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REPRESENTATION OF GENERALIZED MOCK THETA FUNCTIONS AS CONTINUED FRACTION

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Abstract

By using Slater's transformation formula, we express the bilateral generalized mock theta functions of third and eighth order as ${}_2\phi_1$ series and then represent them as continued fraction.

MSC 2020: 33C20, 33D99, 33D15

Key Words and Phrases: continued fraction, bilateral mock theta function, basic hypergeometric series

1. Introduction

In his last letter to G.H. Hardy [11], S. Ramanujan listed seventeen mock theta functions of order three, five and seven. According to Ramanujan, a mock theta function is a function $f(q), |q| < 1$, satisfying the following two conditions:

(0) For every root of unity ξ, there is a θ-function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $q \to \xi$ radially.
(1) There is no single θ-function which works for all ξ, i.e., for every θ-function $\theta(q)$ there is some root of unity ξ for which $f(q) - \theta(q)$ is unbounded as $q \to \xi$ radially.

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The Third Order Mock Theta Functions of Ramanujan's are:

$$f_R(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$
$$\phi_R(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}$$
$$\psi_R(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}$$
$$\chi_R(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)\dots(1-q^n+q^{2n})}$$
$$\omega_R(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}$$
$$\upsilon_R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}$$

and

$$\rho_R(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)\dots(1+q^{2n-1}+q^{4n-2})}.$$

The Eighth Order Mock Theta Functions of Gordon and Mclntosh are:

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}\left(-q;q^2\right)_n}{(-q^2;q^2)_n}$$
$$S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}\left(-q;q^2\right)_n}{(-q^2;q^2)_n}$$
$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}\left(-q^2;q^2\right)_n}{(-q;q^2)_{n+1}}$$
$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}\left(-q^2;q^2\right)_n}{(-q;q^2)_{n+1}}$$
$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}\left(-q;q^2\right)_n}{(-q^4;q^4)_n}$$
$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}\left(-q;q^2\right)_n}{(-q^2;q^4)_{n+1}}$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}\left(-q;q^2\right)_n}{(q;q^2)_n}$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}\left(-q;q^2\right)_n}{(q;q^2)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}\left(-q^4;q^4\right)_n}{(q;q^2)_{2n+2}}.$$

Bilateral Generalized Third Order Mock Theta Functions are:

$$f_c(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2-4n+n\beta}\alpha^n z^{2n}}{(-z;q)_n(-\alpha z/q;q)_n}$$

$$\phi_c(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2-3n+n\beta} z^{2n}}{(-\alpha z^2/q;q^2)_n}$$

$$\psi_c(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2-n+n\beta} z^{2n+1}}{(\alpha z^2/q^2;q^2)_{n+1}}$$

$$v_c(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2-2n+n\beta} z^{2n}}{(-\alpha^2 z^2/q^3;q^2)_{n+1}}$$

$$\omega_c(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{2n^2-5n-4+n\beta}\alpha^{2n} z^{4(n+1)}}{(z^2/q;q^2)_{n+1}\,(\alpha^2 z^2/q^3;q^2)_{n+1}}$$

$$\chi_c(t,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2-3n+n\beta z^{2n}}}{(vz;q)_n\,(-v^2 z;q)_n}$$

$$\rho_c(t,\beta,z;q) = \frac{z^4}{q^4(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{2n^2-3n+n\beta} z^{4n}}{(v^2 z^2/q;q^2)_{n+1}\,(v^{-2} z^2/q;q^2)_{n+1}}.$$

Bilateral Generalized Eighth Order Mock Theta Functions are:

$$S_{0c}(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2-2n+n\alpha} z^n \left(-z^2/q;q^2\right)_n}{(-z^2;q^2)_n}$$

$$S_{1c}(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2+n\alpha} z^n \left(-z^2/q;q^2\right)_n}{(-z^2;q^2)_n}$$

$$T_{0c}(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2+n+n\alpha} z^{n+2} \left(-z^2;q^2\right)_n}{(-z^2/q;q^2)_{n+1}}$$

$$T_{1c}(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty} \frac{(t)_n q^{n^2-n+n\alpha} z^n \left(-z^2;q^2\right)_n}{(-z^2/q;q^2)_{n+1}}$$

$$U_{0c}(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty}\frac{(t)_n q^{n^2-3n+n\alpha} z^{2n}\left(-z^2/q;q^2\right)_n}{(-z^2q^2;q^4)_n}$$

$$U_{1c}(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty}\frac{(t)_n q^{n^2-n+n\alpha} z^{2n+1}\left(-z^2/q;q^2\right)_n}{(-z^2;q^4)_{n+1}}$$

$$V_{0c}(t,\alpha,z;q) = -1+\frac{2}{(t)_\infty}\sum_{n=-\infty}^{\infty}\frac{(t)_n q^{n^2-2n+n\alpha} z^{n}\left(-z;q^2\right)_n}{(z^2/q;q^2)_n}$$

$$= -1+\frac{2}{(t)_\infty}\sum_{n=-\infty}^{\infty}\frac{(t)_n q^{2n^2-3n+n\alpha} z^{2n}\left(-z^2;q^4\right)_n}{(z^2/q;q^2)_{2n+1}}$$

and

$$V_{1c}(t,\alpha,z;q) = \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty}\frac{(t)_n q^{n^2+n\alpha} z^{n+1}\left(-z;q^2\right)_n}{(z^2/q;q^2)_{n+1}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=-\infty}^{\infty}\frac{(t)_n q^{2n^2-n+n\alpha} z^{2n+1}\left(-z^2q^2;q^4\right)_n}{(z^2/q;q^2)_{2n+2}}.$$

2. **Notations**

In this paper we have use the following notations:

$$\left(a;q^k\right)_n = \prod_{m=1}^{n}\left(1-aq^{k(m-1)}\right), \left|q^k\right| < 1,$$

for n, non-negative integer,

$$\left(a;q^k\right)_0 = 1,$$

$$\left(a;q^k\right)_\infty = \prod_{m=1}^{\infty}\left(1-aq^{k(m-1)}\right),$$

$$\left(a_1,a_2,a_3\ldots\ldots a_m;q^k\right)_n = \left(a_1;q^k\right)_n\left(a_2;q^k\right)_n\ldots\ldots\left(a_m;q^k\right)_n,$$

$$ {}_A\phi_{A-1}\left[\begin{array}{c} a_1,a_2,\ldots\ldots,a_A; \\ b_1,b_2,\ldots\ldots,b_{A-1}\end{array};q_1,z\right]$$

$$=\sum_{n=0}^{\infty}\frac{(a_1;q_1)_n\ldots\ldots..(a_A;q_1)_n z^n}{(b_1;q_1)_n\ldots\ldots(b_{A-1};q_1)_n(q_1;q_1)_n},\ |z|<1.$$

The generalized basic hypergeometric function with base q is defined as:

$$ {}_r\phi_s\left[\begin{array}{c} a_1,a_2,\ldots\ldots,a_r; \\ b_1,b_2,\ldots\ldots,b_s\end{array};q,z\right]$$

$$=\sum_{n=0}^{\infty}\frac{(a_1;q)_n\ldots\ldots(a_r;q)_n z^n}{(b_1;q)_n\ldots\ldots(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\frac{n^2-n}{2}}\right]^{1+s-r} z^n.$$

For $r = s + 1$, the above series is convergent for $|z| < 1$, for $r \le s$, the above series is convergent for all z and for $r > s + 1$, the diverges for all z except $z = 0$.

3. Bilateral Generalized Mock Theta Functions as ${}_2\phi_1$ Series

We shall use the following formula of Slater [7, eq.(5.4.5), p.143] with $d = \frac{a_1 a_2}{b_1 b_2}$, to express the generalized bilateral function as ${}_2\phi_1$ series:

$$\begin{aligned}
&\frac{(q/a_1, q/a_2, dz, q/dz; q)_\infty}{(q/b_1, q/b_2; q)_\infty} {}_2\psi_2 \left[\begin{matrix} a_1, a_2 \\ b_1, b_2 \end{matrix} ; q, z \right] \\
&= \frac{q}{b_1} \frac{\left(q, b_1/a_1, b_1/a_2, db_1 z/q, q^2/db_1 z; q\right)_\infty}{(b_1, q/b_1, b_1/b_2; q)_\infty} \\
&\times {}_2\phi_1 \left[\begin{matrix} qa_1/b_1, qa_2/b_1 \\ qb_2/b1 \end{matrix} ; q, z \right] + \text{ idem } (b_1; b_2).
\end{aligned} \tag{3.1}$$

- **Bilateral Generalized Third Order Mock Theta Functions as ${}_2\phi_1$ Series**

  Using (3.1), we now express the seven bilateral generalized third order mock theta functions as a limiting case of a ${}_2\phi_1$ series:

(i) $\frac{\left(q^\beta/q^2, q^3/q^\beta; q\right)_\infty}{(-q/z, -q^2/\alpha z; q)_\infty} f_c(0, \alpha, \beta, z; q)$

$$\begin{aligned}
&= -\frac{q}{z} \frac{\left(q, -zq^\beta/q^3, -q^4/zq^\beta; q\right)_\infty}{(-z, -q/z, q/\alpha; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-2n+n\beta}\alpha^n}{(q;q)_n(\alpha;q)_n} \\
&\quad - \frac{q^2}{\alpha z} \frac{\left(q, -\alpha z q^\beta/q^4, -q^5/\alpha z q^\beta; q\right)_\infty}{(-\alpha z/q, -q^2/\alpha z, \alpha/q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+n\beta}\alpha^{-n}}{(q;q)_n \left(q^2/\alpha;q\right)_n}
\end{aligned} \tag{3.2}$$

$$\left(a_1, a_2 \to \infty, b_1 = -z, b_2 = \frac{-\alpha z}{q}, z = \frac{\alpha z^2 q^\beta}{q^3 a_1 a_2} \text{ in } (3.1) \right).$$

(ii) $\frac{\left(q^\beta/\alpha q, \alpha q^2/q^\beta; q\right)_\infty}{\left(-q^3/\alpha z^2; q^2\right)_\infty} \phi_c(0, \alpha, \beta, z; q)$

$$\begin{aligned}
&= -\frac{iq^{3/2}}{\alpha^{1/2} z} \frac{\left(q, izq^\beta/\alpha^{1/2}q^{5/2}, -i\alpha^{1/2}q^{7/2}/zq^\beta; q\right)_\infty}{\left(i\alpha^{1/2}z/q^{1/2}, -iq^{3/2}/\alpha^{1/2}z, -1; q\right)_\infty} \\
&\qquad \times \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n\beta}}{\alpha^n \left(q^2;q^2\right)_n} \\
&\quad + \frac{iq^{3/2}}{\alpha^{1/2} z} \frac{\left(q, -izq^\beta/\alpha^{1/2}q^{5/2}, i\alpha^{1/2}q^{7/2}/zq^\beta; q\right)_\infty}{\left(-i\alpha^{1/2}z/q^{1/2}, iq^{3/2}/\alpha^{1/2}z, -1; q\right)_\infty} \\
&\qquad \times \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n\beta}}{\alpha^n \left(q^2;q^2\right)_n}
\end{aligned} \tag{3.3}$$

$$\left(a_1, a_2 \to \infty, b_1 = \frac{i\alpha^{1/2}z}{q^{1/2}}, b_2 = -\frac{i\alpha^{1/2}z}{q^{1/2}}, z = \frac{z^2q^\beta}{q^2a_1a_2} \text{ in (3.1)}\right).$$

(iii) $\frac{\left(1-\alpha z^2/q^2\right)\left(-q^\beta/\alpha,-\alpha q/q^\beta;q\right)_\infty}{\left(q^2/\alpha z^2;q^2\right)_\infty}\psi_c(0,\alpha,\beta,z;q)$

$$\begin{aligned}
&= \frac{q}{\alpha^{1/2}}\frac{\left(q, -zq^\beta/\alpha^{1/2}q, -\alpha^{1/2}q^2/zq^\beta;q\right)_\infty}{\left(\alpha^{1/2}z, q/\alpha^{1/2}z, -1;q\right)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n+n\beta}}{\alpha^n\left(q^2;q^2\right)_n} \\
&\quad - \frac{q}{\alpha^{1/2}}\frac{\left(q, zq^\beta/\alpha^{1/2}q, \alpha^{1/2}q^2/zq^\beta;q\right)_\infty}{\left(-\alpha^{1/2}z, -q/\alpha^{1/2}z, -1;q\right)_\infty} \times \sum_{n=0}^{\infty}\frac{q^{n^2+n+n\beta}}{\alpha^n\left(q^2;q^2\right)_n} \qquad (3.4)
\end{aligned}$$

$$\left(a_1, a_2 \to \infty, b_1 = \alpha^{1/2}z, b_2 = -\alpha^{1/2}z, z = \frac{z^2q^\beta}{a_1a_2} \text{ in (3.1)}\right).$$

(iv) $\frac{\left(1+\alpha^2 z^2/q^3\right)\left(q^\beta/\alpha^2 q,\alpha^2 q^2/q^\beta;q\right)_\infty}{\left(-q^3/\alpha^2 z^2;q^2\right)_\infty}v_c(0,\alpha,\beta,z;q)$

$$\begin{aligned}
&= -\frac{iq^{3/2}}{\alpha z}\frac{\left(q, izq^\beta/\alpha q^{3/2}, -i\alpha q^{5/2}/zq^\beta;q\right)_\infty}{\left(i\alpha z/q^{1/2}, -iq^{3/2}/\alpha z, -1;q\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2+n\beta}}{\alpha^{2n}\left(q^2;q^2\right)_n} \\
&\quad + \frac{iq^{3/2}}{\alpha z}\frac{\left(q, -izq^\beta/\alpha q^{3/2}, i\alpha q^{5/2}/zq^\beta;q\right)_\infty}{\left(-i\alpha z/q^{1/2}, iq^{3/2}/\alpha z, -1;q\right)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n^2+n\beta}}{\alpha^{2n}\left(q^2;q^2\right)_n} \qquad (3.5)
\end{aligned}$$

$$\left(a_1, a_2 \to \infty, b_1 = \frac{i\alpha z}{q^{1/2}}, b_2 = -\frac{i\alpha z}{q^{1/2}}, z = \frac{z^2q^\beta}{q^2a_1a_2} \text{ in (3.1)}\right).$$

(v) $\frac{\left(q^\beta/q^3,q^5/q^\beta;q^2\right)_\infty}{\left(q/z^2,q^3/\alpha^2z^2;q^2\right)_\infty}\omega_c(0,\alpha,\beta,z;q)$

$$\begin{aligned}
&= \frac{z^2\left(q^2, z^2q^\beta/q^4, q^6/z^2q^\beta;q^2\right)_\infty}{\left(q^3-\alpha^2z^2\right)\left(z^2/q, q/z^2, q^2/\alpha^2;q^2\right)_\infty} \\
&\qquad\qquad \times \sum_{n=0}^{\infty}\frac{q^{2n^2-3n+n\beta}\alpha^{2n}}{\left(q^2;q^2\right)_n\left(\alpha^2;q^2\right)_n} \\
&\quad + \frac{z^2q^2\left(q^2, \alpha^2z^2q^\beta/q^6, q^8/\alpha^2z^2q^\beta;q^2\right)_\infty}{\alpha^2\left(q^3-\alpha^2z^2\right)\left(\alpha^2z^2/q, q^3/\alpha^2z^2, \alpha^2/q^2;q^2\right)_\infty} \\
&\qquad\qquad \times \sum_{n=0}^{\infty}\frac{q^{2n^2+n+n\beta}\alpha^{-2n}}{\left(q^2;q^2\right)_n\left(q^4/\alpha^2;q^2\right)_n}. \qquad (3.6)
\end{aligned}$$

(Letting $q \to q^2$ and then $a_1, a_2 \to \infty, b_1 = z^2q, b_2 = \frac{\alpha^2z^2}{q}, z = \frac{\alpha^2z^4q^\beta}{q^3a_1a_2}$ in (3.1))

(vi) $\frac{\left(-q^{\beta}/q^2v^3,-q^3v^3/q^{\beta};q\right)_\infty}{(q/vz,-q/v^2z;q)_\infty}\chi_c(0,\beta,z;q)$

$$= \frac{q}{vz}\frac{\left(q,-zq^{\beta}/q^3v^2,-q^4v^2/zq^{\beta};q\right)_\infty}{(vz,q/vz,-1/v;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n+n\beta}v^{-2n}}{(q;q)_n(-vq;q)_n}$$
$$-\frac{q}{v^2z}\frac{\left(q,zq^{\beta}/q^3v,q^4v/zq^{\beta};q\right)_\infty}{(-v^2z,-q/v^2z,-v;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n+n\beta}v^{-4n}}{(q;q)_n(-q/v;q)_n} \quad (3.7)$$

$$\left(a_1,a_2\to\infty, b_1=vz, b_2=-v^2z, z=\frac{z^2q^{\beta}}{q^2a_1a_2}\text{ in (3.1)}\right).$$

(vii) $\frac{\left(q^{\beta}/q^3,q^5/q^{\beta};q^2\right)_\infty}{(q/v^2z^2,q/v^{-2}z^2;q^2)_\infty}\rho_c(0,\beta,z;q)$

$$= \frac{z^2}{v^2q^3}\frac{\left(q^2,z^2q^{\beta}/v^{-2}q^4,v^{-2}q^6/z^2q^{\beta};q^2\right)_\infty}{(1-v^{-2}z^2/q^3)\left(v^2z^2/q,q/v^2z^2,v^4;q^2\right)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{q^{2n^2-n+n\beta}z^{2n}v^{-4n}}{(q^2;q^2)_n\,(q^2/v^4;q^2)_n}$$
$$+\frac{z^2}{v^{-2}q^3}\frac{\left(q^2,z^2q^{\beta}/v^2q^4,v^2q^6/z^2q^{\beta};q^2\right)_\infty}{(1-v^2z^2/q^3)\left(v^{-2}z^2/q,q/v^{-2}z^2,v^{-4};q^2\right)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{q^{2n^2-n+n\beta}z^{2n}v^{4n}}{(q^2;q^2)_n\,(q^2v^4;q^2)_n} \quad (3.8)$$

(Letting $q\to q^2$ and then $a_1,a_2\to\infty, b_1=v^2z^2q, b_2=v^{-2}z^2q, z=\frac{z^4q^{\beta}}{qa_1a_2}$ in (3.1)).

- **Bilateral Generalized Eighth Order Mock Theta Functions as ${}_2\phi_1$ Series**

Using (3.1), we now express the eight bilateral generalized eighth order mock theta functions as a limiting case of a ${}_2\phi_1$ series.

(i) $\frac{\left(1/q,q^3,-q^3/z^2;q^2\right)_\infty}{(-q^2/z^2,-q^3/zq^{\alpha},-zq^{\alpha}/q,q;q^2)_\infty}S_{0,c}(0,\alpha,z;q)$

$$= -\frac{q^2}{z^2}\frac{\left(q^2,-z^2/q^3,-q^5/z^2;q^2\right)_\infty}{(-z^2,-q^2/z^2,zq/q^{\alpha};q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2+n\alpha}z^{-n}}{(q^2;q^2)_n\,(q^{\alpha+1}/z;q^2)_n}$$
$$-\frac{q^3}{zq^{\alpha}}\frac{\left(q^2,-zq^{\alpha}/q^4,-q^6/zq^{\alpha};q^2\right)_\infty}{(-zq^{\alpha}/q,-q^3/zq^{\alpha},q^{\alpha}/zq;q^2)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{q^{2n^2+2n-n\alpha}z^{n}}{(q^2;q^2)_n\,(zq^3/q^{\alpha};q^2)_n} \quad (3.9)$$

(Letting $q \to q^2$ and then $a_1, a_2 \to \infty, b_1 = -z^2, b_2 = -\frac{zq^\alpha}{q}, z = \frac{z^3 q^\alpha}{q^2 a_1 a_2}$ in (3.1))

(ii) $\frac{\left(1/q, q^3, -q^3/z^2; q^2\right)_\infty}{(-q^2/z^2, -q/zq^\alpha, -zq^{\alpha+1}, q; q^2)_\infty} S_{1,c}(0, \alpha, z; q)$

$$= -\frac{q^2}{z^2} \frac{\left(q^2, -z^2/q^3, -q^5/z^2; q^2\right)_\infty}{(-z^2, -q^2/z^2, z/q^{\alpha+1}; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+n\alpha} z^{-n}}{(q^2; q^2)_n (q^{\alpha+3}/z; q^2)_n} - \frac{q}{zq^\alpha} \frac{\left(q^2, -zq^\alpha/q^2, -q^4/zq^\alpha; q^2\right)_\infty}{(-zq^{\alpha+1}, -q/zq^\alpha, q^{\alpha+1}/z; q^2)_\infty} \times \sum_{n=0}^{\infty} \frac{q^{2n^2-n\alpha} z^n}{(q^2; q^2)_n (zq/q^\alpha; q^2)_n} \quad (3.10)$$

(Letting $q \to q^2$ and then $a_1, a_2 \to \infty, b_1 = -z^2, b_2 = -zq^{\alpha+1}, \quad z = \frac{z^3 q^\alpha}{a_1 a_2}$ in (3.1))

(iii) $\frac{\left(1/q, q^3, -q^2/z^2; q^2\right)_\infty}{(-q^3/z^2, -1/zq^\alpha, -zq^{\alpha+2}, q; q^2)_\infty} T_{0,c}(0, \alpha, z; q)$

$$= -\frac{q^2}{z^2} \frac{\left(q^2, -z^2/q^2, -q^4/z^2; q^2\right)_\infty}{(-z^2 q, -q/z^2, z/q^{\alpha+1}; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+n\alpha} z^{-n}}{(q^2; q^2)_n (q^{\alpha+3}/z; q^2)_n} - \frac{q}{zq^\alpha} \frac{\left(q^2, -zq^\alpha/q, -q^3/zq^\alpha; q^2\right)_\infty}{(-zq^{\alpha+2}, -1/zq^\alpha, q^{\alpha+1}/z; q^2)_\infty} \times \sum_{n=0}^{\infty} \frac{q^{2n^2-n\alpha} z^n}{(q^2; q^2)_n (zq/q^\alpha; q^2)_n}. \quad (3.11)$$

(Letting $q \to q^2$ and then $a_1, a_2 \to \infty, b_1 = -z^2 q, b_2 = -zq^{\alpha+2}, z = \frac{z^3 q^{\alpha+2}}{a_1 a_2}$ in (3.1))

(iv) $\frac{\left(1/q, q^3, -q^2/z^2; q^2\right)_\infty}{(-q^3/z^2, -q^2/zq^\alpha, -zq^\alpha, q; q^2)_\infty} T_{1,c}(0, \alpha, z; q)$

$$= -\frac{q^2}{z^4} \frac{\left(q^2, -z^2/q^2, -q^4/z^2; q^2\right)_\infty}{(-z^2 q, -q/z^2, zq/q^\alpha; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+n\alpha} z^{-n}}{(q^2; q^2)_n (q^{\alpha+1}/z; q^2)_n} - \frac{q^3}{z^3 q^\alpha} \frac{\left(q^2, -zq^\alpha/q^3, -q^5/zq^\alpha; q^2\right)_\infty}{(-zq^\alpha, -q^2/zq^\alpha, q^\alpha/zq; q^2)_\infty} \times \sum_{n=0}^{\infty} \frac{q^{2n^2+2n-n\alpha} z^n}{(q^2; q^2)_n (zq^3/q^\alpha; q^2)_n}. \quad (3.12)$$

(Letting $q \to q^2$ and then $a_1, a_2 \to \infty, b_1 = -z^2 q, b_2 = -zq^\alpha, z = \frac{z^3 q^\alpha}{a_1 a_2}$ in (3.1))

(v) $\frac{\left(-q/z,-zq,-q^2/z;q^2\right)_\infty}{\left(q^3/z^2,-q^3/zq^\alpha,-zq^\alpha/q,-z/q;q^2\right)_\infty} V_{0,c}(0,\alpha,z;q)$

$$= \frac{q^3}{z^2}\frac{\left(q^2,-z/q^2,-q^4/z;q^2\right)_\infty}{\left(z^2/q,q^3/z^2,-z/q^\alpha;q^2\right)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2+3n+n\alpha}z^{-2n}}{\left(q^2;q^2\right)_n\left(-q^{\alpha+2}/z;q^2\right)_n}$$
$$-\frac{q^3}{zq^\alpha}\frac{\left(q^2,q^\alpha/q^2,q^4/q^\alpha;q^2\right)_\infty}{\left(-zq^\alpha/q,-q^3/zq^\alpha,-q^\alpha/z;q^2\right)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{q^{2n^2+3n-n\alpha}}{\left(q^2;q^2\right)_n\left(-zq^2/q^\alpha;q^2\right)_n}. \quad (3.13)$$

(Letting $q\to q^2$ and then $a_1,a_2\to\infty, b_1=z^2/q, b_2=-zq^\alpha/q, z=\frac{z^2q^{\alpha-1}}{a_1a_2}$ in (3.1))

(vi) $\frac{\left(-1/zq,-zq^3,-q^2/z;q^2\right)_\infty}{\left(q^3/z^2,-q/zq^\alpha,-zq^{\alpha+1},-zq;q^2\right)_\infty} V_{1,c}(0,\alpha,z;q)$

$$= -\frac{q^2}{z^3}\frac{\left(q^2,-z/q^2,-q^4/z;q^2\right)_\infty}{\left(z^2q,q/z^2,-z/q^\alpha;q^2\right)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2+n+n\alpha}z^{-2n}}{\left(q^2;q^2\right)_n\left(-q^{\alpha+2}/z;q^2\right)_n}$$
$$-\frac{q^2}{z^2q^\alpha}\frac{\left(q^2,q^\alpha/q^2,q^4/q^\alpha;q^2\right)_\infty}{\left(-zq^{\alpha+1},-q/zq^\alpha,-q^\alpha/z;q^2\right)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{q^{2n^2+n-n\alpha}}{\left(q^2;q^2\right)_n\left(-zq^2/q^\alpha;q^2\right)_n}. \quad (3.14)$$

(Letting $q\to q^2$ and then $a_1,a_2\to\infty, b_1=z^2q, b_2=-zq^{\alpha+1}, z=\frac{z^2q^{\alpha+1}}{a_1a_2}$ in (3.1))

4. Continued Fraction Representation for the Bilateral Generalized Third Order Mock Theta Functions

We now represent the bilateral generalized third order mock theta functions as continued fraction.

From equation (3.3) we have

$$\phi_c(0,\alpha,\beta,z;q) = (S_1+T_1)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n\beta}}{\alpha^n\left(q^2;q^2\right)_n}, \quad (4.1)$$

where

$$S_1 = -\frac{iq^{3/2}}{\alpha^{1/2}z}\frac{\left(q,izq^\beta/\alpha^{1/2}q^{5/2},-i\alpha^{1/2}q^{7/2}/zq^\beta,iq^{3/2}/\alpha^{1/2}z;q\right)_\infty}{\left(-1,i\alpha^{1/2}z/q^{1/2},q^\beta/\alpha q,\alpha q^2/q^\beta;q\right)_\infty},$$
$$T_1 = \frac{iq^{3/2}}{\alpha^{1/2}z}\frac{\left(q,-izq^\beta/\alpha^{1/2}q^{5/2},i\alpha^{1/2}q^{7/2}/zq^\beta,-iq^{3/2}/\alpha^{1/2}z;q\right)_\infty}{\left(-1,-i\alpha^{1/2}z/q^{1/2},q^\beta/\alpha q,\alpha q^2/q^\beta;q\right)_\infty}$$

and from equation (3.4) we have

$$\psi_c(0,\alpha,\beta,z;q) = (S_2+T_2)\sum_{n=0}^{\infty}\frac{q^{n^2+n+n\beta}}{\alpha^n\left(q^2;q^2\right)_n}, \tag{4.2}$$

where

$$S_2 = \frac{q}{\alpha^{1/2}z}\frac{\left(q,-zq^{\beta}/\alpha^{1/2}q,-\alpha^{1/2}q^2/zq^{\beta},-q/\alpha^{1/2}z;q\right)_{\infty}}{\left(-1,\alpha^{1/2}z,-q^{\beta}/\alpha,-\alpha q/q^{\beta};q\right)_{\infty}},$$

$$T_2 = -\frac{q}{\alpha^{1/2}z}\frac{\left(q,zq^{\beta}/\alpha^{1/2}q,\alpha^{1/2}q^2/zq^{\beta},q/\alpha^{1/2}z;q\right)_{\infty}}{\left(-1,-\alpha^{1/2}z,-q^{\beta}/\alpha,-\alpha q/q^{\beta};q\right)_{\infty}}.$$

Similarly, from equation (3.5) we have

$$v_c(0,\alpha,\beta,z;q) = (S_3+T_3)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n\beta}}{\alpha^{2n}\left(q^2;q^2\right)_n}, \tag{4.3}$$

where

$$S_3 = -\frac{iq^{3/2}}{\alpha z}\frac{\left(q,izq^{\beta}/\alpha q^{3/2},-i\alpha q^{5/2}/zq^{\beta},iq^{3/2}/\alpha z;q\right)_{\infty}}{\left(-1,i\alpha z/q^{1/2},q^{\beta}/\alpha^2 q,\alpha^2q^2/q^{\beta};q\right)_{\infty}},$$

$$T_3 = \frac{iq^{3/2}}{\alpha z}\frac{\left(q,-izq^{\beta}/\alpha q^{3/2},i\alpha q^{5/2}/zq^{\beta},-iq^{3/2}/\alpha z;q\right)_{\infty}}{\left(-1,-i\alpha z/q^{1/2},q^{\beta}/\alpha^2 q,\alpha^2q^2/q^{\beta};q\right)_{\infty}}.$$

From the above relations we get the following continued fractions:

(i) From (4.1) and (4.2) we have

$$\begin{aligned}\frac{\psi_c(0,-\alpha,\beta,z;q)}{\phi_c(0,\alpha,\beta,z;q)} &= \frac{(S_2+T_2)}{(S_1+T_1)}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n+n\beta}}{\alpha^n(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n\beta}}{\alpha^n(q^2;q^2)_n}}\\ &= \frac{(S_2+T_2)}{(S_1+T_1)}\left[\frac{1}{1}+\frac{-q^{\beta+1}/\alpha}{1+q+}\frac{-q^{\beta+2}/\alpha}{1+q^2+\cdots\cdots}\right].\end{aligned}$$

Here we have used [9, eq.(3.74), p.79].

Special Cases. For $\alpha = q, \beta = 1$ and $z = q$

$$\frac{\phi_c(q)}{\psi_c(0,-q,1,z;q)} = \frac{(S_1+T_1)}{(S_2+T_2)}\left[1+\frac{-q}{1+q+}\frac{-q^2}{1+q^2+\cdots\cdots}\right],$$

where $\phi_c(q)$ is bilateral Ramanujan's third order mock theta function.

(ii) From the relations (4.2) and (4.3):

$$\frac{\psi_c\left(0,-\alpha^2,\beta,z;q\right)}{v_c(0,\alpha,\beta,z;q)} = \frac{(S_2+T_2)}{(S_3+T_3)}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n+n\beta}}{\alpha^{2n}(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n\beta}}{\alpha^{2n}(q^2;q^2)_n}}$$

$$= \frac{(S_2+T_2)}{(S_3+T_3)}\left[\frac{1}{1}+\frac{-q^{\beta+1}/\alpha^2}{1+q+}\frac{-q^{\beta+2}/\alpha^2}{1+q^2+\cdots\cdots}\right].$$

Here we have used [9, eq.(3.74), p.79].

Special Case. For $\alpha = q, \beta = 1$ and $z = q$

$$\frac{v_c(q)}{\psi_c\left(0,-q^2,1,z;q\right)} = \frac{(S_3+T_3)}{(S_2+T_2)}\left[1+\frac{-1}{1+q+}\frac{-q}{1+q^2+\cdots\cdots}\right],$$

where $v_c(q)$ is bilateral Ramanujan's third order mock theta function.

(iii) From equations (4.1) and (4.3), we get

$$\frac{\phi_c\left(0,\alpha^2/q,\beta,z;q\right)}{v_c(0,\alpha,\beta,z;q)} = \frac{(S_1+T_1)}{(S_3+T_3)}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n+n\beta}}{\alpha^{2n}(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n\beta}}{\alpha^{2n}(q^2;q^2)_n}}$$

$$= \frac{(S_1+T_1)}{(S_3+T_3)}\left[\frac{1}{1}+\frac{-q^{\beta+1}/\alpha^2}{1+q+}\frac{-q^{\beta+2}/\alpha^2}{1+q^2+\cdots\cdots}\right].$$

Here we have used [9, eq.(3.74), p.79].

Special Cases. For $\alpha = q, \beta = 1$ and $z = q$

$$\frac{\phi_c(q)}{v_c(q)} = \frac{(S_1+T_1)}{(S_3+T_3)}\left[\frac{1}{1}+\frac{-1}{1+q+}\frac{-q}{1+q^2+\cdots\cdots}\right],$$

where $\phi_c(q)$ and $v_c(q)$ are bilateral Ramanujan's third order mock theta functions.

5. Continued Fraction Representation for Bilateral Generalized Eighth Order Mock Theta Functions

We now represent the bilateral generalized eighth order mock theta functions as continued fractions.

(i) From equation (3.9) we have continued fraction for $S_{0,c}\left(0,\alpha,q^\alpha;q\right)$ as

$$\frac{S_{0,c}\left(0,\alpha,q^\alpha;q\right)}{\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = T_1 + S_1\left[\frac{\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}}{\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^2;q)_{2n}}}\right],$$

where

$$S_1 = -\frac{q^2\left(1+q^{2\alpha}/q^3\right)}{q^{2\alpha}}\frac{\left(q^2,-q^5/q^{2\alpha};q^2\right)_\infty\left(-q^{2\alpha}/q;q^2\right)_\infty^2}{\left(-q^{2\alpha},1/q,q^3;q^2\right)_\infty},$$

$$T_1 = -\frac{q^4}{q^{2\alpha}(1-q)}\frac{\left(q^2,-q^{2\alpha}/q^4,-q^2/q^{2\alpha},-q^6/q^{2\alpha};q^2\right)_\infty}{\left(-q^3/q^{2\alpha},1/q,q^3;q^2\right)_\infty},$$

$$\frac{S_{0,c}\left(0,\alpha,q^\alpha;q\right)}{\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = T_1 + (1-q)S_1\left[1+\frac{q^2-q}{1+q+}\frac{q^4-q}{1+q+\ldots\ldots}\right].$$

Here we have used [9, eq.(3.79), p.82].

Special cases. For $\alpha = 1$:

$$\frac{S_{0,c}(q)}{\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = T_1 + (1-q)S_1 \left[1 + \frac{q^2-q}{1+q+} \frac{q^4-q}{1+q+\ldots\ldots}\right],$$

where $S_{0,c}(q)$ is the bilateral Gordon and McIntosh eighth order mock theta function.

(ii) From equation (3.10) we have

$$\frac{S_{1,c}\left(0,\alpha,q^{\alpha};q\right)}{\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = S_2 + T_2 \left[\frac{\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}}}{\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}}\right],$$

where

$$S_2 = \frac{\left(1+q^{2\alpha-1}\right)}{(1-q)} \frac{\left(q^2,-q/q^{2\alpha};q^2\right)_\infty \left(-q^{2\alpha+1};q^2\right)_\infty^2}{\left(-q^{2\alpha},1/q,q^3;q^2\right)_\infty},$$

$$T_2 = -\frac{q^3\left(1+q^{2\alpha-2}\right)}{q^{4\alpha}} \frac{\left(q^2,-q^{2\alpha}/q^2;q^2\right)_\infty \left(-q^4/q^{2\alpha};q^2\right)_\infty^2}{\left(-q^3/q^{2\alpha},1/q,q^3;q^2\right)_\infty},$$

$$\frac{S_{1,c}\left(0,\alpha,q^{\alpha};q\right)}{\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = S_2 + (1-q)T_2 \left[1 + \frac{q^2-q}{1+q+} \frac{q^4-q}{1+q+\ldots\ldots}\right].$$

Here we have used [9, eq.(3.79), p.82].

Special cases. For $\alpha = 1$

$$\frac{S_{1,c}(q)}{\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = S_2 + (1-q)T_2 \left[1 + \frac{q^2-q}{1+q+} \frac{q^4-q}{1+q+\ldots\ldots}\right],$$

where $S_{1,c}(q)$ is the bilateral Gordon and McIntosh eighth order mock theta function.

(iii) From equation (3.11) we have

$$\frac{T_{0,c}\left(0,\alpha,q^{\alpha};q\right)}{\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = S_3 + T_3 \left[\frac{\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}}}{\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}}\right],$$

where

$$S_3 = \frac{\left(1+q^{2\alpha}\right)}{(1-q)\left(1+q^{2\alpha-1}\right)} \frac{\left(q^2, -1/q^{2\alpha}; q^2\right)_\infty \left(-q^{2\alpha+2}; q^2\right)_\infty^2}{\left(-q^{2\alpha+1}, 1/q, q^3; q^2\right)_\infty},$$

$$T_3 = -\frac{q}{q^{2\alpha}} \frac{\left(q^2, -q^{2\alpha}/q; q^2\right)_\infty \left(-q^3/q^{2\alpha}; q^2\right)_\infty^2}{\left(-q^2/q^{2\alpha}, 1/q, q^3; q^2\right)_\infty},$$

$$\frac{T_{0,c}\left(0, \alpha, q^\alpha; q\right)}{\sum_{n=0}^\infty \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = S_3 + (1-q)T_3 \left[1 + \frac{q^2-q}{1+q+} \frac{q^4-q}{1+q+\ldots\ldots}\right].$$

Here we have used [9, eq.(3.79), p.82].

Special cases. For $\alpha = 1$

$$\frac{T_{0,c}(q)}{\sum_{n=0}^\infty \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = S_3 + (1-q)T_3 \left[1 + \frac{q^2-q}{1+q+} \frac{q^4-q}{1+q+\ldots\ldots}\right],$$

where $T_{0,c}(q)$ is the bilateral Gordon and McIntosh eighth order mock theta function.

(iv) From equation (3.12) we have

$$\frac{T_{1,c}\left(0, \alpha, q^\alpha; q\right)}{\sum_{n=0}^\infty \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = T_4 + S_4 \left[\frac{\sum_{n=0}^\infty \frac{q^{2n^2}}{(q;q)_{2n}}}{\sum_{n=0}^\infty \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}}\right],$$

where

$$S_4 = -\frac{q\left(1+q^{2\alpha-2}\right)}{q^{2\alpha}\left(1+q^{2\alpha-1}\right)} \frac{\left(q^2, -q^{2\alpha}/q^2; q^2\right)_\infty \left(-q^{2\alpha}; q^2\right)_\infty^2}{\left(-q^{2\alpha+1}, 1/q, q^3; q^2\right)_\infty},$$

$$T_4 = \frac{q^4\left(1+q^3/q^{2\alpha}\right)}{q^{4\alpha}(1-q)} \frac{\left(q^2, -q^{2\alpha}/q^3; q^2\right)_\infty \left(-q^5/q^{2\alpha}; q^2\right)_\infty^2}{\left(-q^2/q^{2\alpha}, 1/q, q^3; q^2\right)_\infty},$$

$$\frac{T_{1,c}\left(0, \alpha, q^\alpha; q\right)}{\sum_{n=0}^\infty \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = T_4 + (1-q)S_4 \left[1 + \frac{q^2-q}{1+q+} \frac{q^4-q}{1+q+\ldots\ldots}\right].$$

Here we have used [9, eq.(3.79), p.82].

Special cases. For $\alpha = 1$

$$\frac{T_{1,c}(q)}{\sum_{n=0}^\infty \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}} = T_4 + (1-q)S_4 \left[1 + \frac{q^2-q}{1+q+} \frac{q^4-q}{1+q+\ldots\ldots}\right],$$

where $T_{1,c}(q)$ is the bilateral Gordon and McIntosh eighth order mock theta function.

(v) From equation (3.13) we have

$$\frac{V_{0,c}\left(0,\alpha,q^{\alpha-1};q\right)}{\sum_{n=0}^{\infty}\frac{q^{2n^2+5n-n\alpha}}{(q^2;q^2)_n(-q^3;q^2)_n}} = S_5 + T_5\left[\frac{\sum_{n=0}^{\infty}\frac{q^{2n^2+3n-n\alpha}}{(q^2;q^2)_n(-q;q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{2n^2+5n-n\alpha}}{(q^2;q^2)_n(-q^3;q^2)_n}}\right],$$

where

$$S_5 = \frac{q^2\left(1+q^{\alpha-2}\right)}{q^{\alpha}\left(1+q^{\alpha-3}\right)}\frac{\left(q^2,-q^{\alpha}/q^3,-q^{2\alpha}/q^2,-q^4/q^{2\alpha};q^2\right)_{\infty}}{\left(q^{2\alpha}/q^3,-1/q,-q^2/q^{\alpha};q^2\right)_{\infty}},$$

$$T_5 = -\frac{q^4}{q^{2\alpha}}\frac{\left(q^2,q^{\alpha}/q^2,q^5/q^{2\alpha},q^4/q^{\alpha};q^2\right)_{\infty}}{\left(-q^2/q^{\alpha},-q;q^2\right)_{\infty}} \times \frac{V_{0,c}\left(0,\alpha,q^{\alpha-1};q\right)}{\sum_{n=0}^{\infty}\frac{q^{2n^2+5n-n\alpha}}{(q^2;q^2)_n(-q^3;q^2)_n}}$$

$$= S_5 + (1+q)T_5\left[1+\frac{q+q^{5-\alpha}}{1-q+}\,\frac{q+q^{7-\alpha}}{1-q+\ldots\ldots}\right]. \quad (1)$$

Here we have used [9, eq.(3.79), p.82].

Special cases. For $\alpha = 0$

$$\frac{V_{0,c}(0,0,1/q;q)}{\sum_{n=0}^{\infty}\frac{q^{2n^2+5n}}{(q^2;q^2)_n(-q^3;q^2)_n}} = S_5 + (1+q)T_5\left[1+\frac{q^5+q}{1-q+}\,\frac{q^7+q}{1-q+\ldots\ldots}\right].$$

(vi) From equation (3.14) we have

$$\frac{V_{1,c}\left(0,\alpha,q^{\alpha-1};q\right)}{\sum_{n=0}^{\infty}\frac{q^{2n^2+3n-n\alpha}}{(q^2;q^2)_n(-q^3;q^2)_n}} = S_6 + T_6\left[\frac{\sum_{n=0}^{\infty}\frac{q^{2n^2+n-n\alpha}}{(q^2;q^2)_n(-q;q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{2n^2+3n-n\alpha}}{(q^2;q^2)_n(-q^3;q^2)_n}}\right],$$

where

$$S_6 = \frac{q}{\left(1+q^{\alpha-3}\right)\left(1-q^{2\alpha-3}\right)}\frac{\left(q^2,-q^{\alpha}/q^3,-q^{2\alpha},-q^2/q^{2\alpha};q^2\right)_{\infty}}{\left(q^{2\alpha}/q,-1/q,-1/q^{\alpha};q^2\right)_{\infty}},$$

$$T_6 = \frac{q^5\left(1+q^{\alpha}\right)}{q^{3\alpha}(1-q)}\frac{\left(q^2,q^{\alpha}/q^2,q^4/q^{\alpha},q^5/q^{2\alpha};q^2\right)_{\infty}}{(-q;q^2)_{\infty}^2\left(-q^3/q^{\alpha};q^2\right)_{\infty}} \times \frac{V_{1,c}\left(0,\alpha,q^{\alpha-1};q\right)}{\sum_{n=0}^{\infty}\frac{q^{2n^2+3n-n\alpha}}{(q^2;q^2)_n(-q^3;q^2)_n}}$$

$$= S_6 + (1+q)T_6\left[1+\frac{q+q^{3-\alpha}}{1-q+}\,\frac{q+q^{5-\alpha}}{1-q+\ldots\ldots}\right].$$

Here we have used [9, eq.(3.79), p.82].

Special cases. For $\alpha = 0$

$$\frac{V_{1,c}(0,0,1/q;q)}{\sum_{n=0}^{\infty} \frac{q^{2n^2+3n}}{(q^2;q^2)_n(-q^3;q^2)_n}}$$
$$= S_6 + (1+q)T_6 \left[1 + \frac{q^3+q}{1-q+} \frac{q^5+q}{1-q+\ldots\ldots}\right].$$

(vii) From equations (3.13) and (3.14), we get

$$\frac{V_{0,c}\left(0,\alpha,q^{\alpha};q\right)}{V_{1,c}\left(0,\alpha,q^{\alpha};q\right)} = \frac{S_7+T_7}{S_8+T_8}\left[\frac{\sum_{n=0}^{\infty}\frac{q^{2n^2+3n-n\alpha}}{(q^4;q^4)_n}}{\sum_{n=0}^{\infty}\frac{q^{2n^2+n-n\alpha}}{(q^4;q^4)_n}}\right],$$

where

$$S_7 = \frac{q}{q^{\alpha}\left(1+q^{\alpha-2}\right)} \frac{\left(q^2,-q^{\alpha}/q^2,-q^{2\alpha}/q,-q^3/q^{2\alpha};q^2\right)_{\infty}}{\left(q^{2\alpha}/q,-1,-q/z;q^2\right)_{\infty}},$$
$$T_7 = -\frac{q^3\left(1+q^{\alpha-1}\right)}{q^{2\alpha}} \frac{\left(q^2,q^{\alpha}/q^2,q^4/q^{\alpha},q^3/q^{2\alpha};q^2\right)_{\infty}}{\left(-q^2/q^{\alpha},-1,-q/z;q^2\right)_{\infty}},$$
$$S_8 = \frac{\left(1+q^{\alpha+1}\right)}{q\left(1+q^{\alpha-2}\right)\left(1-q^{2\alpha-1}\right)} \frac{\left(q^2,-q^{\alpha}/q^2,-q^{2\alpha+1},-q/q^{2\alpha};q^2\right)_{\infty}}{\left(q^{2\alpha+1},-1,-1/q^{\alpha+1};q^2\right)_{\infty}},$$
$$T_8 = -\frac{q^2\left(1+q^{\alpha+1}\right)}{q^{3\alpha}} \frac{\left(q^2,q^{\alpha}/q^2,q^4/q^{\alpha},q^3/q^{2\alpha};q^2\right)_{\infty}}{\left(-q^2/q^{\alpha},-1,-1/q^{\alpha+1};q^2\right)_{\infty}},$$
$$\frac{V_{0,c}\left(0,\alpha,q^{\alpha};q\right)}{V_{1,c}\left(0,\alpha,q^{\alpha};q\right)} = \frac{S_7+T_7}{S_8+T_8}\left[\frac{1}{1} + \frac{q^{3-\alpha}}{1+q^2+} \frac{q^{5-\alpha}}{1+q^4+\ldots\ldots}\right]$$

Here we have used [9, eq.(3.74), p.79].

Special cases. For $\alpha = 1$

$$\frac{V_{0,c}(q)}{V_{1,c}(q)} = \frac{S_7+T_7}{S_8+T_8}\left[\frac{1}{1} + \frac{q^2}{1+q^2+} \frac{q^4}{1+q^4+\ldots\ldots}\right],$$

where $V_{0,c}(q)$ and $V_{1,c}(q)$ are bilateral Gordon and McIntosh eighth order mock theta functions.

Conclusion

Mock theta functions are mysterious functions. In this study we have given Slater's transformation formula to express bilateral generalized mock theta functions of third and eighth order as ${}_2\phi_1$ series and also continued fractions for these bilateral generalized mock theta functions of order third and eighth.

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