

MATHEMATICS OF THE 2023 LOCAL ELECTIONS IN BULGARIA

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Abstract: The paper is devoted to the mathematical aspects of the nation-wide local elections held in Bulgaria on October 29 and November 5, 2023. In particular Pearson correlation between voting for municipal mayors and municipal councilors in large municipalities is analyzed. Mobilization coefficients of candidates are introduced and studied for both local and presidential elections. Eligibility margins of independent candidates, reversals in runoffs and paradoxes in local elections are also studied. Most of these issues are considered for the first time. Thus the aphorism “Elections are cross point of law and mathematics” is illustrated.

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1. Introduction

Voting theory is often mathematically trivial but has great political impact. In this paper we study mathematical aspects of nationwide local elections held in Bulgaria in 2023. The structure of these elections is complicated. The country is divided into 28 geographical districts, 27 of which contain one large municipality (Lm). In turn, a Lm includes the main city in the district.

Three Lm are divided into administrative regions (Ar) as follows: Varna has 5 Ar, Plovdiv has 6 Ar and Sofia–Capital has 24 Ar. Lm is the largest municipality in the corresponding geographical district. For example, Varna district contains 12 municipalities (including Varna municipality itself) and 159 villages. Sofia district, different from Sofia–Capital, is not a municipality itself, contains 22 smaller municipalities, and is not analyzed further on.

As a whole, the country is divided into 265 municipalities which differ up to 2 000 times by population(!). A voter may vote for up to five things, including members of municipal council, preference for a member of municipal council, mayor of municipality, mayor of Ar (in Varna and Plovdiv municipalities and in Sofia–Capital) and mayor of small settlement.

Applying statistical methods (including elementary ones) to the analysis of the results of local elections may help political and economical leaders to better understand the underlying processes and tendencies that govern the politics and economics of the country. To present such a brief analysis is the aim of this paper.

2. Notations

We use the following notations: Ar – administrative region; IC – independent candidate; Lm – large municipality; MC – municipal councilor; N – number of MC (varies from 11 to 61); MM – municipal mayor; Q – municipal quota; C_k – candidate k for MM; $Mco(C)$ – mobilization coefficient of candidate C ; $\mu(C) = u/Q$ – eligibility margin of the independent candidate C for MC with u votes; Π_k – political party k .

We also use the following mathematical abbreviations: $\text{flo}(a)$ – rounded value of a in machine arithmetic; $\mathbf{u} = [u_1, u_2, \dots, u_n]$ – vector of restricted votes $u_k = u(k)$ cast for MC; $\mathbf{u} + s$ – vector with elements $u_k + s$, where s is scalar; $\mathbf{v} = [v_1, v_2, \dots, v_n]$ – vector of restricted votes $v_k = v(k)$ cast for MM at first tour; w_k – votes cast for candidate k at runoff; $(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 + \dots + u_nv_n$ – scalar product of vectors $\mathbf{u}$ and $\mathbf{v}$; $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$ – norm of $\mathbf{u}$; $\mathcal{S}(\mathbf{u}) =$

$u_1 + u_2 + \dots + u_n$ – sum of elements of $\mathbf{u}$; $\mathcal{E}(\mathbf{u}) = \mathcal{S}(\mathbf{u})/n$ – mean of $\mathbf{u}$; $\tilde{\mathbf{u}} = \mathbf{u} - \mathcal{E}(\mathbf{u})$ – centered vector with elements $\tilde{u}_k = u_k - \mathcal{E}(\mathbf{u})$; $\mathcal{D}(\mathbf{u}) = \|\tilde{\mathbf{u}}\|/\sqrt{n}$ – standard deviation of $\mathbf{u}$; $H(N, \mathbf{u})$ – vector of seats in N –member municipal council corresponding; $\mathbf{x}$ and $\mathbf{y}$ – m –vectors of total votes for MC and MM, respectively.

The correlation coefficient of vectors $\mathbf{u}$ and $\mathbf{v}$ is $C(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v})/(\|\mathbf{u}\| \|\mathbf{v}\|)$, while Pierson correlation coefficient of $\mathbf{u}$ and $\mathbf{v}$ is $P(\mathbf{u}, \mathbf{v}) = C(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$. The coefficient K of linear regression $\tilde{\mathbf{v}} \approx K\tilde{\mathbf{u}}$ for which $\|\tilde{\mathbf{v}} - K\tilde{\mathbf{u}}\|$ is minimum is $K(\mathbf{v}, \mathbf{u}) = (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})/\|\tilde{\mathbf{u}}\|^2$.

3. Electoral system for MC

3.1. The system

The system for electing MC is a variant of the method of largest remainder, or Hamilton method [1], with threshold $1/N$ and municipal quota Q . Here Q is the upper integer bound of the sum of all votes for MC, divided by N .

A party, or an IC, with u_k votes participates in the distribution of seats in a municipal council if and only if $u_k \geq Q$. If there are $p \leq N$ elected IC then the parties compete for the remaining $N - p$ seats. If more than m parties with equal remainders compete for m seats then a lot is drawn between these parties.

The presence of threshold leads to voting paradoxes [1, 3]. These are true statements that contradict the “common sense” in voting matters. Another paradox is connected to the automatic use of finite arithmetic in voting computations [4], e.g. binary floating-point arithmetic obeying the IEEE Standard [2]. These two paradoxes are considered in the next two subsections.

3.2. Effects of threshold

Voting theory deals with transforming votes cast into seats of collective organs such as national assemblies and municipal councils. One paradox here is that seat distributions can be more sensitive to perturbations in votes when a proportional electoral system with threshold is used rather than when a majority electoral system is applied. In majority systems a change of one vote can change one seat, while in proportional systems a change of one vote can change all seats of the collective body.

Consider for example a municipal council with $N = 61$ seats as in Sofia–

Capital. Suppose that 62 parties Π_k with votes u_k ($k = 1, 2, \dots, 62$) take part in the elections. Let the votes cast be $u_1 = 11$, $u_2 = u_3 = \dots = u_{61} = 10$ and $u_{62} = 0$. We have $\mathcal{S}(\mathbf{u}) = 611$ and the quota Q , the upper integer bound of $611/61 \simeq 10.0164$, is 11. Thus each of the parties $\Pi_2, \Pi_3, \dots, \Pi_{61}$ with 10 votes is under the quota and gets no seats. Party Π_{62} with no votes gets no seats as well. Hence party Π_1 with 11 votes takes all 61 seats.

Let now (say, after a recount) it is established that 1 vote of party Π_{62} was wrongly given to party Π_1 and the correct distribution of votes is $u_1 = u_2 = \dots = u_{61} = 10$ and $u_{62} = 1$. The quota remains 11 but now all parties are below the quota and there is no MC elected! Thus a change of one vote destroyed the whole N -member municipal council and the famous ideal “One Man, One Vote”, considered in [1], became “One Vote, No Man Elected”.

3.3. Effects of rounding

Effects of rounding in automatic voting computations are usually unknown even among specialists and are thus neglected. For example, one of the authors of this paper had been surprised in 2013 to discover such an effect when experimenting with automatic codes for seat distribution in parliamentary elections. The corresponding result was published in [4].

A motivation for such frivolous behavior may be that voting computations include integer data (votes), less than 10^8 , while the rounding unit in double precision floating-point arithmetic is $2^{-53} \simeq 10^{-16}$. It seems that about $8 = 16 - 8$, or at least 7 decimal digits in the computed results should be true. And these digits are indeed true but this is not enough. Because rounded values of equal quantities such as $4/3 - 1$ and $1/3$, which appear in voting computations, may be different and this leads to wrong conclusions about the distribution of seats computed by the method of largest remainder.

Probably such an error occurred in the Bulgarian electoral practice before 2003 when one MC in a small municipality had been determined wrongly. This error went unnoticed and had been discovered after the municipal council was dismissed! At present, all voting computations are done, except automatically, also “by hand” to avoid any error and the severe consequences afterward.

Consider for example the MATLAB[®] code `max` [6] for finding the maximum element of a vector a with elements $a(k)$. If applied as `[A,k] = max(a)` the code returns the maximum element $A = a(k)$ of a and its index k . If there are several elements equal to A , the code returns the smallest index k such that $A = a(k)$. Thus the command `[A,k] = max([1/3,1/3])` returns `A = 0.3333`, `k = 1` as expected. If a has several maximum elements, the code `max` produces

results depending on the way these elements are written! For example, the code $[A, k] = \max([1/3, 4/3-1])$ gives $A = 0.3333$, $k = 1$ as promised.

We may expect that the code $[A, k] = \max([4/3-1, 1/3])$ will give the same result but it *does not*. The computed result is $A = 0.3333$, $k = 2$ and it tells us that the second element is maximal (true) and has index 2 (untrue!). The reason for this wrong conclusion is that the rounded values $\text{flo}(4/3 - 1)$ and $\text{flo}(1/3)$ of the equal quantities $4/3 - 1$ and $1/3$ are different, namely

$$\begin{aligned}\text{flo}(4/3 - 1) &= 6004799503160660 \times 2^{-54} \\ \text{flo}(1/3) &= 6004799503160661 \times 2^{-54}\end{aligned}$$

and $\text{flo}(1/3) = \text{flo}(4/3 - 1) + 2^{-54} > \text{flo}(4/3 - 1)$.

The implementation of the code $[B, k] = \min(a)$ for finding the minimum element $B = a(k)$ of a and its index k meets similar problems. To overcome this phenomenon one must have in mind the following conclusion [5].

Conclusion 1. *Integer modulo $\mathcal{S}(a)$ remainders must be used rather than fractional remainders in the program realization of the Hamilton method.*

4. Correlation of votes for MM and MC

4.1. Basic correlation

Consider a given Lm, where n parties take part in the elections for both MM and MC. We shall analyze correlation between votes cast for MM and MC. For this purpose we consider parties $\Pi_1, \Pi_2, \dots, \Pi_n$, $n \geq 5$ (the last inequality is imposed for statistical credibility), which satisfy the following restrictive condition:

- The party Π_k has won at least one MC and has raised a candidate for MM

Thus parties which have no MC elected and/or no candidate for MM, are not considered in this particular analysis. This requirement seems reasonable but it may be too restrictive as well. For example, in Targovishte municipality there are only 2 parties, satisfying the above restriction, see Table 1.

As mentioned above, there are 27 Lm which elect both MC and MM. One district (Sofia district) consists of 22 smaller municipalities and elects neither common MC nor MM. The smaller municipalities, in turn, elect their own municipal councils and municipal mayors. Thus Sofia district is excluded from our analysis.

Denote votes cast for MC and MM of party Π_k satisfying the restriction as u_k and v_k and let $\mathbf{u} = [u_1, u_2, \dots, u_n]$ and $\mathbf{v} = [v_1, v_2, \dots, v_n]$ be the vectors of votes for MC and MM, respectively. The correlation coefficient of vectors $\mathbf{u}, \mathbf{v}$ is $C(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) / (\|\mathbf{u}\| \|\mathbf{v}\|)$. In general $-1 \leq C(\mathbf{u}, \mathbf{v}) \leq 1$ but here we have $0 < C(\mathbf{u}, \mathbf{v})$ since the elements u_k, v_k are positive.

In a more general statement we consider parties Π_i which have either won at least one MC but have not raised a candidate for MM (i.e. $v_i = 0$), or have raised a candidate for MM but have no MC elected (including the case $u_i = 0$). The votes for MC and MM in this case are called extended votes and are denoted as x_i and y_i , respectively, where $i = 1, 2, \dots, m$ and $m > n$. For example, in Sofia–Capital we have $n = 7$ and $m = 27$.

Another measure for correlation between $\mathbf{u}$ and $\mathbf{v}$ is the Pierson correlation coefficient $P(\mathbf{u}, \mathbf{v}) = C(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$, see [7]. The coefficient $P(\mathbf{u}, \mathbf{v})$ is more suitable than $C(\mathbf{u}, \mathbf{v})$ for voting calculations. It is computed in MATLAB® by the command `Puv = corrcoef(u, v)`, while $C(\mathbf{u}, \mathbf{v})$ is computed as `Cuv = dot(u, v) / norm(u) / norm(v)`. Pierson coefficient varies theoretically between -1 and 1 but in our case only positive values of $P(\mathbf{u}, \mathbf{v})$ had been observed. Hence our next conclusion is as follows.

Conclusion 2. *The Pierson correlation between vectors $\mathbf{u}$ and $\mathbf{v}$ is always positive.*

It is accepted that a positive correlation between $\mathbf{u}$ and $\mathbf{v}$ is weak if $0 < P(\mathbf{u}, \mathbf{v}) \leq 0.3$, moderate if $0.3 < P(\mathbf{u}, \mathbf{v}) \leq 0.7$, strong if $0.7 < P(\mathbf{u}, \mathbf{v}) \leq 0.95$ and very strong if $0.95 < P(\mathbf{u}, \mathbf{v}) \leq 1$. Our next observation confirms that there is strong correlation between voting for municipal councils and municipal mayors in Lm.

Conclusion 3. *The correlation between the vectors $\mathbf{u}$ and $\mathbf{v}$ in almost all large municipalities is strong and very strong.*

For example, in Bourgas municipality we have $P(\mathbf{u}, \mathbf{v}) = 0.9683$ with $n = 9$. In Blagoevgrad municipality we have $P(\mathbf{u}, \mathbf{v}) = 0.9501$ with $n = 3$ although 16 parties have MC elected. Here the inequality $n \geq 5$ is violated and the result in Blagoevgrad cannot be accepted as reliable.

Finally, it is interesting to find a linear approximation of the correlation relation between the centered vectors $\tilde{\mathbf{u}} = \mathbf{u} - \mathcal{E}(\mathbf{u})$ and $\tilde{\mathbf{v}} = \mathbf{v} - \mathcal{E}(\mathbf{v})$ in the form

$$\tilde{\mathbf{v}} \approx K(\mathbf{v}, \mathbf{u})\tilde{\mathbf{u}}$$

Here the quantity $K = K(\mathbf{v}, \mathbf{u})$ minimizes the norm of $\tilde{\mathbf{v}} - K\tilde{\mathbf{u}}$, i.e.

$$K(\mathbf{v}, \mathbf{u}) = \frac{(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})}{\|\tilde{\mathbf{u}}\|^2} = P(\mathbf{u}, \mathbf{v}) \frac{\|\tilde{\mathbf{v}}\|}{\|\tilde{\mathbf{u}}\|}$$

and $K(\mathbf{v}, \mathbf{u})K(\mathbf{u}, \mathbf{v}) = P^2(\mathbf{u}, \mathbf{v})$. For example, in Sofia–Capital we have $P(\mathbf{u}, \mathbf{v}) = 0.9446$ and $K(\mathbf{v}, \mathbf{u}) = 1.0828$.

The correlation coefficients $P(\mathbf{u}, \mathbf{v})$, $C(\mathbf{u}, \mathbf{v})$ and $K(\mathbf{v}, \mathbf{u})$ are presented at Table 1. In five Lm, marked in square brackets, $n \leq 4$ and the computed results may not lead to meaningful conclusions.

4.2. Other correlation relations

Statistical analysis based on other methods such as maximum likely-hood estimation confirm the above conclusions as a whole. A work to be done is to analyze in detail the correlation between voting for mayors of administrative regions (in Sofia-Capital, Plovdiv and Varna) and for municipal council in the corresponding municipality. This should be done not only for the 2023 Bulgarian nationwide local elections but also for the local elections held in 2019 and 2015.

Another interesting phenomenon with political consequences is observed in the largest three municipalities with mayors of administrative regions. It is the correlation between the votes for MC, for MM and the sum of votes for mayors of administrative regions. This will be a subject of future statistical and political analysis.

4.3. If mayors were councilors

Let the vector of votes $\mathbf{u}$ for municipal council with N seats produces a vector $H(N, \mathbf{u})$ of seats $h_k(N, \mathbf{u})$ according to the Hamilton method with natural quota Q . It is interesting to see what would be the distribution of seats in an “artificial” municipal council if we replace the vector $\mathbf{u}$ of votes for MC by the vector $\mathbf{v}$ of votes for MM. Due to the strong correlation between $\mathbf{u}$ and $\mathbf{v}$ we may expect that the corresponding vectors of seats would be close. It turns out that this is not exactly the case for some Lm.

For example, in Sofia–Capital we have

$$\begin{aligned} \mathbf{u} &= [112\,087, 65\,070, 45\,291, 36\,388, 15\,210, 14\,675, 5\,746] \\ \mathbf{v} &= [119\,121, 66\,792, 80\,875, 29\,052, 21\,326, 11\,861, 3\,462] \end{aligned}$$

while the quota is 4828 votes. The official distribution of seats in the 61-member municipal council, corresponding to $\mathbf{u}$, is

$$H(61, \mathbf{u}) = [23, 14, 9, 8, 3, 3, 1]$$

At the same time the “artificial” distribution of seats is

$$H(61, \mathbf{v}) = [22, 12, 15, 6, 4, 2, 0]$$

The last party gets no seats in the artificial distribution because the sum of its “votes” 3462 is now below the new quota of 5911 votes. As expected, the Pearson coefficient 0.9350 of vectors $H(61, \mathbf{u})$ and $H(61, \mathbf{v})$ is less than the Pearson coefficient 0.9450 of $\mathbf{u}$ and $\mathbf{v}$.

An important political conclusion here is that the personal contribution of the candidate V. Grigorova with $v_3 = 80\,875$ votes is

$$h_3(61, \mathbf{v}) - h_3(61, \mathbf{u}) = 15 - 9 = 6$$

seats for the corresponding party.

Conclusion 4. *The result of V. Grigorova as candidate for municipal mayor in Sofia–Capital may be an argument in the inter-party struggle for existence.*

4.4. Total correlation of votes

Correlation of votes for parties which had won at least one MC and had raised a candidate for MM may represent a restrictive picture of the political situation in the given Lm. A less restrictive, or relaxed, approach is to consider all parties which had raised at least one candidate for either MC or MM, or both.

As mentioned above, in Sofia–Capital the number of parties satisfying the restrictive condition is $n = 7$, while the relaxed condition is satisfied by much larger number $m = 27$ of parties. The statistical characteristics in the latter case are interpreted as total correlation of votes.

Let $\mathbf{x}$ and $\mathbf{y}$ be the 27-vectors of votes for MC and MM. The total correlation is characterized by the coefficients

$$P(\mathbf{x}, \mathbf{y}) = 0.9729, \quad C(\mathbf{x}, \mathbf{y}) = 0.9775, \quad K(\mathbf{y}, \mathbf{x}) = 1.1095$$

The corresponding coefficients for the vectors $\mathbf{u}$ and $\mathbf{v}$ are

$$P(\mathbf{u}, \mathbf{v}) = 0.9446, \quad C(\mathbf{u}, \mathbf{v}) = 0.9778, \quad K(\mathbf{v}, \mathbf{u}) = 1.0828$$

Thus in Sofia–Capital the correlation coefficients for data $\mathbf{x}, \mathbf{y}$ and data $\mathbf{u}, \mathbf{v}$ are close. This is not the case in some other Lm.

For example, in Blagoevgrad municipality with $N = 41$ we have $n = 3$ but $m = 24$ with 16 parties having MC. The corresponding relaxed coefficients are much smaller, namely

$$P(\mathbf{x}, \mathbf{y}) = 0.2123, \quad C(\mathbf{x}, \mathbf{y}) = 0.3807, \quad K(\mathbf{y}, \mathbf{x}) = 0.3590$$

and they differ considerably from the restricted ones

$$P(\mathbf{u}, \mathbf{v}) = 0.9951, \quad C(\mathbf{u}, \mathbf{v}) = 0.9501, \quad K(\mathbf{v}, \mathbf{u}) = 4.0700$$

These drastic differences deserve a special political analysis.

5. Mobilization coefficients. Reversals

5.1. Rules for runoff

Let n candidates $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_n$ for mayor in a given electoral region have $v_1 \geq v_2 \geq \dots \geq v_n$ votes, respectively, where $n \geq 1$. The case $n = 1$ is exceptional although it happens in Bulgaria. Here the only candidate $\mathbf{C}_1$ with $v_1 \geq 1$ votes is elected. This case is not interesting and hence we shall suppose that $n \geq 2$.

The candidate $\mathbf{C}_1$ is elected at the first tour if and only if $2v_1 > \mathcal{S}(\mathbf{v})$, or $v_1 > v_2 + v_3 + \dots + v_n$. Equivalently, there is no candidate elected at first tour if and only if $2v_1 \leq \mathcal{S}(\mathbf{v})$. In the latter case there is runoff among the first p candidates $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_p$. The standard runoff is when $p = 2$ and it happens either if $n = 2$ and $v_1 = v_2$, or if $n \geq 3$ and $v_2 > v_3$.

There are also two nonstandard runoffs with $p \geq 3$ candidates. They correspond to distribution of votes either $v_1 > v_2 = v_3 = \dots = v_p > v_{p+1}$, or $v_1 = v_2 = \dots = v_p > v_{p+1}$ (if $n = 3$ then $p = 3$ and inequalities $v_p > v_{p+1}$ above are not applicable).

If there is no candidate elected at runoff then new elections are carried out. In Kingdom of Bulgaria, existing up to 1946, in such case there was a lot. Once in the new history of Bulgaria runoffs with three (or more) candidates were legally possible when e.g. $v_3 > v_4$. This social experiment was later declared as unsuccessful.

5.2. Mco of candidates for MM

Let a candidate $\mathbf{C}$ with v votes goes to runoff and receives there w votes. Note that usually, but not always, we have $w > v$, see Table 4 for two exceptions.

Then the ratio

$$\text{Mco}(\mathbf{C}) = \frac{w}{v}$$

is said to be *mobilization coefficient* of the candidate $\mathbf{C}$.

For example, at 2001 presidential elections, Mco of the winner G. Parvanov and its rival, then acting president P. Stoyanov, were

$$\text{Mco}(\text{GP}) = \frac{2\,043\,443}{1\,032\,665} = 1.9788, \quad \text{Mco}(\text{PS}) = \frac{1\,731\,676}{991\,680} = 1.7462$$

The Mco of G. Parvanov of almost 2 is very high which is typical for candidates of the Bulgarian socialist party. It is worth mentioning that the Mco of almost 1.75 of P. Stoyanov is also decently high.

Conclusion 5. *The mobilization coefficient 1.7462 of P. Stoyanov in 2001 is the highest ever achieved at presidential elections by a democratic candidate.*

Obviously, in the eve of the fatal turnoff on November 18 of 2001, democratic forces in Bulgaria have felt the hazard but it was too late. History had made its next turn.

Another instructive example are the elections for MM in Sofia–Capital when the candidate V. Terziev with 119 121 votes and its rival V. Grigorova with 80 875 votes at first tour went to runoff and received there 175 044 and 170 258 votes, respectively. Thus Mco of V. Terziev and V. Grigorova are

$$\text{Mco}(\text{VT}) = \frac{175\,044}{119\,121} = 1.4695, \quad \text{Mco}(\text{VG}) = \frac{170\,258}{80\,875} = 2.1052$$

V. Grigorova has very high $\text{Mco}(\text{VG}) \approx 2.11$ which confirms our next thesis.

Conclusion 6. *The Bulgarian Socialist Party has the strongest mobilization capability among Bulgarian system parties (among all parties, to be more exact).*

Winners at first tour in eight Lm are shown at Table 2 (here Mco is not relevant). Often, but not always, candidates for MM with higher Mco win runoffs. Candidates with higher Mco may also lose runoffs, see Vidin municipality and Sofia–Capital at Table 3 as well as eight cases at Table 4 (Mco of these candidates are given in *italic*). The same is valid about candidates for president of the Republic, see Table 5.

5.3. Reversals in runoffs

Let candidates $\mathbf{C}_1$ and $\mathbf{C}_2$ for MM with votes v_1 and v_2 at first tour go to runoff. Let the votes for $\mathbf{C}_1$ and $\mathbf{C}_2$ at second tour be w_1 and w_2 . We say that there is

a *reversal* in the voting result if

$$(v_1 - v_2)(w_1 - w_2) < 0$$

Obviously, there is a reversal when either $v_1 > v_2$ and $w_1 < w_2$, or $v_1 < v_2$ and $w_1 > w_2$. In elections for MM both pairs of inequalities above have been observed in contrast to presidential elections.

Conclusion 7. *In presidential elections there are no reversals during the whole period 1992–2021.*

Otherwise speaking, the leader at the first tour becomes president of the Republic of Bulgaria.

For 2023 Bulgarian nationwide local elections, reversals in Lm are shown at Tables 3 and 4.

6. Independent candidates

An independent candidate $\mathbf{C}$ for MC is elected if it has votes u not less than the municipal quota Q , i.e. when

$$\mu(\mathbf{C}) = \frac{u}{Q} \geq 1$$

The quantity $\mu(\mathbf{C})$ is called *eligibility margin* of the candidate $\mathbf{C}$.

IC for MC are rarely elected in Bulgarian local elections. Other elections deserve special mentioning.

Conclusion 8. *In Bulgarian parliamentary elections and in elections for members of the European Parliament from Bulgaria no independent candidate has ever been elected.*

Moreover, such IC receive very small portion (less than 10%) of the votes v necessary for election, where in parliamentary elections v is approximately 10-12 thousands and in European elections v is about 100 to 120 thousands. In contrast, IC for MM and for president are often elected. Another quite different issue is whether or not these IC are really independent. Many are not.

For example, in 2023 nationwide local elections there were five IC for MC in Lm and only one has been elected. The elected candidate P. Pavlov from Shumen municipality has eligibility margin $\mu(\mathbf{PP}) = 2.6293$. The other four IC have eligibility margins varying from 0.6764 to 0.2987.

The problem with IC for MC with u votes is either when μ is slightly less than 1, or when μ is considerably larger than 1. In both cases there is a heavy

loss of votes (if $\mu \ll 1$ then only a small number of votes is “lost”). If μ is slightly less than 1 then all u votes for this IC, almost equal to the quota, are lost.

If μ is slightly more than 1 then the IC is elected and almost no votes are lost. When finally μ is considerably larger than 1 then actually all $u - Q$ votes for this IC are lost. For example, the IC B. Bonev in Sofia–Capital in 2019 local elections had lost votes enough for electing seven more MC!

In contrast to elections for collective bodies, IC are more successful in majority elections for all kinds of mayors as well as for president. The reason is that in such elections party candidates are often disguised as independent. Although this practice is spread worldwide, it is hard to consider it morally justified.

7. Presidential elections

There had been seven presidential elections in Bulgaria in the thirty year period 1992–2021, see Table 5. Problems arising in presidential elections are, to a certain extent, similar to these in elections for MM. According to the Constitution and the Electoral Code of Bulgaria a candidate for president is elected if 1) he/she has received more than half of the votes cast, and 2) more than half of the voters have voted. If there is no candidate elected at the first tour then there is a runoff, where rule 2) is not applied according to a Decision of the Constitutional Court. In 2006 presidential elections there was runoff because of violation of condition 2) although the then leading candidate had more than half of the votes cast at first tour.

A reversal in presidential elections is defined as in elections for mayors. It must be stressed that no such reversal occurred in all seven presidential elections in Bulgaria. So the rule, up to now, is according to the next conclusion.

Conclusion 9. *The one who leads at the first tour becomes a president at the second.*

Mco of presidential candidates are given at Table 5. The winner had higher Mco in 2001, 2006 and 2016 elections. In other presidential elections (1992, 1996, 2011 and 2021) the defeated candidate had higher Mco. Thus there is no clear relation between the Mco of candidates and the final results of presidential elections.

It is a common practice in Bulgarian presidential elections to declare party candidates as independent. We have already commented this morally controversial practice.

8. Data analysis

8.1. The data

Bulgaria is among the countries with best officially documented results for all kinds of elections: local, parliamentary (including elections for European parliament), presidential and national referendums. As a whole there are eight types of such elections. The data encompasses all voting sections and all candidates since 1991 and is available from both paper editions and internet sources, see e.g. `portal.cik.bg` for elections since 2003.

Official data for our investigation is given in [8] for first tour of 2023 nationwide local elections, and in [9] for runoffs. Below we give only names of candidates and rarely mention their party affiliation.

8.2. Brief analysis

The only successful IC for MC is P. Pavlov from Shumen municipality. His votes correspond to 2.63 times the quota and would give 3 seats in the municipality council if he had participated “as a party”. Hence more than half of his votes are actually lost.

However, the most remarkable loss of votes ever had been achieved by the IC for MC B. Bonev in 2019. In Sofia–Capital he received 42 759 votes (corresponding to 4th place among parties) which would give him 8 seats had he “participated as a party”. Thus he lost votes enough for electing 7 MC in Sofia–Capital! Had he registered differently, fate of Sofia could also be different in the period 2019–2023.

Table 1 presents correlation coefficients $P(\mathbf{u}, \mathbf{v})$, $C(\mathbf{u}, \mathbf{v})$ and coefficients $K(\mathbf{v}, \mathbf{u})$ for all 27 Lm. There are five Lm with insufficient data (Targovishte, Silistra, Pernik, Blagoevgrad and Haskovo municipalities, marked in square brackets) because of small samples $n \leq 4$, which are excluded from consideration. We have

$$\begin{aligned} 0.8199 \text{ (Pleven)} &\leq P(\mathbf{u}, \mathbf{v}) \leq 0.9982 \text{ (Vratsa)} \\ 0.9374 \text{ (Pleven)} &\leq C(\mathbf{u}, \mathbf{v}) \leq 0.9988 \text{ (Yambol)} \\ 1.0708 \text{ (Razgrad)} &\leq K(\mathbf{v}, \mathbf{u}) \leq 3.0658 \text{ (Stara Zagora)} \end{aligned}$$

Table 2 gives results for eight Lm, where MM are elected at first tour. The best result belongs to S. Vladimirov in Pernik municipality with 79.32% of votes. Almost as high is the second result of 79.15% of V. Revanski in Yambol municipality.

Denote the 8-vector of percents of winners at first tour as

$$\mathbf{p} = [79.32, 79.15, 69.09, 67.01, 62.70, 60.49, 60.25, 55.64]$$

The average and standard deviation of $\mathbf{p}$ are $\mathcal{E}(\mathbf{p}) = 66.71\%$ and $\mathcal{D}(\mathbf{p}) = 8.77\%$. Thus we can make the following conclusion.

Conclusion 10. *The winners at first tour of the elections for mayor of municipality have an average of about 2/3 of the votes.*

Table 3 contains data for eight Lm, where $\text{Mco} > 2$ for either the winner (six cases) or its rival (two cases). The astonishing result $\text{Mco}(\text{PK}) = 3.0442$ of P. Kulenski in Pazardjik municipality needs special attention. Also very impressive is the second higher $\text{Mco}(\text{OA}) = 2.6833$ of O. Atanasov in Kyustendil municipality.

These two results shall be analyzed elsewhere from political viewpoint. We only mention that defeated ex-mayors T. Popov and P. Paunov in these municipalities have had long but controversial terms of office. They managed to mobilize powerful protest votes at runoffs following the principle “All Versus Old Mayor”. Similar effect in Sofia–Capital appeared as “All Versus Old Party”. As a result the candidate A. Hekimyan of the Old Party (with Bulgarian abbreviation GERB) went third.

It must be stressed that the high $\text{Mco}(\text{AM}) = 1.7499$ of A. Mateev in Vidin municipality and $\text{Mco}(\text{VG}) = 2.1052$ of V. Grigorova in Sofia–Capital were not enough for reversal due to their relatively low initial performance at first tour.

In six cases of eight (exceptions are Vidin municipality and Sofia–Capital) the candidate with higher Mco had won the runoff. In most discussed case Sofia–Capital the winner V. Terziev has $\text{Mco}(\text{VT}) = 1.4695$ which is considerably less than $\text{Mco}(\text{VG}) = 2.1052$ of its rival V. Georgieva. In Lm at Table 3 there are four reversals in Pazardjik, Kyustendil, Razgrad and Lovech municipalities due to high Mco of the corresponding candidates.

Table 4 presents data for Mco for the rest 11 Lm. Mco of candidates in these Lm are less than 2 for both winners and their rivals. Here the tendency that winners at runoffs have larger Mco is reversed. Only three of all 11 winners (in Varna, Blagoevgrad and Silistra municipalities) at runoffs have larger Mco . Larger Mco of defeated candidates are marked in *italic*. These eight defeated candidates have Mco from 1.9729 to 1.4055.

Another interesting phenomenon is that Mco of winners in Veliko Tarnovo and Haskovo municipalities, marked in **bold**, are less than 1 which is rarely met. In Lm shown at Table 4 there are reversals only in Varna and Blagoevgrad

municipalities. As a whole, there are six reversals in all 19 Lm with runoffs, or in about 1/3 of all cases.

Mco of candidates for president are given at Table 5. In all cases there was runoff, one due to low voting rate (2006). The cases when losing candidate has higher Mco are marked in *italic*. The 2001 elections have been already discussed.

The 2006 elections present a special end very educative case. Due to weakness of democratic opposition its candidate went third. So the candidate V. Siderov went to runoff and got weakest result ever (24% of votes in runoff and $\text{Mco}(\text{VS}) \approx 1.09$), while acting president won 76% of votes which is the best result ever. Remarkably, leaders of so called “democratic community” then appealed in favor of the socialist candidate G. Parvanov.

Highest $\text{Mco}(\text{RR}) \approx 2.12$ ever was achieved by R. Radev in 2016. The second weakest result (32% of votes in the runoff and $\text{Mco}(\text{AG}) \approx 1.20$) belongs to A. Gerdjikov in 2021.

In six cases of seven, with the exception of 2021 elections, the Mco of the socialist candidate has been higher. As was in 2023 elections for MM in Sofia–Capital, to pinpoint again our thesis of *excellent mobilization capacity of the Bulgarian Socialist Party*.

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Municipality	n	$P(\mathbf{u}, \mathbf{v})$	$C(\mathbf{u}, \mathbf{v})$	$K(\mathbf{v}, \mathbf{u})$
[Targovishte]	[2]	[1.0000]	[0.9839]	[3.5594]
[Silistra]	[3]	[1.0000]	[0.9938]	[1.9049]
[Pernik]	[4]	[0.9983]	[0.9821]	[4.2426]
Vratsa	5	0.9982	0.9890	2.4462
Montana	5	0.9973	0.9984	1.5303
Yambol	5	0.9972	0.9988	2.0039
Kardjali	5	0.9964	0.9979	1.1417
Sliven	6	0.9956	0.9902	1.7459
[Blagoevgrad]	[3]	[0.9951]	[0.9501]	[4.0700]
Gabrovo	7	0.9943	0.9956	1.4401
Razgrad	6	0.9925	0.9972	1.0708
Pazardjik	10	0.9877	0.9816	2.3179
[Haskovo]	[4]	[0.9841]	[0.9926]	[1.7682]
Stara Zagora	10	0.9807	0.9552	3.0658
Veliko Tarnovo	6	0.9795	0.9849	1.9045
Shumen	8	0.9723	0.9863	1.7321
Bourgas	9	0.9683	0.9952	2.3026
Vidin	8	0.9669	0.9795	1.8188
Dobrich	9	0.9592	0.9839	1.3199
Smolyan	6	0.9582	0.9911	1.5587
Plovdiv	8	0.9577	0.9671	1.7529
Varna	7	0.9458	0.9785	1.9349
Sofia–Capital	7	0.9446	0.9778	1.0828
Ruse	9	0.9353	0.9510	2.0424
Kyustendil	7	0.9306	0.9766	1.3812
Lovech	7	0.8969	0.9713	1.4037
Pleven	9	0.8199	0.9374	1.3072

Table 1: Correlation coefficients

Municipality	Mayor elected	Votes	% of votes
Pernik	S. Vladimirov	28 514	79.32
Yambol	V. Revanski	16 813	79.15
Targovishte	D. Dimitrov	13 119	69.09
Vratsa	K. Kamenov	17 543	67.01
Montana	Z. Jivkov	11 701	62.70
Bourgas	D. Nikolov	37 552	60.49
Kardjali	E. Myumyun	15 588	60.25
Stara Zagora	J. Todorov	28 007	55.64

Table 2: First round winners

Municipality	Candidates	w	v	Mco
Pazardjik (reversal)	P. Kulenski	17 702	5 815	3.0442
	T. Popov	14 425	14 131	1.0208
Kyustendil (reversal)	O. Atanasov	15 319	5 709	2.6833
	P. Paunov	6 609	6 250	1.0574
Pleven	V. Hristov	21 824	8 288	2.6332
	G. Spartanski	11 494	7 833	1.4674
Razgrad (reversal)	D. Dobrev	8 009	3 250	2.4643
	M. Grancharov	6 374	4 097	1.0208
Vidin	Ts. Tsenkov	9 389	7 294	1.2872
	A. Mateev	6 543	3 739	<i>1.7499</i>
Shumen	H. Hristov	15 574	6 814	2.2856
	G. Kolev	5 646	5 495	1.0275
Lovech (reversal)	S. Petkov	7 425	3 262	2.2762
	V. Nedyalkova	6 541	5 096	1.2836
Sofia–Capital	V. Terziev	175 044	119 121	1.4695
	V. Grigorova	170 258	80 975	<i>2.1052</i>

Table 3: Runoffs with Mco > 2

Municipality	Candidates	w	v	Mco
Varna (reversal)	B. Kocev	45 395	22 893	1.9829
	I. Portnih	35 519	28 206	1.2593
Gabrovo	T. Hristova	8 570	7 765	1.1037
	D. Petrov	3 715	1 883	<i>1.9729</i>
Smolyan	N. Melemov	6 258	4 045	1.5471
	S. Sabrutev	6 109	3 154	<i>1.9369</i>
Blagoevgrad (reversal)	M. Baykushev	13 611	7 498	1.8153
	I. Stoynov	12 805	9 333	1.3720
Dobrich	Y. Yordanov	9 071	5 854	1.5495
	K. Nikolov	8 055	4 708	<i>1.7109</i>
Plovdiv	K. Dimitrov	37 041	32 570	1.1373
	I. Staribratov	28 511	17 295	<i>1.6485</i>
V. Tarnovo (Mco < 1)	D. Panov	11 065	11 406	0.9701
	Y. Terzyiski	8 798	5 645	<i>1.5585</i>
Sliven	S. Radev	18 271	13 395	1.3640
	P. Stoyanov	11 669	7 730	<i>1.5096</i>
Haskovo (Mco < 1)	S. Dechev	16 241	16 429	0.9886
	N. Stavrev	10 053	6 912	<i>1.4544</i>
Ruse	P. Milkov	17 662	15 267	1.1569
	I. Ivanov	8 852	6 298	<i>1.4055</i>
Silistra	A. Sabanov	9 231	8 388	1.1005
	I. Ivanov	6 778	6 383	1.0619

Table 4: Runoffs with Mco < 2

Year	Candidates	% at runoff	Mco
1992	J. Jelev	52.85	1.2045
	V. Valkanov	47.15	<i>1.5764</i>
1996	P. Stoyanov	59.73	1.3242
	I. Marazov	40.27	<i>1.4568</i>
2001	G. Parvanov	54.06	1.9788
	P. Stoyanov	45.94	1.7462
2006	G. Parvanov	75.95	1.1519
	V. Siderov	24.05	1.0874
2011	R. Plevneliev	52.58	1.2584
	I. Kalfin	47.42	<i>1.5716</i>
2016	R. Radev	62.15	2.1186
	Ts. Tsacheva	37.85	1.4947
2021	R. Radev	67.72	1.1643
	A. Gerdjikov	32.28	<i>1.2012</i>

Table 5: Mco of candidates for president

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