stable.*

## 3. The 3−dimensional fractional Volterra model with two controls

The Volterra model were defined to describe population evolution in a hierarchical system by competing individuals ([5], [3]).

The $n$-dimensional fractional-order Volterra model on $\mathbf{R}^n$ is defined by the following fractional differential equations:

$$D_t^q x^i(t) = x^i(t)(x^{i+1}(t) - x^{i-1}(t)), \quad i = \overline{1,n}, \qquad (11)$$

where $x^0(t) = x^{n+1}(t) = 0, x^n \geq 0$, $x^i, i = \overline{1,n}$, are state variables and $t$ is the time.

In particular, the *3-dimensional fractional Volterra model* is defined by the following three differential equations on $\mathbf{R}^3$:

$$\begin{cases} D_t^q x^1(t) &= x^1(t)x^2(t), \\ D_t^q x^2(t) &= x^2(t)(x^3(t) - x^1(t)), \qquad q \in (0,1), \\ D_t^q x^3(t) &= -x^2(t)x^3(t). \end{cases} \qquad (12)$$

In this section we introduce the *(3−dimensional) fractional Volterra model with two controls* around axes $Ox^2$ and $Ox^3$. This fractional model is associated to system (12) and it is defined by:

$$\begin{cases} D_t^q x^1(t) &= x^1(t)x^2(t), \\ D_t^q x^2(t) &= -x^1(t)x^2(t) + x^2(t)x^3(t) - ax^2(t), \qquad q \in (0,1), \\ D_t^q x^3(t) &= -x^2(t)x^3(t) - bx^3(t), \end{cases} \qquad (13)$$

where $a, b \in \mathbf{R}^*$ are parameters.

If we substitute $q = 1$ and $a = b = 0$ in the fractional model (13), then the $3-$dimensional Volterra model is obtained.

**Remark 1.** Differential systems of the Volterra model type have been studied from various research directions by many authors. From the point of view of Poisson geometry, Volterra model was researched in [22], and as metriplectic system it was investigated in [12].

In the following, the existence and uniqueness of solution of initial value problem for the fractional system (13) are proved.

The initial value problem of the fractional system (13) can be represented in the following matrix form:

$$D_t^\alpha x(t) = x^2(t)Ax(t) + Bx(t), \qquad x(0) = x_0, \tag{14}$$

where $0 < q < 1$, $x(t) = (x^1(t), x^2(t), x^3(t))^T$, $t \in (0, \tau)$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -b \end{pmatrix}.$$

**Proposition 2.** *The initial value problem of the fractional Volterra model with two controls* (13) *has a unique solution.*

*Proof.* Let $f(x(t)) = x^2(t)Ax(t) + Bx(t)$. It is continuous and bounded on $D = \{x \in \mathbf{R}^3 | \ x^i \in [x_0^i - \delta, x_0^i + \delta]\}, i = \overline{1,3}$ for any $\delta > 0$. We have $f(x(t)) - f(y(t)) = x^2(t)Ax(t) - y^2(t)Ay(t) + Bx(t) - By(t) = g(t) + h(t)$, where $g(t) = x^2(t)Ax(t) - y^2(t)Ay(t)$ and $h(t) = Bx(t) - By(t)$. Then

$(a)$ $|f(x(t)) - f(y(t))| \leq |g(t)| + |h(t)|$.

Using reasoning analogous to that in the proof of Proposition 2.1 in [10], we can show that:

$(b)$ $|g(t)| \leq (\|A\| + |y^2(t)|) \cdot |x(t) - y(t)|$ and $|h(t)| \leq \|B\| \cdot |x(t) - y(t)|$.

According to $(b)$ the relation $(a)$ becomes

$(c)$ $|f(x(t)) - f(y(t))| \leq (\|A\| + \|B\| + |y^2(t)|) \cdot |x(t) - y(t)|$.

Replacing $\|A\| = 2$, $\|B\| = \sqrt{a^2 + b^2}$ and using the inequalities $|y^i(t)| \leq |x_0| + \delta$, $i = 1, 3$ from the relation $(c)$, we deduce that

$(d)$ $|f(x(t)) - f(y(t))| \leq L \cdot |x(t) - y(t)|$, $L = 2 + \sqrt{a^2 + b^2} + 2|x_0| + \delta > 0$.

The inequality $(d)$ shows that $f(x(t))$ satisfies a Lipschitz condition. Based on the results of Theorem 1, we can conclude that the initial value problem of the system (13) has a unique solution. $\square$

For the fractional system (13) we introduce the following notations:

$$f_1(x) = x^1x^2, \quad f_2(x) = -x^1x^2 + x^2x^3 - ax^2, \quad f_3(x) = -x^2x^3 - bx^3. \quad (15)$$

**Proposition 3.** *The equilibrium states of the fractional Volterra model with two controls* (13) *are given as the following family:*

$$E := \{e_0 = (0,0,0)\} \cup \{e = (0,-b,a)\} \cup \{e_1^m = (m,0,0) \in \mathbf{R}^3 |\ m \in \mathbf{R}^*\}.$$

*Proof.* The equilibrium states are solutions of the equations $f_i(x) = 0, i = \overline{1,3}$ where $f_i,\ i = \overline{1,3}$ are given by (15). □

## 4. Main results

### 4.1. Stability analysis of the fractional Volterra model (13)

In this subsection we present the study of asymptotic stability for the equilibrium states of the fractional system (13). For this study we apply the Matignon's test (Proposition 1).

The Jacobian matrix associated to system (13) is:

$$J(x,a,b) = \begin{pmatrix} x^2 & x^1 & 0 \\ -x^2 & -x^1 + x^3 - a & x^2 \\ 0 & -x^3 & -x^2 - b \end{pmatrix}.$$

**Proposition 4.** *Let be the fractional system* (13) *and the equilibrium state* $e = (0,-b,a) \in E$.

**1.** $b > 0$.

*(i) If* $a > 0$, *then* $e$ *is unstable* $(\forall) q \in (0,1)$.

*(ii) If* $a < 0$, *then* $e$ *is asymptotically stable* $(\forall) q \in (0,1)$.

**2.** $b < 0$. *If* $a \in \mathbf{R}^*$, *then* $e$ *is unstable* $(\forall) q \in (0,1)$.

*Proof.* The characteristic polynomial of

$$J(e,a,b) = \begin{pmatrix} -b & 0 & 0 \\ b & 0 & -b \\ 0 & -a & 0 \end{pmatrix}$$

is $p_{J(e,a,b)}(\lambda) = \det(J(e,a,b) - \lambda I) = -(\lambda + b)(\lambda^2 - ab)$. The equation $p_{J(e,a,b)}(\lambda) = 0$ has the roots $\lambda_1 = -b$ and $\lambda_{2,3} = \pm\sqrt{ab}$.

**1. Case** $b > 0$ and $q \in (0,1)$. Then $\lambda_1 = -b < 0$.

$(i)$ We suppose $a > 0$. Then $\lambda_{2,3} = \pm\sqrt{ab} \in \mathbf{R}$. In this case, $J(e, a, b)$ has a positive eigenvalue and by Lemma 1(i), $e = (0, -b, a)$ is unstable.

$(ii)$ We suppose $a < 0$. Then $\lambda_{2,3} = \pm i\sqrt{-ab}$. We have $|arg(\lambda_1)| = \pi > \frac{q\pi}{2}$ and $|arg(\lambda_{2,3})| = \frac{\pi}{2} > \frac{q\pi}{2}$ for all $q \in (0, 1)$. By Proposition 1(i), it implies that $e = (0, -b, a)$ is asymptotically stable. Hence, the assertions $1(i) - (ii)$ hold.

**2. Case** $b < 0$, $a \in \mathbf{R}^*$ and $q \in (0, 1)$. Since $\lambda_1 = -b > 0$, $J(e, a, b)$ has at least a positive eigenvalue. Then, by Lemma 1(i), $e = (0, -b, a)$ is unstable. Hence, the assertion 2 holds. □

**Proposition 5.** *The equilibrium states* $e_0$ *and* $e_1^m \in E$ *are unstable* $(\forall) q \in (0, 1)$.

*Proof.* We have $J(e_1^m, a, b) = \begin{pmatrix} 0 & m & 0 \\ 0 & -m-a & 0 \\ 0 & 0 & -b \end{pmatrix}$. Its characteristic polynomial, for $m \in \mathbf{R}$, is

$$p_{J(e_1^m,a,b)}(\lambda) = \det(J(e_1^m, a, b) - \lambda I) = -\lambda(\lambda + a + m)(\lambda + b).$$

The equation $p_{J(e_1^m,a,b)}(\lambda) = 0$ has the root $\lambda_1 = 0$. By Lemma 1(i), follows that $e_0$ and $e_1^m$ are unstable for all $q \in (0, 1)$. □

**4.2. Controllability of chaotic behaviors of the fractional Volterra model** (13)

In this subsection we will discuss how to stabilize the unstable equilibrium states of the fractional system (13) via Caputo fractional derivative. For this purpose we will apply the general method for to control the stability of (13 at an equilibrium point [6].

Let $x_e$ be an unstable equilibrium point of the 3−dimensional fractional Volterra model with two controls (13). We associate to (13) a new fractional-order system with (external) controls and given by:

$$\begin{cases} D_t^q x^1(t) &= x^1(t)x^2(t) + u_1(t), \\ D_t^q x^2(t) &= -x^1(t)x^2(t) + x^2(t)x^3(t) - ax^2(t) + u_2(t),\ q \in (0,1), \\ D_t^q x^3(t) &= -x^2(t)x^3(t) - bx^3(t) + u_3(t), \end{cases} \tag{16}$$

where $u_i(t), i = \overline{1,3}$ are control functions.

We take the control functions $u_i(t), i = \overline{1,3}$, given by:

$$u_1(t) = k(x^1(t) - x_e^1), \quad u_2(t) = 0, \quad u_3(t) = 0, \quad k \in \mathbf{R}^*. \tag{17}$$

With the control functions (17), the system (16) becomes:

$$\begin{cases} D_t^q x^1(t) &=& x^1(t)x^2(t) + k(x^1(t) - x_e^1), \\ D_t^q x^2(t) &=& -x^1(t)x^2(t) + x^2(t)x^3(t) - ax^2(t), \quad q \in (0,1), \\ D_t^q x^3(t) &=& -x^2(t)x^3(t) - bx^3(t), \end{cases} \tag{18}$$

where $k \in \mathbf{R}^*$ is a control parameter.

The fractional system (18) is called the *controlled fractional Volterra model associated to* (13) *at* $x_e$.

If one selects the appropriate parameters $a, b, k \in \mathbf{R}^*$ which then make the eigenvalues of the linearized equation of (18) satisfy one of the conditions from Proposition 1, then the trajectories of (18) asymptotically approaches the unstable equilibrium state $x_e$ in the sense that $\lim_{t\to\infty} \|x(t) - x_e\| = 0$. In the case when $x_e$ is unstable, then fractional model (18) may exhibit chaotic behavior.

The Jacobian matrix of the controlled fractional model (18) is:

$$J(x,a,b,k) = \begin{pmatrix} x^2 + k & x^1 & 0 \\ -x^2 & -x^1 + x^3 - a & x^2 \\ 0 & -x^3 & -x^2 - b \end{pmatrix}.$$

**Theorem 2.** *Let be the fractional system* (18) *and* $e_0 = (0,0,0)$.

**1.** $k < 0$.

(*i*) *If* $a > 0$ *and* $b > 0$, *then* $e_0$ *is asymptotically stable* $(\forall) q \in (0,1)$.

(*ii*) *If* $a < 0$ *and* $b < 0$ *or* $ab < 0$, *then* $e_0$ *is unstable* $(\forall) q \in (0,1)$.

**2.** $k > 0$. *If* $a, b \in \mathbf{R}^*$, *then* $e_0$ *is unstable* $(\forall) q \in (0,1)$.

*Proof.* The characteristic polynomial of

$$J(e_0,a,b,k) = \begin{pmatrix} k & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -b \end{pmatrix}$$

is $p_{J(e_0,a,b,k)}(\lambda) = \det(J(e_0,a,b,k) - \lambda I) = -(\lambda - k)(\lambda + a)(\lambda + b)$. The roots of equation $p_{J(e_0,a,b,k)}(\lambda) = 0$ are $\lambda_1 = k$, $\lambda_2 = -a$, $\lambda_3 = -b$.

**1. Case** $k < 0$ and $q \in (0,1)$. Then $\lambda_1 < 0$.

(*i*) We have $\lambda_2 < 0$ and $\lambda_3 < 0 \Leftrightarrow a > 0$ and $b > 0$. Then $\lambda_i < 0, i = \overline{1,3}$ and by Lemma 1 (ii), $e_0$ is asymptotically stable.

(*ii*) We suppose $a < 0$ and $b < 0$ or $ab < 0$. Then $J(e_0,a,b,k)$ has at least a positive eigenvalue and by Lemma 1(i), $e_0$ is unstable. Hence, $1(i) - (ii)$ hold.

**2. Case** $k > 0$ and $q \in (0,1)$. Since $\lambda_1 > 0$, $J(e_0, a, b, k)$ has at least a positive eigenvalue $(\forall) a, b \in \mathbf{R}^*$. Then $e_0$ is unstable. Hence, the assertion 2 holds. $\square$

**Theorem 3.** *Let be the system* (18) *and* $e_1^m = (m, 0, 0) \in E$.

**1.** $k < 0$ *and* $q \in (0,1)$.

$(i)$ *If* $b > 0$, *then* $e_1^m$ *is asymptotically stable for all* $m \in (-a, \infty)$ *and unstable for all* $m \in (-\infty, -a)$.

$(ii)$ *If* $b < 0$, *then* $e_1^m$ *is unstable for all* $a, m \in \mathbf{R}^*$.

**2.** $k > 0$ *and* $q \in (0,1)$. *The equilibrium state* $e_1^m$ *is unstable for all* $a, b, m \in \mathbf{R}^*$.

*Proof.* We have $J(e_1^m, a, b, k) = \begin{pmatrix} k & m & 0 \\ 0 & -a-m & 0 \\ 0 & 0 & -b \end{pmatrix}$. Its characteristic polynomial is $p_{J(e_1^m,a,b,k)}(\lambda) = -(\lambda - k)(\lambda + a + m)(\lambda + b)$. The roots of equation $p_{J(e_1^m,a,b,k)}(\lambda) = 0$ are $\lambda_1 = k$, $\lambda_2 = -a - m$, $\lambda_3 = -b$.

**1. Case** $k < 0$ and $q \in (0,1)$. Then $\lambda_1 < 0$.

$(i)$ We suppose $b > 0$. Then $\lambda_3 < 0$. We have $\lambda_i < 0, i = \overline{1,3}$ if and only if $m \in (-a, \infty)$. By Lemma 1(ii), $e_1^m$ is asymptotically stable. Also, for $m \in (-\infty, -a)$ it follows that $\lambda_2 > 0$ and $J(e_1^m, a, b, k)$ has a positive eigenvalue. According to Lemma 1(i), $e_1^m$ is unstable $(\forall) m \in (-\infty, -a)$.

$(ii)$ We suppose $b < 0$. Then $\lambda_3 > 0$ and $J(e_1^m, a, b, k)$ has at least a positive eigenvalue. By Lemma 1(i), $e_1^m$ is unstable $(\forall) a, m \in \mathbf{R}^*$. Hence, $1(i) - (ii)$ hold.

**2. Case** $k > 0$ and $q \in (0,1)$. Since $\lambda_1 > 0$, $J(e_1^m, a, b, k)$ has a positive eigenvalue $(\forall) a, b, m \in \mathbf{R}^*$. Then $e_1^m$ is unstable. Hence, the assertion 2 holds. $\square$

### 4.3. Examples

**Example 1.** Let be the 3-dimensional fractional-order Volterra model with controls $k, a, b$ described by (18).

(1) We select $k = -0.65$, $a = 0.15$ and $b = 0.32$. According to Theorem 2.1$(i)$, it follows that $e_0 = (0,0,0)$ is asymptotically stable for all $q \in (0,1)$.

(2) We consider $k = -1, a = 1$ and $b = -0.7$. Applying Theorem 2.1$(ii)$, it follows that $e_0 = (0,0,0)$ is unstable for all $q \in (0,1)$. In other words, in this case the fractional model (18) behaves chaotically around the equilibrium point $e_0$.

(3) We consider $k = 1, a = -2$ and $b = -1.5$. Applying Theorem 2.2, it follows that $e_0 = (0,0,0)$ is unstable for all $q \in (0,1)$.

**Example 2.** Let be the 3-dimensional fractional-order Volterra model with controls $k, a, b$ described by (18).

(1) We select $k = -0.6$, $b = 0.9$, $a = -2$ and $q = 0.73$. According to Theorem 3.1($i$), $e_{11} = (2.01, 0, 0)$ is asymptotically stable and $e_{12} = (1.98, 0, 0)$ is unstable.

(2) We consider $k = 1.2$, $a = 0.4$ and $b = -0.25$. Applying Theorem 3.2, $e_{13} = (-0.8, 0, 0)$ is unstable for all $q \in (0, 1)$.

**Remark 2.** The study of the dynamics of metriplectic systems and fractional systems associated with Volterra-type or Toda-type lattices has been addressed in a series of papers, such as ([3], [12], [8]).

## 5. Numerical integration of the controlled fractional Volterra model

In this section we apply the fractional Euler's method (FEM) to numerically integrate the controlled fractional Volterra model (18). For the description and application of FEM's can be consulted ([18], [23]).

Consider the following general form of the initial value problem (IVP) with Caputo derivative [18]:

$$D_t^q y(t) = f(t, y(t)), \quad y(0) = y_0, \quad t \in I = [0, T],\ T > 0, \tag{19}$$

where $y : I \to \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function and $q \in (0, 1)$.

Every solution of the initial value problem given by (19) is also a solution of the following *Volterra fractional integral equation*:

$$y(t) = y(0) + \ I_t^q f(t, y(t)), \tag{20}$$

where $I_t^q$ is the $q-$order Riemann-Liouville integral operator. Moreover, every solution of (20) is a solution of the (IVP) (19).

To integrate the fractional equation (19), means to find the solution of (20) over the interval $[0, T]$. In this context, a set of points $(t_j, y(t_j))$ are produced which are used as approximated values. The interval $[0, T]$ is partitioned into $n$ subintervals $[t_j, t_{j+1}]$ each equal width $h = \frac{T}{n}$, $t_j = jh$ for $j = 0, 1, ..., n$. It computes an approximation denoted as $y_{j+1}$ for $y(t_{j+1})$, $j = 0, 1, ....$

The *general formula of the fractional Euler's method* for to compute the elements $y_j$, is

$$y_{j+1} = y_j + \frac{h^q}{\Gamma(q+1)} f(t_j, y(t_j)), \qquad t_{j+1} = t_j + h, \quad j = 0, 1, ..., n. \tag{21}$$

We will now apply the above considerations to the controlled fractional-order Volterra model (18). For this, consider the following fractional differential equations

$$\begin{cases} D_t^q x^i(t) & = & F_i(x^1(t), x^2(t), x^3(t)),\ i=\overline{1,3},\ t\in(t_0,\tau),\ q\in(0,1) \\ x(t_0) & = & (x^1(t_0), x^2(t_0), x^3(t_0)), \end{cases} \tag{22}$$

where

$$\begin{cases} F_1(x(t)) & = & x^1(t)x^2(t) + k(x^1(t) - x_e^1), \\ F_2(x(t)) & = & -x^1(t)x^2(t) + x^2(t)x^3(t) - ax^2(t), \quad a,b,k \in \mathbf{R}^* \\ F_3(x(t)) & = & -x^2(t)x^3(t) - bx^3(t). \end{cases} \tag{23}$$

Since the functions $F_i(x(t)), i=\overline{1,3}$ are continuous, the initial value problem (22) is equivalent to system of Volterra integral equations, which is given as follows:

$$x^i(t) \ = \ x^i(0) + \ I_t^q F_i(x^1(t), x^2(t), x^3(t)), \qquad i=\overline{1,3}. \tag{24}$$

The system (24) is called the *Volterra integral equations associated to controlled fractional-order Volterra model* (18).

For the numerical integration of the system (22) one can use the fractional Euler's method (the formula (21) ), which is expressed as follows:

$$x^i(j+1) = x^i(j) + \frac{h^q}{\Gamma(q+1)} F_i(x^1(j), x^2(j), x^3(j)), \quad i=\overline{1,3}, \tag{25}$$

where $j=0,1,2,...,N, h=\dfrac{T}{N}, T>0, N>0$.

More precisely, the numerical integration of the fractional system (22) is given by:

$$\begin{cases} x^1(j+1) & = x^1(j) + h^q \ \dfrac{1}{\Gamma(q+1)}(x^1(j)x^2(j) + k(x^1(j) - x_e^1)) \\ x^2(j+1) & = x^2(j) + h^q \ \dfrac{1}{\Gamma(q+1)}(-x^1(j)x^2(j) \\ & \qquad\qquad +x^2(j)x^3(j) - ax^2(j)) \\ x^3(j+1) & = x^3(j) + h^q \ \dfrac{1}{\Gamma(q+1)}(-x^2(j)x^3(j) - bx^3(j)) \\ x^i(0) & = x_e^i + \varepsilon, \quad i=\overline{1,3}. \end{cases} \tag{26}$$

**Example 3.** Let us we present the numerical integration of the controlled fractional-order Volterra model with two controls which has considered in Example 2.1($i$). For this we apply the algorithm (26) and software Maple or

Mathematica. Then, in (26) we take: $a = -2, b = 0.9$ and $k = -0.6$. It is known that the equilibrium state $e_{11} = (2.01, 0, 0)$ is asymptotically stable for $q = 0.73$.

**Remark 3.** Appyling (26) and the Mathematica package for the numerical simulation of solution of fractional model (18) for each set of values for parameters $a, b, k$, given in the Examples 1 and 2, it will be found that the results obtained are valid.

## 6. Conclusions

This paper presents a 3-dimensional fractional-order Volterra model with two controls, denoted by (13). The fractional model (13) was studied from fractional differential equations theory point of view: asymptotic stability, determining of sufficient conditions on parameters $a, b, k$ to control the chaos in the controlled fractional system associated to (13) and numerical integration of the fractional model (18). The study of chaotic fractional systems has appplications in theory of chaos synchronization and secure communications. In this context, by choosing the right $a, b$ and $k$ in the fractional model (18), this work offers a series of chaotic and non-chaotic fractional differential systems.

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