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ASYMPTOTICS OF ZEROS OF AN ENTIRE FUNCTION WITH AN INTEGRAL REPRESENTATION, CONNECTED BY A REGULAR LOADED DIFFERENTIATION OPERATOR ON AN INTERVAL

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Abstract: In this paper, we construct a characteristic determinant of the spectral problem for a loaded first-order differential equation on an interval with a periodic boundary value condition, which is an entire analytical function of spectral parameter. Based on the formula of the characteristic determinant, conclusions about the asymptotic behavior of the spectrum of the loaded first-order differential equation are drawn on an interval. Adjoint operator is constructed. Moreover, we show that the spectral questions of the adjoint operator have a similar structure. A special feature of the considered operator is the non-self-adjointness of the operator in $L_2(-1, 1)$.

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1. Introduction

Necessity of studying the asymptotics and distributions of zeros of entire functions arises when studying the spectrum of a differential operator [1], [2, Chapter 5]. The works [3] - [8] are devoted to the study of zeros of entire functions that have an integral representation.

Asymptotic properties of entire functions with a given distribution law of roots were deeply studied in the doctoral dissertation of V.B. Sherstyukov, based on which the paper [9] was published.

Meromorphic functions of completely regular growth in the upper half-plane with respect to the growth function are studied in [10]. The issues of the location of zeros of an entire function: on one ray, on a straight line, on several rays, in an angle or arbitrarily in the complex plane have been studied in numerous works [1], [11] [19].

Connection between zeros of entire functions of exponential type with spectral problems is reflected in [5] - [8], [16] - [21].

2. Statement and discussion of the problem

We consider the question of distribution of zeros of an entire function of the form:

$$\Delta_1(\lambda) = e^{-\lambda} - e^{\lambda} - \int_{-1}^{1} e^{\lambda t} \cdot \Phi(t)\, dt, \tag{1}$$

where $\Phi(t)$ is a continuous function and satisfies the condition:

$$\Phi(-1) = \Phi(1) = 1. \tag{2}$$

Eigenvalue problems for certain classes of differential operators on an interval are reduced to a similar problem. In particular, the following eigenvalue problem in the space $W_2^1(-1, 1)$ for a loaded first-order differential operator on the interval leads to the studied question:

$$L_1 y = y'(t) + \Phi(t)\, y(1) = \lambda y(t), \tag{3}$$

with the boundary value problem

$$y(-1) = y(1), \tag{4}$$

where $\Phi(t)$ is a continuous function and satisfies the condition (2).

Spectral questions, more precisely, questions on basis property of root functions of loaded differential operators were studied in the works [22] - [24], which the method of spectral decompositions by V.A. Ilyin [25] are extended to the case of loaded differential operators. The main fundamental feature of such problems is their non-self-adjointness. This causes the main difficulties in studying perturbed differential operators. Another variant of perturbation of regular, but not strongly regular, boundary value problems was studied in [26] - [30], [7], [21]. In the monograph by B.E. Kanguzhin and M.A. Sadybekov [7], the basic properties of the system of root vectors of Sturm-Liouville boundary value problems on a finite segment were studied in the space of quadratically summable functions, irregular by Birkhoff, where the effect was noted when the same Sturm-Liouville boundary value problem, depending on properties of the potential, can have discrete or continuous spectrum.

The essence of the problem statement is that it is required to find those complex values λ, for which the operator equation (3) has non-zero solutions.

3. Characteristic determinant of the spectral problem (3) - (4)

Assuming $y(1)$ to be some independent constant, we make sure that the general solution of the equation (3), for $\lambda \neq 0$, can be represented in the form:

$$y(t) = Ce^{\lambda t} - e^{\lambda} \cdot y(1) \cdot \int\limits_{-1}^{t} e^{\lambda \xi} \cdot \Phi(\xi)\, d\xi, \tag{5}$$

Thus, assuming first $t = -1$, and then satisfying (5) with the boundary value condition (4), we obtain the system of two equations, which can be represented in vector-matrix form:

$$\begin{bmatrix} e^{\lambda} - e^{-\lambda} & e^{\lambda} \int\limits_{-1}^{1} e^{\lambda \xi} \cdot \Phi(\xi)\, d\xi \\ e^{-\lambda} & -1 \end{bmatrix} \cdot \begin{bmatrix} C \\ y(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By using simple calculations, we find that the characteristic determinant $\Delta_1(\lambda)$ of the spectral problem (3) - (4) is represented in the form (1), which is an entire analytical function of the variable $\lambda = x + iy$, $\mathrm{Re}\lambda = x$, $Jm\lambda = y$, $i = \sqrt{-1}$. Further, we research zeros of the entire function $\Delta_1(\lambda)$, which adequately determine eigenvalues of the loaded differential operator L_1.

4. The main result

Due to the well-known theorem of Rouchet [31], we introduce the function in (1):

$$\Delta_1(\lambda) = \Delta_0(\lambda) - f(\lambda),$$

where

$$\Delta_0(\lambda) = e^{-\lambda} - e^{\lambda}, f(\lambda) = \int_{-1}^{1} e^{\lambda t}\Phi(t)\,dt,$$

which each of these functions are entire analytic functions. We estimate the function $\Delta_0(\lambda)$ from below $|\Delta_0(\lambda)| \geq e^{|\lambda|} - e^{-|\lambda|} \geq e^{x} - e^{-x}$.

We study distribution of zeros of the entire function $f(\lambda)$ separately. We divide the segment $[-1, 1]$ into $2m$ equal parts.

Hence, the function $f(\lambda)$ has the following form:

$$f(\lambda) = \int_{-1}^{1} e^{\lambda t}\Phi(t)\,dt$$

$$= \int_{-1}^{\frac{-2(m-1)}{2m}} e^{\lambda t}\Phi(t)\,dt + \int_{\frac{-2(m-1)}{2m}}^{\frac{2(2-m)}{2m}} e^{\lambda t}\Phi(t)\,dt + \int_{\frac{2(2-m)}{2m}}^{\frac{2(3-m)}{2m}} e^{\lambda t}\Phi(t)\,dt + \ldots$$

$$+ \int_{\frac{2(m-1)}{2m}}^{1} e^{\lambda t}\Phi(t)\,dt = \sum_{P=-m+1}^{m} \int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{\lambda t}\Phi(t)\,dt.$$

Transform the function $f(\lambda)$:

$$f(\lambda) = \sum_{P=-m+1}^{m} \int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{\lambda t}\Phi(t)\,dt$$

$$= \sum_{P=-m+1}^{m} \int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{\lambda t}\left[\Phi(t) - \Phi\left(\frac{P}{m}\right) + \Phi\left(\frac{P}{m}\right)\right]dt$$

$$= \sum_{P=-m+1}^{m} \int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{\lambda t} \Phi\left(\frac{P}{m}\right) dt + \sum_{P=-m+1}^{m} \int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{\lambda t} \left(\Phi\left(t\right) - \Phi\left(\frac{P}{m}\right)\right) dt.$$

We show that $f\left(\lambda\right)$ does not have zeros out of the domain ($|x| \leq n \cdot r \cdot \omega\left(\frac{1}{r}\right)$, for some n). Due to Rouchet's theorem [22], we introduce:

$$h\left(\lambda\right) = \sum_{P=-m+1}^{m} \int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{\lambda t} \Phi\left(\frac{P}{m}\right) dt,$$

and

$$G\left(\lambda\right) = \sum_{P=-m+1}^{m} \int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{\lambda t} \left(\Phi\left(t\right) - \Phi\left(\frac{P}{m}\right)\right) dt.$$

Let $\mathrm{Re}\lambda > 0$. We calculate the integrals included in the function $h\left(\lambda\right)$.

$$h\left(\lambda\right) = \sum_{P=-m+1}^{m} \Phi\left(\frac{P}{m}\right) \frac{1}{\lambda}\left(e^{\lambda\frac{P}{m}} - e^{\lambda\frac{P-1}{m}}\right)$$

$$= \frac{1}{\lambda}\left[\Phi\left(-1+\frac{1}{m}\right) \cdot \left(e^{\lambda\left(-1+\frac{1}{m}\right)} - e^{-\lambda}\right)\right.$$

$$+\Phi\left(-1+\frac{2}{m}\right) \cdot \left(e^{\lambda\left(-1+\frac{2}{m}\right)} - e^{\lambda\left(-1+\frac{1}{m}\right)}\right) + \ldots$$

$$+\Phi\left(1-\frac{2}{m}\right) \cdot \left(e^{\lambda\left(1-\frac{2}{m}\right)} - e^{\lambda\left(1-\frac{3}{m}\right)}\right)$$

$$\left.+\Phi\left(1-\frac{1}{m}\right) \cdot \left(e^{\lambda\left(1-\frac{1}{m}\right)} - e^{\lambda\left(1-\frac{2}{m}\right)}\right)\right]$$

$$= \frac{1}{\lambda}\left[e^{\lambda\left(-1+\frac{1}{m}\right)}\left(\Phi\left(-1+\frac{1}{m}\right) - \Phi\left(-1+\frac{2}{m}\right)\right)\right.$$

$$+e^{\lambda\left(-1+\frac{2}{m}\right)}\left(\Phi\left(-1+\frac{2}{m}\right) - \Phi\left(-1+\frac{3}{m}\right)\right)$$

$$+e^{\lambda\left(-1+\frac{3}{m}\right)}\left(\Phi\left(-1+\frac{3}{m}\right) - \Phi\left(-1+\frac{4}{m}\right)\right)$$

$$-\Phi\left(-1+\frac{1}{m}\right)e^{-\lambda}+...$$
$$\left.+\,e^{\lambda\left(1-\frac{1}{m}\right)}\left(\Phi\left(1-\frac{1}{m}\right)-\Phi\left(1\right)\right)+\Phi\left(1\right)\cdot e^{\lambda}\right].$$

Grouping the exponentials in pairs, we have:

$$h\left(\lambda\right)=\frac{1}{\lambda}\left[e^{\lambda\left(-1+\frac{1}{m}\right)}\left(\Phi\left(-1+\frac{1}{m}\right)-\Phi\left(-1+\frac{2}{m}\right)\right)\right.$$
$$-\Phi\left(-1+\frac{1}{m}\right)e^{-\lambda}+\Phi\left(-1\right)e^{-\lambda}-\Phi\left(-1\right)e^{-\lambda}+...$$
$$\left.+e^{\lambda\left(1-\frac{1}{m}\right)}\left(\Phi\left(1-\frac{1}{m}\right)-\Phi\left(1\right)\right)+\Phi\left(1\right)\cdot e^{\lambda}\right]$$
$$=\frac{1}{\lambda}\left[e^{\lambda\left(-1+\frac{1}{m}\right)}\left(\Phi\left(-1+\frac{1}{m}\right)-\Phi\left(-1+\frac{2}{m}\right)\right)\right.$$
$$+e^{-\lambda}\left(\Phi\left(-1\right)-\Phi\left(-1+\frac{1}{m}\right)\right)-\Phi\left(-1\right)\cdot e^{-\lambda}+...$$
$$\left.+\,e^{\lambda\left(1-\frac{1}{m}\right)}\left(\Phi\left(1-\frac{1}{m}\right)-\Phi\left(1\right)\right)+\Phi\left(1\right)e^{\lambda}\right].$$

Denote:

$$h_1\left(\lambda\right)=\frac{1}{\lambda}\left[\Phi\left(1\right)e^{\lambda}-\Phi\left(-1\right)e^{-\lambda}\right]=\frac{1}{\lambda}\left[e^{\lambda}-e^{-\lambda}\right],$$

and

$$g\left(\lambda\right)=\frac{1}{\lambda}\left[e^{\lambda\left(-1+\frac{1}{m}\right)}\cdot\left(\Phi\left(-1+\frac{1}{m}\right)-\Phi\left(-1+\frac{2}{m}\right)\right)\right.$$
$$+e^{-\lambda}\cdot\left(\Phi\left(-1\right)-\Phi\left(-1+\frac{1}{m}\right)\right)+...$$
$$\left.+e^{\lambda\left(1-\frac{1}{m}\right)}\cdot\left(\Phi\left(1-\frac{1}{m}\right)-\Phi\left(1\right)\right)\right]$$
$$=\frac{1}{\lambda}\sum_{P=-m+1}^{m}e^{\lambda\frac{P-1}{m}}\cdot\left(\Phi\left(\frac{P-1}{m}\right)-\Phi\left(\frac{P}{m}\right)\right).$$

Then

$$\mu\left(\lambda\right)=G\left(\lambda\right)+g\left(\lambda\right)$$

$$= \sum_{P=-m+1}^{m} \left(\int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{\lambda t} \left(\Phi(t) - \Phi\left(\frac{P}{m}\right)\right) dt \right.$$
$$\left. + \frac{e^{\lambda\left(\frac{P-1}{m}\right)}}{\lambda} \left(\Phi\left(\frac{P-1}{m}\right) - \Phi\left(\frac{P}{m}\right)\right)\right).$$

We will estimate the function $h_1(\lambda)$ from below, at the same time as the remaining terms, that is, the function $\mu(\lambda)$ is estimated from above:

$$|h_1(\lambda)| = \left|\frac{1}{\lambda} \cdot \left(e^{\lambda} - e^{-\lambda}\right)\right| \geq \frac{1}{|\lambda|} e^{\lambda} - \frac{1}{|\lambda|} \cdot \left| \underset{=}{O}\left(e^{-\lambda}\right)\right|. \tag{6}$$

We estimate from above the function $\mu(\lambda)$:

$$|\mu(\lambda)| \leq \sum_{P=-m+1}^{m} \left[\int_{\frac{P-1}{m}}^{\frac{P}{m}} \left|e^{\lambda t}\right| \cdot \left|\Phi(t) - \Phi\left(\frac{P}{m}\right)\right| dt \right.$$
$$\left. + \frac{e^{\lambda\left(\frac{P-1}{m}\right)}}{|\lambda|} \cdot \left| \Phi\left(\frac{P-1}{m}\right) - \Phi\left(\frac{P}{m}\right)\right|\right]$$
$$\leq \sum_{P=-m+1}^{m} \left[\int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{xt} \cdot \sup_{\frac{P-1}{m} \leq t \leq \frac{P}{m}} \left|\Phi(t) - \Phi\left(\frac{P}{m}\right)\right| dt \right.$$
$$\left. + \frac{e^{x\left(\frac{P-1}{m}\right)}}{|\lambda|} \cdot \sup_{|t-\tau| \leq \frac{1}{m}} |\Phi(t) - \Phi(\tau)| \right].$$

We introduce the modulus of continuity of the function $\Phi(t)$, by using the formula:

$$\omega\left(\frac{1}{m}\right) < \sup_{|t-\tau| \leq \frac{1}{m}} |\Phi(t) - \Phi(\tau)|.$$

Then

$$|\mu(\lambda)| \leq \sum_{P=-m+1}^{m} \left[\int_{\frac{P-1}{m}}^{\frac{P}{m}} e^{xt} \cdot \omega\left(\tfrac{1}{m}\right) dt + \frac{e^{x \cdot \frac{P-1}{m}}}{x} \cdot \omega\left(\tfrac{1}{m}\right)\right] \tag{7}$$
$$= \omega\left(\tfrac{1}{m}\right) \cdot \tfrac{e^{x} - e^{-x}}{x}$$

Therefore, due to (6), (7), we come to the estimation:

$$|f(\lambda)| \geq \frac{1}{|\lambda|}e^{x} - \frac{1}{|\lambda|}\left|\overset{=}{O}\left(\frac{1}{|\lambda|}\right)\right| - \omega\left(\frac{1}{m}\right)\cdot\left(\frac{e^{x}+e^{-x}}{x}\right).$$

.

Considering that $|\lambda| = r$, $m = [r]$, we get:

$$|\lambda f(\lambda)| = |f_1(\lambda)| \geq e^{x} - \frac{e^{x}\cdot\omega\left(\frac{1}{r}\right)\cdot r}{x} - e^{-x} - \frac{e^{-x}\cdot r\cdot\omega\left(\frac{1}{r}\right)}{x}. \qquad (8)$$

For the final statement, we choose n, so that

$$\left|\frac{\omega\left(\frac{1}{r}\right)\cdot r}{x}\right| + e^{-2x} + e^{-2x}\cdot\frac{r\cdot\omega\left(\frac{1}{r}\right)}{x} < \frac{1}{2}$$

when $x > n\cdot r\cdot\omega\left(\frac{1}{r}\right)$. This is possible, because value of the left-hand side of the last inequality is determined in the main first term.

According to the conditions of Rouchet's theorem [31], determining the main part of the function $\Delta_1(\lambda)$, taking into account the lower bounds of the functions $\Delta_0(\lambda)$ and $f_1(\lambda)$ in (8), that is $|\Delta_0(\lambda)| > |f_1(\lambda)|$, we get the following theorem.

Theorem 1. *If the function $\Phi(t)$ is continuous on the segment $[-1,\ 1]$ and satisfies the condition (2), then all zeros of the entire function $\Delta_1(\lambda)$, that is, all eigenvalues of the loaded first-order differential operator L_1 belong to the strip $|\mathrm{Re}\lambda| < n\cdot r\cdot\omega\left(\frac{1}{r}\right)$ for some n, where $\lambda = x + iy$, $\mathrm{Re}\lambda = x$, $\omega(\delta)$ is the modulus of continuity of the function $\Phi(t)$, $r = |\lambda|$.*

Remark 2. If $\Phi(t)$ is continuous on the segment $[-1,\ 1]$ and $\Phi(-1) = \Phi(1) = 1$, all zeros of the entire function $\Delta_1(\lambda)$, that is, all the eigenvalues of the loaded differentiation operator L_1 are concentrated in some vertical strip on the complex plane λ, which expands depending on the properties of the $\omega(\delta)$-modulus of continuity of the function $\Phi(t)$.

Theorem 3. *Let $\Phi(t)$ be a continuous function on $[-1,\ 1]$ and the condition (2) hold. Then the set of zeros of the entire function $\Delta_1(\lambda)$, that is, all the eigenvalues of the loaded ("perturbed") operator L_1 of the spectral problem (3) - (4) form a countable set and the asymptotic formula for the (zeros $\Delta_1(\lambda)$) eigenvalues of the loaded operator L_1 is valid as $n\to\infty$, $\lambda_n = i\pi n + \underset{=}{O}\left(n\omega\left(\frac{1}{n}\right)\right)$, where $\omega(h)$ is the modulus of continuity of $\Phi(t)$.*

Proof. In the proof of Theorem 1, we introduced two functions $h_1(\lambda)$ and $\mu(\lambda)$, such that $f(\lambda) = h_1(\lambda) + \mu(\lambda)$. Zeros of the functions $\Delta_0(\lambda)$ and $h_1(\lambda)$ have the form $\lambda_n^0 = i\pi n$, $n = \pm 1, \pm 2, \ldots$. We consider a square T with a side 2ε, centered at the point λ_n^0 on the complex plane λ. We assume that the sides of T are parallel to the real and imaginary axes of the variable λ. The proof of Theorem 3 consists in choosing ε such that the conditions of Rouchet's theorem [31] are satisfied for the functions $\Delta_0(\lambda)$, $h_1(\lambda)$ and $\mu(\lambda)$ on the sides of the square T. First, we consider the right half of the square T, that is, in this case $\mathrm{Re}\lambda \geq 0$. Now, we divide the side of the square T into two parts $0 \leq \mathrm{Re}\lambda \leq C$ and $C \leq \mathrm{Re}\lambda \leq \varepsilon$, where $C > 0$, the choice of which will be made later.

4.1 case. Let $0 \leq \mathrm{Re}\lambda \leq C$. Since zeros of the functions $\Delta_0(\lambda)$ and $h_1(\lambda)$ identically coincide and these functions are equal to each other, so it is enough to estimate the function $h_1(\lambda)$. Let's compare modules of the functions $h_1(\lambda) \cdot e^{-\lambda}$ and $\mu(\lambda) \cdot e^{-\lambda}$. Taking boundedness of the corresponding derivative into account, we obtain the estimate:

$$\left|h_1(\lambda) \cdot e^{-\lambda}\right| = \left|h_1(\lambda) \cdot e^{-\lambda} - h_1\left(\lambda_n^0\right) \cdot e^{-\lambda_n^0}\right| =$$

$$= \left|\frac{d}{d\lambda} h_1(\lambda) \cdot e^{-\lambda}\right| \cdot \left|\lambda - \lambda_n^0\right| \geq \frac{C_1}{|\lambda|} \cdot \varepsilon.$$

According to boundedness of modules of the exponents, included in $\mu(\lambda)$ we write the inequality:

$$\left|\mu(\lambda) \cdot e^{-\lambda}\right| \leq C_2 \cdot \omega\left(\frac{1}{n}\right).$$

Therefore, for conditions of Rouchet's theorem it is enough to take ε from the requirement:

$$\varepsilon = \underset{=}{\mathrm{O}}\left(n\omega\left(\frac{1}{n}\right)\right),$$

since module λ behaves like $\lambda = n\left(1 + \overset{=}{\mathrm{O}}(1)\right)$.

4.2 case. Let $C \leq \mathrm{Re}\lambda \leq \varepsilon$. For $C > 0$ the module $h_1(\lambda)$ is estimated by the module of one of the exponents included in $h_1(\lambda)$:

$$|h_1(\lambda)| = \left|\frac{e^{\lambda} - e^{-\lambda}}{\lambda}\right| \geq \frac{1}{2} \cdot \frac{e^{x}}{|\lambda|}.$$

.

Note, that C should be chosen from the inequality $C > \ln\varphi$. Since the modules of the exponents included in the function $\mu(\lambda)$ are overestimated by the next exponent e^x, then

$$|\mu(\lambda)| \le e^x \cdot \omega\left(\frac{1}{n}\right) \cdot C_3.$$

.

It implies that for conditions of Rouchet's theorem [31] it is enough to take ε from the constraint of the form:

$$\varepsilon = \underset{=}{\mathrm{O}}\left(n\omega\left(\frac{1}{n}\right)\right).$$

Thus, Theorem 3 is completely proved. $\diamond$

Remark 4. One of the features of the considered problem is that the adjoint to (3) - (4) is the spectral problem for the differentiation operator with a "perturbed" integral boundary value condition:

$$L_1^* v \equiv v'(t) = \lambda v(t), \ -1 \le t \le 1 \tag{9}$$

$$v(-1) = v(1) + \int_{-1}^{1} v(t)\,\Phi(t)\,dt, \tag{10}$$

where $\Phi(t)$ is a continuous function on the segment $[-1, 1]$ and satisfies the condition (2).

5. Case $\Phi(t) \equiv 0$

If $\Phi(t) \equiv 0$, we get that $\Delta_0(\lambda) = e^{-\lambda} - e^{\lambda}$ is a characteristic determinant of the "unperturbed" spectral problem:

$$L_0 y = y'(t) = \lambda y(t), \ -1 \le t \le 1, \ y(-1) = y(1). \tag{11}$$

Eigenvalues of the "unperturbed" operator L_0 are the numbers $\lambda_n^0 = in\pi$, $n = \pm 1, \ \pm 2, \ \pm 3, ...$, and the corresponding eigenfunctions are $y_{n0}^0(t) = C \cdot e^{in\pi t}$, $\forall C > 0$, which form a complete orthonormal system and Riesz basis in the space $L_2(-1, 1)$.

In [21], [32], it is studied the case when $\Phi(t)$ is a function of bounded variation and satisfies the condition (2), where the spectral parameter λ is included in the boundary value condition for integral perturbations, which proves that the systems of eigenfunctions of the "unperturbed" operator $\left\{y_{n0}^0(t)\right\}$ and the "perturbed" operator $\{y_{n1}(t)\}$ are quadratically close in $L_2(-1,\ 1)$, therefore the system $\{y_{n1}(t)\}$ forms a Riesz basis in $L_2(-1,\ 1)$, but are not orthonormal.

The fundamental difference of this work is that the function $\Phi(t)$ is continuous, which causes certain difficulties in the proof.

In the case of $\lambda = 0$, we have $y(t) = C \neq 0$, that is, $\lambda_0 = 0$ is the eigenvalue of the loaded differentiation operator L_1.

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