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GENERALIZED RELAXATION FRACTIONAL DIFFERENTIAL MODEL OF FLUID FILTRATION IN A POROUS MEDIUM

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Abstract

The process of anomalous filtration of a homogeneous liquid in a porous medium is modeled by differential equations with a fractional derivative. Fractional derivatives are used as defined by Caputo. The problem of filtration in a finite homogeneous reservoir is posed and numerically solved. The influence of process abnormality on filtration characteristics was estimated. It is shown that a decrease in the exponent of the derivative in the relaxation term with respect to pressure leads to the decrease of the pressure distribution up to a certain distance from the beginning of the medium, and then to an increase. Reducing the order of the derivative in the relaxation term with respect to the filtration velocity acts inversely. The corresponding dynamics with decreasing orders of derivatives las the filtration velocity. As a special case, the case with the predominance of the filtration velocity relaxation time over the pressure relaxation time is singled out, in particular, when the latter is equal to zero. In this case, the solution of the filtration equation acquires a wave character. With an increase in the difference between relaxation times in terms of filtration velocity and pressure, the propagation velocity of pressure waves decreases.

Math. Subject Classification: 76T99

Key Words and Phrases: fractional derivative, finite difference method, relaxation time, anomalous filtration, porous medium

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1. **Introduction**

Mathematical modeling of the dynamics of geofiltration processes in complex conditions of their flow is one of the topical areas of geomathematics, geoinformatics, geomechanics, which develops mainly within the framework of classical problem formulations based on generally accepted methods and approaches of the continuum theory [13, 24]. At the same time, most of the known mathematical models of the processes of motion of media in geoporous media are based on classical balance laws, which are inadequate under conditions of a significant deviation of the system from the equilibrium state [12, 18, 27]. In addition, classical filtration models postulate such a very strict restriction on processes as an infinite perturbation propagation velocity, which contradicts real physical concepts.

Attempts to theoretically take into account the effects of nonequilibrium (in particular, memory effects) in nonstationary filtration in a porous medium led to the creation of the theory of relaxation filtration, the first most complete exposition of which, apparently, is contained in the well-known work [18].

An effective modern approach to describing transport processes in systems for which it is important to take into account nonlocal space-time properties is associated with the use of non-integer-order integro-differentiation apparatus [4,5,9,10,19,26]. So, for example, in [4], mathematical models were developed and solutions were obtained for some filtration boundary value problems on modeling the fractional-differential dynamics of relaxation filtration processes in porous and fractured-porous massifs of finite thickness, and in [5] the problem of modeling fractional differential dynamics of the relaxation filtration process in the presence of nonlocal boundary conditions. We also note the work [6] on mathematical modeling of the fractional differential dynamics of relaxation processes of convective diffusion of soluble substances in underground seepage flows.

The use of the classical theory of filtration of homogeneous liquids in an elastic regime based on Darcy′s law sometimes leads to a discrepancy with real data in an elastic regime [1, 18]. Inconsistencies are observed especially in strong non-stationary filtration modes, in the filtration of high-viscosity oil, oil and gas in loamy low-permeability rocks, etc. Under these conditions, the equilibrium character of Darcy′s law is usually violated [1,20,25]. In these works, relaxation phenomena are taken into account in the Darcy′s law during filtration in porous media. Based on these studies, the influence of relaxation parameters on filtration characteristics, such as filtration velocity, pressure, etc, was established.

Many natural porous media have a fractal structure, the modeling of filtration processes in which requires the use of new approaches, methodologies and methods of analysis that differ significantly from traditional ones. As applied to the oil and gas industry, this means that the classical methods of field development based on the theory of fluid flow through homogeneous porous media are insufficient in this case [1, 18].

The relaxation theory of fluid filtration as a non-classical anomalous filtration was developed in [2,18]. In relaxation models of filtration, the apparatus of fractional derivatives was used in [9, 27].

In this work, in contrast to [7], we consider a generalized relaxation fractional differential model, which simultaneously takes into account relaxation phenomena both in the filtration velocity and in the pressure gradient. On the basis of such a generalized model, filtration equations are derived. The filtration problem for this equation is posed and numerically solved. The influence of the orders of fractional derivatives on the distribution of pressure and filtration velocity in the medium at different moments of time is estimated.

2. **Statement of the problem**

The filtration model with double relaxation in the one-dimensional case has the form ([2])

$$\upsilon + \lambda_\upsilon \frac{\partial \upsilon}{\partial t} = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x} + \lambda_p \frac{\partial^2 p}{\partial x \partial t}\right), \tag{1}$$

λ_υ,λ_p - relaxation times of filtration velocity, υ and pressure p, k-medium permeability, μ-liquid viscosity.

The continuity equation is written as

$$\frac{\partial \upsilon}{\partial x} + \beta^* \frac{\partial p}{\partial t} = 0, \tag{2}$$

where β^* is the coefficient of reservoir elasticity.

Equation (1) is written here in a generalized form

$$\upsilon + \lambda_\upsilon D_t^\beta \upsilon = -\frac{k}{\mu}\frac{\partial}{\partial x}\left(p + \lambda_p D_t^\alpha p\right), \tag{3}$$

where D_t^β , D_t^α are Caputo fractional derivative operators [9] .

Differentiating equation (3) with respect to the coordinate x, we obtain

$$\frac{\partial \upsilon}{\partial x} + \lambda_\upsilon \frac{\partial}{\partial x} D_t^\beta \upsilon = -\frac{k}{\mu}\frac{\partial^2}{\partial x^2}\left(p + \lambda_p D_t^\alpha p\right). \tag{4}$$

Taking from equation (2) the derivative of order β with respect to time, we obtain

$$D_t^\beta \frac{\partial \upsilon}{\partial x} + \beta^* D_t^{\beta+1} p = 0, \tag{5}$$

and using equation (5), we write equation (4) in the following form

$$\frac{\partial p}{\partial t} + \lambda_\upsilon D_t^{\beta+1} p = \kappa\left(\frac{\partial^2 p}{\partial x^2} + \lambda_p D_t^\alpha\left(\frac{\partial^2 p}{\partial x^2}\right)\right), \tag{6}$$

where $\kappa = \frac{k}{\mu\beta^*}$ is the coefficient of piezoconductivity, $0 < \alpha \leq 1$, $0 < \beta \leq 1$.

For $\alpha = 1$, $\beta = 1$ from (6) we obtain the relaxation filtration equation [2, 18]

$$\frac{\partial p}{\partial t} + \lambda_v \frac{\partial^2 p}{\partial t^2} = \kappa \left(\frac{\partial^2 p}{\partial x^2} + \lambda_p \frac{\partial^3 p}{\partial t \partial x^2} \right). \tag{7}$$

The initial and boundary conditions for filtration in the finite medium $[0, L]$ are taken in the following form

$$p(0, x) = 0, \tag{8}$$

$$p(t, 0) = p_0,\ p_0 = const,\ p(t, L) = 0. \tag{9}$$

For (6), for $\beta > 0$ the initial condition (8) is not enough. We need to add one more condition, for example, for

$$\frac{\partial p(0, x)}{\partial t} = 0. \tag{10}$$

Equation (6) is solved under conditions (8), (9), (10).

3. Numerical solution of the problem.

For the numerical solution of the problem (6), (8), (9), (10) we use the method of finite differences. In the domain, $\Omega = \{0 \le x \le L, 0 \le t \le T\}$ we introduce a uniform grid $\omega_{h\tau} = \{(x_i, t_j), x_i = ih, i = \overline{0, N}, h = L/N, t_j = j\tau, j = \overline{0, M}, \tau = T/M\}$, where h is the grid step in coordinate x,τ is the grid step in time [8]. We denote the grid function at a point (x_i, t_j) by p_i^j.

The finite difference approximation of equation (6) has the form

$$\frac{p_i^{j+1} - p_i^{j+1}}{\tau} + \frac{\lambda_v \tau^{2-\beta}}{\Gamma(3-\beta)} \sum_{k=0}^{j-2} \frac{p_i^{k+1} - 2p_i^k - p_i^{k-1}}{\tau^2}$$

$$\frac{\lambda_v \tau^{2-\beta}}{\Gamma(3-\beta)} (((j-k+1)^{2-\beta} - (k-1)^{2-\beta}) + \frac{p_i^{j+1} - 2p_i^j + p_i^{j-1}}{\tau^2})$$

$$= \kappa \left(\frac{p_{i+1}^{j+1} - 2p_i^{j+1} + p_{i-1}^{j+1}}{h^2} + \frac{\lambda_p \tau^{1-\alpha}}{\Gamma(2-\alpha) h^2} (S_1 - 2S_2 + S_3) \right), \tag{11}$$

$$S_1 = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j-2} \frac{p_{i+1}^{k+1} - p_{i+1}^k}{\tau} \left((j-k+1)^{1-\alpha} - (j-k)^{1-\alpha} \right)$$

$$+ \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \frac{p_{i+1}^{j+1} - p_{i+1}^j}{\tau},$$

$$S_2 = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j-2} \frac{p_i^{k+1} - p_i^k}{\tau} \left((j-k+1)^{1-\alpha} - (j-k)^{1-\alpha} \right)$$

$$+ \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \frac{p_i^{j+1} - p_i^j}{\tau},$$

$$S_3 = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j-2} \frac{p_{i-1}^{k+1} - p_{i-1}^{k}}{\tau} \left((j-k+1)^{1-\alpha} - (j-k)^{1-\alpha}\right) + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \frac{p_{i-1}^{j+1} - p_{i-1}^{j}}{\tau},$$

where $\Gamma(\cdot)$ is the Gamma function.

When approximating the fractional derivatives in (11), the methodology of [11,13-17,21-23,28,29] was used.

With a known pressure field, from (3) it is possible to determine the field of filtration velocity. Since the elastic filtration regime is considered, i.e. the compressibility of the porous medium and liquid is taken into account, it is natural to expect a non-uniform distribution of the filtration velocity field in time. Relaxation phenomena, as shown above, have a significant effect on the pressure field. Accordingly, the filtration velocity field changes depending on the relaxation properties of the filtration law.

With known $p(t,x)$ the filtration velocity is determined from (3), after discretization, it takes the form

$$S_4 = \sum_{k=0}^{j-1} (v_i^{k+1} - v_i^{k})((j-k+1)^{1-\beta} - (j-k)^{1-\beta})$$

$$v_i^{j+1} + \frac{\lambda_v}{\Gamma(2-\beta)\tau^{\beta}} (S_4 + (v_i^{j+1} - v_i^{j})) = -\frac{k}{\mu h} (p_{i+1}^{j+1} - p_i^{j+1} + \lambda_p (S_1 - S_2)). \quad (12)$$

The filtration velocity on the upper time layer from (12) is defined as

$$v_i^{j+1} = \frac{\Gamma(2-\beta)\tau^{\beta}}{(\Gamma(2-\beta)\tau^{\beta} + \lambda_v)} \left(-\frac{k}{\mu h} (p_{i+1}^{j+1} - p_i^{j+1} - \lambda_v (S_1 - S_2)) \right) + \frac{\Gamma(2-\beta)\tau^{\beta}}{(\Gamma(2-\beta)\tau^{\beta} + \lambda_v)} \left(\frac{\lambda_v v_i^{j}}{\Gamma(2-\beta)\tau^{\beta}} - \frac{\lambda_v S_4}{\Gamma(2-\beta)\tau^{\beta}} \right). \quad (13)$$

Some results of calculations according to (13) with the same initial parameters used in determining the pressure field are given below.

4. **Results and discussion**

Some results of numerical calculations according to (11) are shown in Figs. 1-18. The following values of the initial parameters were used in the calculations: $k = 10^{-13} m^2$; $\mu = 10^{-4} Pa \cdot s$; $p_0 = 5 \cdot 10^5 Pa$; $\beta^* = 3 \cdot 10^{-8} Pa^{-1}$; $L = 40m$.

Figure 1a shows the pressure profiles for different values of β when $\alpha = 0.7$. As the order of the derivative β decreases from 1, a relatively slow propagation of the pressure profile is observed. Moreover, such a slowdown is noted starting from a certain distance from the beginning of the medium (from the entrance to the medium where pressure p_0 is applied). For small x values for reduced values β (from 1), slightly overestimated pressure values are obtained. Consequently, when the order of the derivative β decreases, the pressure distribution changes from overestimated to underestimated. We also analyzed the case of a decrease in the order of the derivative α from 1 for a fixed value of (Fig. 1b). In this case, the reverse dynamics of change occurs compared to the decrease of β at a fixed value of α. Up to a certain point x,the pressure takes on lower values when α decreases, and then higher values are obtained. Here, too, there is a change in the mode of pressure change, only from low to high. Comparing Fig. 1a with Fig. 1b, one can see that such a regime change occurs at a relatively large x with decrease in α than with a decrease in β.So in the first case (Fig. 1a) if the mode change occurs at $x \approx 5 - 6m$, then in the second case (Fig. 1b) it occurs at $x \approx 10m$.

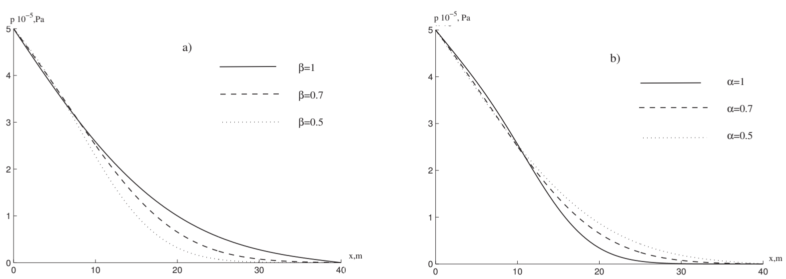

Fig 1. Pressure profiles at various β, α and $t = 3600s, \lambda_v = 1000s, \lambda_p = 500s$, a) $\alpha = 0.7$, b) $\beta = 0.7$.

The pressure field was also studied for various values of the relaxation parameters λ_v and λ_p for given values of the orders of the derivatives α and β. Figure 2a shows the pressure profiles for various λ_v. As it can be seen from the graphs, an increase in the relaxation time with respect to the filtration velocity leads, starting from some x to a slowdown in the development of pressure profiles. In this respect, the action λ_p is completely analogous to the action of the order of the derivative β. Only this effect is observed with a decrease β from 1 and here with an increase in λ_v. The change in the mode of change p occurs approximately at the same x (compare Fig. 1a and Fig. 2a). Pressure profiles at various λ_p and fixed values of other parameters are shown in Fig. 2b. The same changes occur in the pressure profiles as the order of the derivative α from 1 decreases. In this case, the change in the mode of pressure change has the same character as when the values of β decrease from 1. With an increase λ_p a more progressive pressure distribution is observed. The case $\lambda_p = 0$ with $\lambda_v \neq 0$ significantly differs from the cases $\lambda_p \neq 0, \lambda_v \neq 0$ (Fig. 2b). At $\lambda_p = 0$ equation (7) transforms into a wave equation with resistance. Naturally, in this case, the nature of the solution will change greatly, the solution, acquiring a wave character, has a finite propagation velocity in the region. The propagation speed of the leading edge of the solution for the case $\alpha = 1, \beta = 1$ is determined by the value $\sqrt{\frac{\kappa}{\lambda_v}}$. In Fig. 2b, this case corresponds to a solid line, which has a finite propagation velocity, when $t = 3600s$ the leading edge of the wave reaches - 22m, i.e. the pressure front does not reach the end of the medium. Of course, at $\alpha \neq 1, \beta \neq 1$ the nature of the propagation of the pressure wave, in particular, its speed, will change.

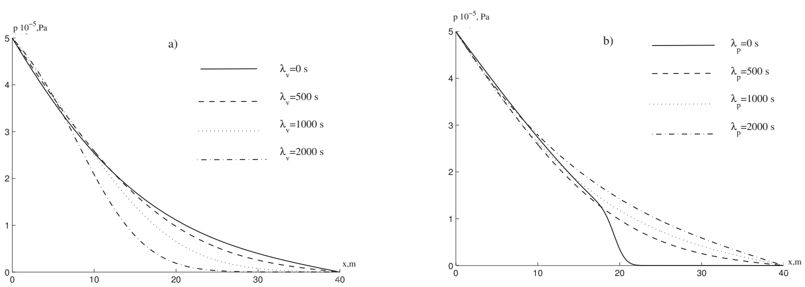

Fig 2. Pressure profiles $t = 3600s, \alpha = 0.7, \beta = 0.7$ and various λ_v, λ_p, a) $\lambda_p = 500s$ b) $\lambda_v = 500s$.

In order to indicate the development of pressure profiles over time, calculations were carried out for given parameters of the problem for various values of time. Some of the results are shown in Figure 3. Obviously, within the framework of the task set, with increasing time t the pressure field will develop, taking on ever greater values. For long times, when all relaxation and non-stationary transient processes are completed, a stationary linear pressure distribution is established.

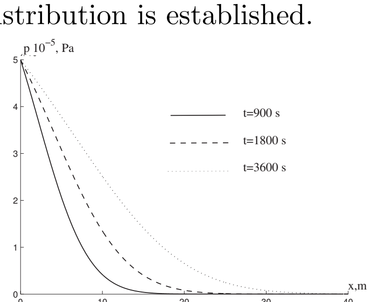

Fig 3. Pressure profile at $\lambda_v = 1000s, \lambda_p = 500s, \alpha = 0.7, \beta = 0.7$ at different times.

We also analyzed the case of a decrease in the order of the derivative α from 1 for a fixed value of β and $\lambda_v = 0$ (Fig. 4a). It is clear that in this case there are only relaxation effects in pressure. In contrast to Fig. 1b, a more progressive distribution of pressure profiles is obtained here. This is due to the fact that here $\lambda_v = 0$. As shown above, as $\lambda_v = 0$ increases the velocity of propagation of pressure waves (or pressure profiles) decreases. Therefore, the case $\lambda_v = 0$ corresponds to the least influence of the inhibitory effect λ_v. In this case, a decrease α from 1 acts as a favorable factor for the propagation of pressure profiles. Similarly, the results on the action of filtration velocity relaxation alone in the absence of pressure relaxation are shown in Fig. 4b. Here, the final velocity of propagation of pressure waves is clearly traced. With decreasing β the speed of wave propagation decreases. In this case, up to certain x, the pressure is lower at relatively large β, and then lower. Therefore, a decrease in β from 1 leads to a slowdown in the propagation velocity of pressure waves. With a decrease in β from 1, the mode change from larger to smaller relative to the result $\beta = 1$ occurs at smaller x (Fig. 4b). The calculations performed for other combinations of values of the relaxation times λ_v and λ_p show a similar effect of the change in α and β on the filtration characteristics.

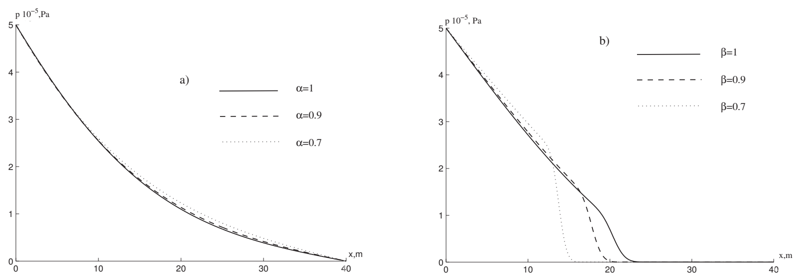

Fig 4. Pressure profiles at various α, β and $t = 3600s$ a) $\lambda_v = 0s, \lambda_p = 1000s, \beta = 1$ b) $\lambda_v = 1000s, \lambda_p = 0s, \alpha = 1$

Calculations for the equal values λ_v and λ_p with decreasing values of α and β from 1 show (Figs. 5a, 5b) that the influence of the parameters is similar to the previous cases. However, they do not change profiles as much as in the $\lambda_p \neq \lambda_v$. The case $\alpha = 1$ (Fig.5a) and $\beta = 1$ (Fig. 5b) give the same results as the classical case, i.e. the case of the absence of relaxation effects. The profiles shown by the solid line in Figs. 5a and 5b coincide with the results of $\lambda_p = \lambda_v = 0, \alpha = \beta = 1$. This suggests that when the relaxation times are equal and the orders of the derivatives α and β are equal, relaxation effects do not appear. However, if $\lambda_p = \lambda_v$ and $\alpha \neq \beta$, then due to different values of derivatives $D_t^{\beta+1}p$ and $D_t^{\alpha}\left(\frac{\partial^2 p}{\partial x^2}\right)$ relaxation effects appear. In this case, the predominant influence is naturally exerted by that term from $D_t^{\beta+1}p$ and $D_t^{\alpha}\left(\frac{\partial^2 p}{\partial x^2}\right)$, which has large values.

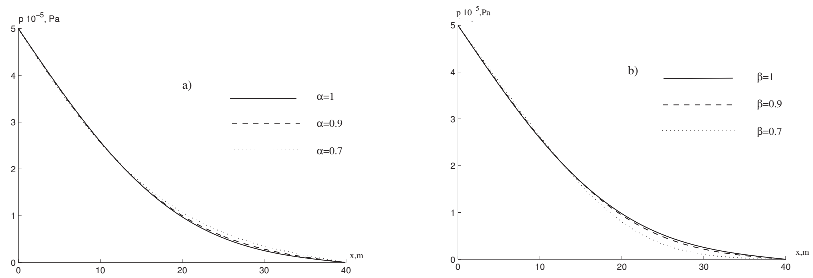

Fig 5. Pressure profiles at various α, β and $t = 3600s, \lambda_v = 500s, \lambda_p = 500s$ a) $\beta = 1$ b) $\alpha = 1$.

Figs. 6a, 6b show the distribution of the filtration velocity is shown when decreasing β from 1 and α from 1, respectively. Comparison of Figs. 6a, 6b with Figs. 1a,1b shows that the nature of the change in the filtration velocity is the same as the pressure field. In this case, a decrease in the values of β and α acts on the filtration velocity field in the same way as when the pressure field acts on the field. The filtration velocity decreases as x increases. This nature of the change in the filtration velocity is explained by the elastic regime, which is the basis for the derivation of equation (6). The pressure p_0 applied to $x = 0$, creating fluid motion in the medium, leads to a pressure distribution, the values of which are much greater at small x, than at large x. This occurs due to the non-stationarity of the process and the compressibility of the porous medium and liquid. Of course, this distribution is significantly affected by relaxation phenomena. At large t, when all relaxation and non-stationary phenomena are completed, a linear pressure distribution and a constant filtration velocity will be obtained.

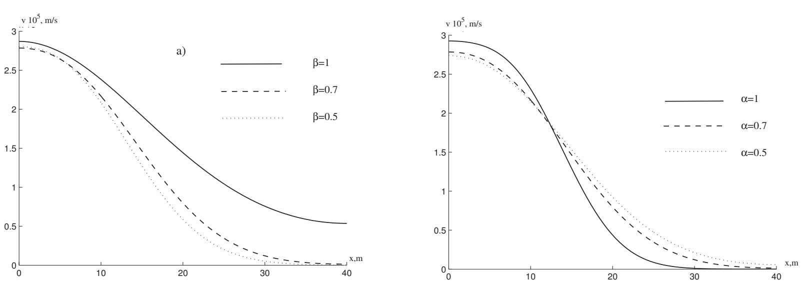

Fig 6. Profiles of filtration velocity at various β, α and $t = 3600s, \lambda_v = 1000s, \lambda_p = 500s$, a) $\alpha = 0.7$, b) $\beta = 0.7$.

The effect of relaxation times λ_p and λ_v on the distribution of the filtration velocity field is shown in Figs. 7a,7b. Comparison of these data with Figs. 2a,2b shows that a similar distribution is obtained for the filtration velocity, as for the pressure field. As for pressure, the manifestation of a finite perturbation propagation velocity (pressure and filtration velocity) is explained by the compressibility of the porous medium and liquid, as well as the relaxation properties of the filtration law. As in the case of a change in pressure, an increase λ_v in general leads to a decrease, and an increase λ_p, in general, to an increase (for large x) in the filtration velocity.

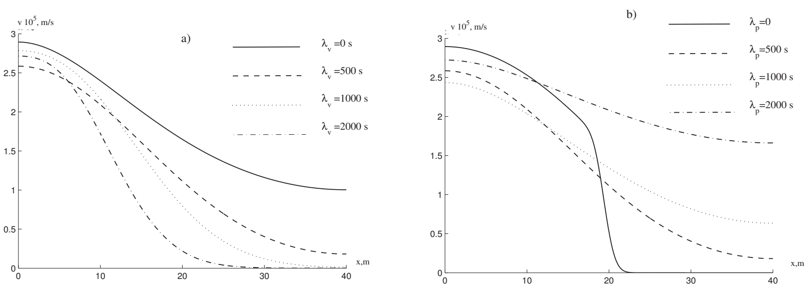

Fig 7. Profiles of filtration velocity at $t = 3600s, \alpha = 0.7, \beta = 0.7$ and various λ_v, λ_p, a) $\lambda_p = 500s$ b) $\lambda_v = 500s$.

The development of filtration velocity profiles over time is shown in Fig. 8. Comparison with Fig. 3 shows that the filtration velocity decreases with increasing time for small x, and increases for large ones x. Such a non-monotonous dynamics of the development of profiles is explained by the formation of various pressure gradients and the relaxation properties of the filtration law. From Fig. 3 it can be seen that at small t in the initial parts of the medium (small x) large pressure gradients are formed, which correspond to large v. As time increases, this gradient decreases, resulting in a decrease in v. Of course, the development of profiles v is determined not only by the pressure gradient, but also by other filtration parameters, since the filtration law is non-equilibrium.

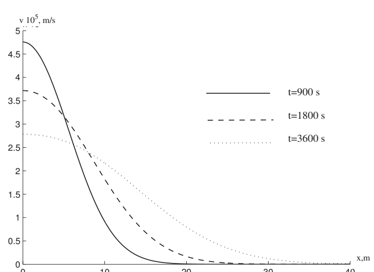

Fig 8. Profiles of filtration velocity at $\lambda_v = 1000s, \lambda_p = 500s, \alpha = 0.7, \beta = 0.7$ at various points of time.

A separate effect of relaxation phenomena in pressure and filtration velocity on the distribution of filtration velocity is shown in Figs. 9a,9b. In the absence of relaxation in filtration velocity, a decrease α from 1 leads to an increase v (Fig. 9a). In the absence of relaxation in pressure (Fig.9b), with a decrease β from 1 the propagation zone v decreases. Considering that in this case we have a wave solution, this means a decrease in the propagation velocity of the perturbation in the medium, both pressure and filtration velocity. At the same time, with a decrease β from 1 in the initial parts of the medium (small x), the filtration velocity takes large values, and then (relatively large x)- smaller values. This distribution v is fully consistent with the distribution p (Fig. 4b).

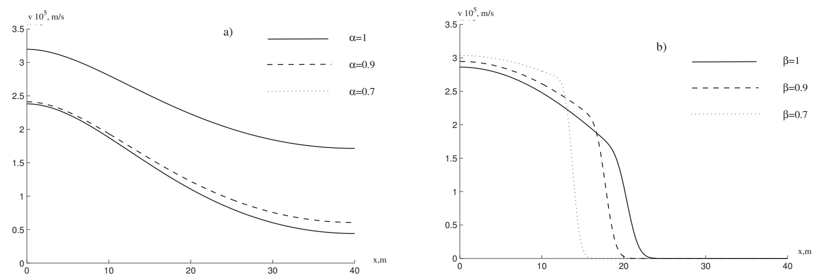

Fig 9. Profiles of filtration velocity at various α, β and $t = 3600s$ a) $\lambda_v = 0s, \lambda_p = 1000s, \beta = 1$ b) $\lambda_v = 1000s, \lambda_p = 0s, \alpha = 1$.

The results for the case of equal relaxation parameters $\lambda_p = \lambda_v$ are shown in Figs. 10a,10b. The nature of the change in the filtration velocity with a change in α and β is quite similar to the change in the pressure field (Figs. 5a,5b). With a decrease α from 1, the filtration velocity increases (Fig. 10a). A decrease β from 1 leads to a complex change dynamics of v. At the same time, up to certain x values x becomes larger, then smaller. As noted above in the analysis of Figs. 5a,5b, the solid lines in Figs. 10a,10b correspond to the solution of the classical piezoconductivity equation derived from the linear equilibrium Darcy$'$s law. Therefore, in the cases of $\lambda_p = \lambda_v = 500, \alpha = 1, \beta = 1$ and $\lambda_p = \lambda_v = 0$ for arbitrary α, β are identical.

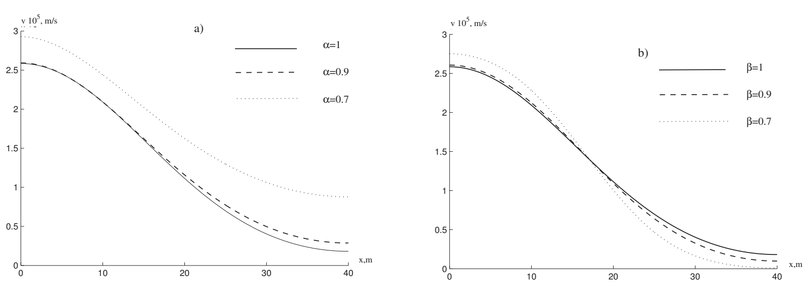

Fig 10. Profiles of filtration velocity at various α, β and $t = 3600s, \lambda_v = 500s, \lambda_p = 500s$ a) $\beta = 1$ b) $\alpha = 1$.

Conclusions. The paper considers the problem of anomalous non-stationary filtration of a homogeneous liquid in a homogeneous porous medium on the basis of a relaxation fractional-differential filtration law that takes into account relaxation phenomena both in pressure and in filtration velocity. Using this law, the piezoconductivity equation is derived, for which the problem of filtration in a finite one-dimensional medium is set. The problem is solved numerically with the approximation of fractional derivatives in the piezoconductivity equation based on their Caputo definition. On the basis of numerical calculations, the effect of relaxation times on pressure λ_p and filtration velocity λ_v, as well as the orders of fractional derivatives in the fractional-differential, relaxation Darcy′s law, α and β, on the filtration characteristics was established. The fields of pressure and filtration velocity are determined for various values of the parameters $\lambda_v, \lambda_p, \alpha$ and β. It is shown that a decrease β from 1 leads to a slight increase in pressure values on p up to certain values of x, and then to a decrease. The effect of decreasing α from 1 is the opposite: pressure up to certain values$x$ takes smaller values, and then larger ones. The effect of an increase λ_v on the pressure field is similar to the effect of a decrease in β. Accordingly, the effect of an increase λ_p on the pressure field is similar to the effect of a decrease in α. The following feature of the increase in piezoconductivity is revealed: at relatively large $\lambda_p - \lambda_v$, in particular at $\lambda_v > 0, \lambda_p = 0$, the solution of the equation acquires a wave character. At, $\alpha = \beta = 1$ the propagation velocity of pressure waves is determined by the quantity $\sqrt{\kappa/\lambda_v}$. With the change in α, β the nature of the propagation of pressure waves, in particular, its speed, will change. For example, as values of β the nature of the propagation of pressure waves, in particular, its speed, will change. For example, as values of Eq.(4a) with, $\alpha = \beta = 1$ and $\lambda_p = \lambda_v$ classical solution of the equation is obtained without taking into account relaxation effects. However, in (6) at, $\alpha \neq \beta$ the relaxation effects are preserved even in the case $\lambda_p = \lambda_v$. The filtration velocity field as a whole has similar characteristics with the pressure field. Changes in $\lambda_p, \lambda_v, \alpha$ and β lead to the same changes in the filtration velocity field as in the pressure field. However, in the dynamics of the development of filtration velocity profiles , non-monotonicity up to certain values of x can be observed. This is due to the change in the pressure gradient in this zone with increasing time. In general, both for pressure and for the filtration velocity, non-uniform fields of their distribution are obtained, which is associated with the effects of elasticity of the porous medium and liquid, as well as the relaxation nature of the filtration law.

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