$\sup_t^\alpha$-HESITANT FUZZY UP-SUBALGEBRAS AND UP-IDEALS OF UP-ALGEBRAS

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Abstract

As a general concept of a sup-hesitant fuzzy UP-subalgebra (resp., UP-filter, UP-ideal, strong UP-ideal) of a UP-algebra, the concept of a $\sup_t^\alpha$-hesitant fuzzy UP-subalgebra (resp., UP-filter, UP-ideal, strong UP-ideal) is introduced, and the generalizations of these hesitant fuzzy sets are discussed. Moreover, we discuss the relations between the $\sup_t^\alpha$-hesitant fuzzy UP-subalgebras (resp., UP-filters, UP-ideals, strong UP-ideals) and their level subsets.

1. Introduction

As a generalization of a fuzzy set [22], Torra and Narukawa [21, 20] introduced the concept of a hesitant fuzzy set, which is a mapping from a reference set to a power set of the unit interval. After that, hesitant fuzzy sets are applied to algebraic structures; for examples, UP-algebras [10, 12, 11], BCK/BCI-algebras [13, 14, 15], semigroups [6, 18], Γ-semigroups [1, 8], ternary semigroups [7, 19], BE-algebras [16], groups [2, 9].

In 2018, Mosrijai et al. [10] introduced concepts of sup-hesitant fuzzy UP-subalgebras, sup-hesitant fuzzy UP-filters, sup-hesitant fuzzy UP-ideals, and sup-hesitant fuzzy strong UP-ideals of UP-algebras and investigated some properties and generalizations of these concepts. They considered the relations between the concepts and their level subsets.

As previously stated, it motivated us to study hesitant fuzzy set theory on UP-algebras. We introduce concepts of $\sup_t^\alpha$-hesitant fuzzy UP-subalgebras, $\sup_t^\alpha$-hesitant fuzzy UP-filters, $\sup_t^\alpha$-hesitant fuzzy UP-ideals and $\sup_t^\alpha$-hesitant fuzzy strong UP-ideals of UP-algebras when $\alpha \in \{-, +\}$ and $t \in [0, 1]$. These concepts are general forms of the paper by Mosrijai et al. ([10]). Some properties and generalizations of these hesitant fuzzy sets are discussed. Moreover, we investigate the relations between the $\sup_t^\alpha$-hesitant fuzzy UP-subalgebras (resp., UP-filters, UP-ideals, strong UP-ideals) and their level subsets. In particular, it is shown that if we take $t = 0$, then the paper's main results [10] hold.

2. Preliminaries

Before we start our study, we recall some definitions and facts about UP-algebras.

An algebra $\mathcal{A} = (\mathcal{A}, *, 0)$ of type $(2, 0)$ is called a *UP-algebra* [4] where $\mathcal{A}$ is a nonempty set, $*$ is a binary operation on $\mathcal{A}$, and 0 is a fixed element of $\mathcal{A}$ if it satisfies the following assertions:

(UP-1): $(\forall p, q, r \in \mathcal{A})((q * r) * ((p * q) * (p * r)) = 0)$,
(UP-2): $(\forall p \in \mathcal{A})(0 * p = p)$,
(UP-3): $(\forall p \in \mathcal{A})(p * 0 = 0)$, and
(UP-4): $(\forall p, q \in \mathcal{A})(p * q = 0, q * p = 0 \Rightarrow p = q)$.

In a UP-algebra $\mathcal{A} = (\mathcal{A}, *, 0)$, the following assertions are valid (see [4, 5]):

$$(\forall p \in \mathcal{A})(p * p = 0), \tag{2.1}$$

$$(\forall p, q, r \in \mathcal{A})(p * q = 0, q * r = 0 \Rightarrow p * r = 0), \tag{2.2}$$

$$(\forall p, q, r \in \mathcal{A})(p * q = 0 \Rightarrow (r * p) * (r * q) = 0), \tag{2.3}$$

$$(\forall p, q, r \in \mathcal{A})(p * q = 0 \Rightarrow (q * r) * (p * r) = 0), \tag{2.4}$$

$$(\forall p, q \in \mathcal{A})(p * (q * p) = 0), \tag{2.5}$$

$$(\forall p, q \in \mathcal{A})((q * p) * p = 0 \Leftrightarrow p = q * p), \tag{2.6}$$

$$(\forall p, q \in \mathcal{A})(p * (q * q) = 0), \tag{2.7}$$

$$(\forall p, q, r, a \in \mathcal{A})((p * (q * r)) * (p * ((a * q) * (a * r))) = 0), \tag{2.8}$$

$$(\forall p, q, r, a \in \mathcal{A})((((a * p) * (a * q)) * r) * ((p * q) * r) = 0), \tag{2.9}$$

$$(\forall p, q, r \in \mathcal{A})(((p * q) * r) * (q * r) = 0), \tag{2.10}$$

$$(\forall p, q, r \in \mathcal{A})(p * q = 0 \Rightarrow p * (r * q) = 0), \tag{2.11}$$

$$(\forall p, q, r \in \mathcal{A})(((p * q) * r) * (p * (q * r)) = 0), \text{ and} \tag{2.12}$$

$$(\forall p, q, r, a \in \mathcal{A})(((p * q) * r) * (q * (a * r)) = 0). \tag{2.13}$$

We assume from now on that $\mathcal{A}$ will always denote a UP-algebra $(\mathcal{A}, *, 0)$.

DEFINITION 2.1. [4] A subset S of $\mathcal{A}$ is called a *UP-subalgebra* of $\mathcal{A}$ if the constant 0 of $\mathcal{A}$ is in S, and $(S, *, 0)$ itself forms a UP-algebra.

Iampan [4] proved the useful criteria that a nonempty subset S of $\mathcal{A}$ is a UP-subalgebra of $\mathcal{A}$ if and only if S is closed under the $*$ multiplication on $\mathcal{A}$.

DEFINITION 2.2. ([17, 4, 3]) A subset S of $\mathcal{A}$ is called:
(1) a *UP-filter* of $\mathcal{A}$ if $0 \in S$, and

$$(\forall p, q \in \mathcal{A})(p * q \in S, p \in S \Rightarrow q \in S),$$

(2) a *UP-ideal* of $\mathcal{A}$ if $0 \in S$, and

$$(\forall p, q, r \in \mathcal{A})(p * (q * r) \in S, q \in S \Rightarrow p * r \in S),$$

(3) a *strong UP-ideal* of $\mathcal{A}$ if $0 \in S$, and

$$(\forall p, q, r \in \mathcal{A})((r * q) * (r * p) \in S, q \in S \Rightarrow p \in S).$$

Guntasow et al. [3] showed that the concept of a UP-subalgebra is a general concept of a UP-filter, the concept of a UP-filter is a general concept of a UP-ideal, and the concept of a UP-ideal is a general concept of a strong UP-ideal. Furthermore, they also proved that a UP-algebra $\mathcal{A}$ is the only one strong UP-ideal of itself.

A *fuzzy subset* (FS) [22] of $\mathcal{A}$ is a function from the set $\mathcal{A}$ into the unit interval $[0,1]$. A *hesitant fuzzy set* (HFS) [20, 21] on $\mathcal{A}$ is defined in term of a function $\widehat{\xi}$ that when applied to $\mathcal{A}$ return a subset of $[0,1]$, that is, $\widehat{\xi}\colon \mathcal{A} \to \wp([0,1])$ where $\wp([0,1])$ is the power set of $[0,1]$. Note that the concept of HFSs is a generalization of the concept of FSs. For a HFS $\widehat{\xi}$ on $\mathcal{A}$ and an element p of $\mathcal{A}$, define $\mathrm{SUP}\widehat{\xi}(p)$ by

$$\mathrm{SUP}\widehat{\xi}(p) = \begin{cases} \sup \widehat{\xi}(p) \text{ if } \widehat{\xi}(p) \neq \emptyset, \\ \quad 0 \quad\;\; \text{otherwise.} \end{cases}$$

DEFINITION 2.3. ([10]) A HFS $\widehat{\xi}$ on $\mathcal{A}$ is called:

(1) a sup-*hesitant fuzzy UP-subalgebra* (sup-HFUPs) of $\mathcal{A}$ if

$$(\forall p, q \in \mathcal{A})(\mathrm{SUP}\widehat{\xi}(p * q) \geq \min\{\mathrm{SUP}\widehat{\xi}(p), \mathrm{SUP}\widehat{\xi}(q)\}),$$

(2) a sup-*hesitant fuzzy UP-filter* (sup-HFUPf) of $\mathcal{A}$ if
 (i) $(\forall p \in \mathcal{A})(\mathrm{SUP}\widehat{\xi}(0) \geq \mathrm{SUP}\widehat{\xi}(p))$, and
 (ii) $(\forall p, q \in \mathcal{A})(\mathrm{SUP}\widehat{\xi}(q) \geq \min\{\mathrm{SUP}\widehat{\xi}(p * q), \mathrm{SUP}\widehat{\xi}(p)\})$,

(3) a sup-*hesitant fuzzy UP-ideal* (sup-HFUPi) of $\mathcal{A}$ if
 (i) $(\forall p \in \mathcal{A})(\mathrm{SUP}\widehat{\xi}(0) \geq \mathrm{SUP}\widehat{\xi}(p))$, and
 (ii) $(\forall p, q, r \in \mathcal{A})(\mathrm{SUP}\widehat{\xi}(p * r) \geq \min\{\mathrm{SUP}\widehat{\xi}(p * (q * r)), \mathrm{SUP}\widehat{\xi}(q)\})$,

(4) a sup-*hesitant fuzzy strong UP-ideal* (sup-HFSUPi) of $\mathcal{A}$ if
 (i) $(\forall p \in \mathcal{A})(\mathrm{SUP}\widehat{\xi}(0) \geq \mathrm{SUP}\widehat{\xi}(p))$, and
 (ii) $(\forall p, q, r \in \mathcal{A})(\mathrm{SUP}\widehat{\xi}(p) \geq \min\{\mathrm{SUP}\widehat{\xi}((r * q) * (r * p)), \mathrm{SUP}\widehat{\xi}(q)\})$.

Mosrijai et al. [10] showed that the concept of a sup-HFUPs is a general concept of a sup-HFUPf, the concept of a sup-HFUPf is a general concept of a sup-HFUPi, and the concept of a sup-HFUPi is a general concept of a sup-HFSUPi. Moreover, they also proved that a HFS on a UP-algebra is a sup-HFSUPi if and only if it is a constant function.

3. sup_t^α-**hesitant fuzzy UP-substructures**

In this section, we will introduce concepts of sup_t^α-hesitant fuzzy UP-subalgebras, sup_t^α-hesitant fuzzy UP-filters, sup_t^α-hesitant fuzzy UP-ideals and sup_t^α-hesitant fuzzy strong UP-ideals of UP-algebras when α is an element of $\{-, +\}$ and t is an element of $[0,1]$, prove their generalizations and investigate some of their important properties.

Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$, $x \in \mathcal{A}$ and $t \in [0,1]$. We define

$$\mathrm{SUP}_t^-\widehat{\xi}(x) = \max\{\mathrm{SUP}\widehat{\xi}(x) - t, 0\},$$

and

$$\mathrm{SUP}_t^+\widehat{\xi}(x) = \min\{\mathrm{SUP}\widehat{\xi}(x) + t, 1\}.$$

Clearly, we have $\mathrm{SUP}_0^- \widehat{\xi}(x) = \mathrm{SUP}_0^+ \widehat{\xi}(x) = \mathrm{SUP}\widehat{\xi}(x)$. From now on throughout this paper, α is an element of $\{-, +\}$ and t is an element of $[0,1]$ unless otherwise. In the following definition, we introduce $\sup_t^+$-hesitant fuzzy UP-subalgebras, $\sup_t^+$-hesitant fuzzy UP-filters, $\sup_t^+$-hesitant fuzzy UP-ideals, and $\sup_t^+$-hesitant fuzzy strong UP-ideals of UP-algebras.

DEFINITION 3.1.   A HFS $\widehat{\xi}$ on $\mathcal{A}$ is called:

(1) a $\sup_t^+$-*hesitant fuzzy UP-subalgebra* ($\sup_t^+$-HFUPs) of $\mathcal{A}$ if
    $\mathrm{SUP}_t^+ \widehat{\xi}(p * q) \geq \min\{\mathrm{SUP}_t^+ \widehat{\xi}(p), \mathrm{SUP}_t^+ \widehat{\xi}(q)\}$ for all $p, q \in \mathcal{A}$,
(2) a $\sup_t^+$-*hesitant fuzzy UP-filter* ($\sup_t^+$-HFUPf) of $\mathcal{A}$ if
    (i) $\mathrm{SUP}_t^+ \widehat{\xi}(0) \geq \mathrm{SUP}_t^+ \widehat{\xi}(p)$ for all $p \in \mathcal{A}$, and
    (ii) $\mathrm{SUP}_t^+ \widehat{\xi}(q) \geq \min\{\mathrm{SUP}_t^+ \widehat{\xi}(p * q), \mathrm{SUP}_t^+ \widehat{\xi}(p)\}$ for all $p, q \in \mathcal{A}$,
(3) a $\sup_t^+$-*hesitant fuzzy UP-ideal* ($\sup_t^+$-HFUPi) of $\mathcal{A}$ if
    (i) $\mathrm{SUP}_t^+ \widehat{\xi}(0) \geq \mathrm{SUP}_t^+ \widehat{\xi}(p)$ for all $p \in \mathcal{A}$, and
    (ii) $\mathrm{SUP}_t^+ \widehat{\xi}(p * r) \geq \min\{\mathrm{SUP}_t^+ \widehat{\xi}(p * (q * r)), \mathrm{SUP}_t^+ \widehat{\xi}(q)\}$ for all
          $p, q, r \in \mathcal{A}$,
(4) a $\sup_t^+$-*hesitant fuzzy strong UP-ideal* ($\sup_t^+$-HFSUPi) of $\mathcal{A}$ if
    (i) $\mathrm{SUP}_t^+ \widehat{\xi}(0) \geq \mathrm{SUP}_t^+ \widehat{\xi}(p)$ for all $p \in \mathcal{A}$, and
    (ii) $\mathrm{SUP}_t^+ \widehat{\xi}(p) \geq \min\{\mathrm{SUP}_t^+ \widehat{\xi}((r * q) * (r * p)), \mathrm{SUP}_t^+ \widehat{\xi}(q)\}$ for all
          $p, q, r \in \mathcal{A}$.

If $\widehat{\xi}$ is a $\sup_t^+$-HFUPs of $\mathcal{A}$, then it follows from (2.1), we have that $\mathrm{SUP}_t^+ \widehat{\xi}(0) \geq \mathrm{SUP}_t^+ \widehat{\xi}(p)$ for all $p \in \mathcal{A}$.

EXAMPLE 3.1.   Let $\mathcal{A} = \{0, p, q, r\}$ be a UP-algebra with a binary operation $*$ defined by the following Cayley table:

$*$	0	p	q	r
0	0	p	q	r
p	0	0	r	r
q	0	p	0	0
r	0	p	q	0

(1) Define a HFS $\widehat{\xi}_1$ on $\mathcal{A}$ as follows: $\widehat{\xi}_1(0) = [0, 0.7]$, $\widehat{\xi}_1(p) = \{0.2, 0.3\}$, $\widehat{\xi}_1(q) = \{0.1, 0.2, 0.4\}$, $\widehat{\xi}_1(r) = (0, 0.8)$, we get that $\widehat{\xi}_1$ is a $\sup_t^+$-HFUPs of $\mathcal{A}$ for all $t \in [0.3, 1]$.
(2) Define a HFS $\widehat{\xi}_2$ on $\mathcal{A}$ as follows: $\widehat{\xi}_2(0) = \{0.6, 0.7, 0.8\}$, $\widehat{\xi}_2(p) = [0.3, 0.9]$, $\widehat{\xi}_2(q) = \emptyset$, $\widehat{\xi}_2(r) = \{0\}$, we have that $\widehat{\xi}_2$ is a $\sup_t^+$-HFUPf of $\mathcal{A}$ for all $t \in [0.2, 1]$.

(3) Define a HFS $\widehat{\xi}_3$ on $\mathcal{A}$ as follows: $\widehat{\xi}_3(0) = \{0.8, 0.9\}$, $\widehat{\xi}_3(p) = [0, 0.5]$, $\widehat{\xi}_3(q) = \{0.3, 0.5, 0.7\}$, $\widehat{\xi}_3(r) = (0, 0.8)$, we get that $\widehat{\xi}_3$ is a $\sup_t^+$-HFUPi of $\mathcal{A}$ for all $t \in [0, 1]$.

(4) Define a HFS $\widehat{\xi}_4$ on $\mathcal{A}$ as follows: $\widehat{\xi}_4(0) = (0, 0.6)$, $\widehat{\xi}_4(p) = [0.3, 0.9]$, $\widehat{\xi}_4(q) = (0.1, 0.8]$, $\widehat{\xi}_4(r) = \{0.5, 0.6, 0.7\}$, we have that $\widehat{\xi}_4$ is a $\sup_{0.4}^+$-HFSUPi of $\mathcal{A}$ for all $t \in [0.4, 1]$.

PROPOSITION 3.1. *Every sup-HFUPs (resp., sup-HFUPf, sup-HFUPi, sup-HFSUPi) of $\mathcal{A}$ is a $\sup_t^+$-HFUPs (resp., $\sup_t^+$-HFUPf, $\sup_t^+$-HFUPi, $\sup_t^+$-HFSUPi) of $\mathcal{A}$.*

P r o o f. Assume that $\widehat{\xi}$ is a sup-HFUPs of $\mathcal{A}$ and $p, q \in \mathcal{A}$. Then $\mathrm{SUP}\widehat{\xi}(p*q) \geq \min\{\mathrm{SUP}\widehat{\xi}(p), \mathrm{SUP}\widehat{\xi}(q)\}$ which implies that $\mathrm{SUP}\widehat{\xi}(p*q) \geq \mathrm{SUP}\widehat{\xi}(p)$ or $\mathrm{SUP}\widehat{\xi}(p*q) \geq \mathrm{SUP}\widehat{\xi}(q)$. Now, we consider the following two cases:

Case 1: Suppose that $\mathrm{SUP}\widehat{\xi}(p*q) \geq \mathrm{SUP}\widehat{\xi}(p)$. Then $\mathrm{SUP}\widehat{\xi}(p*q) + t \geq \mathrm{SUP}\widehat{\xi}(p) + t$ and so

$$\begin{aligned}
\mathrm{SUP}_t^+\widehat{\xi}(p*q) &= \min\{\mathrm{SUP}\widehat{\xi}(p*q) + t, 1\} \\
&\geq \min\{\mathrm{SUP}\widehat{\xi}(p) + t, 1\} \\
&= \mathrm{SUP}_t^+\widehat{\xi}(p) \\
&\geq \min\{\mathrm{SUP}_t^+\widehat{\xi}(p), \mathrm{SUP}_t^+\widehat{\xi}(q)\}.
\end{aligned}$$

Thus,

$$\mathrm{SUP}_t^+\widehat{\xi}(p*q) \geq \min\{\mathrm{SUP}_t^+\widehat{\xi}(p), \mathrm{SUP}_t^+\widehat{\xi}(q)\}.$$

Case 2: Suppose that $\mathrm{SUP}\widehat{\xi}(p*q) \geq \mathrm{SUP}\widehat{\xi}(q)$. By proving similarly as Case 1, we have

$$\mathrm{SUP}_t^+\widehat{\xi}(p*q) \geq \min\{\mathrm{SUP}_t^+\widehat{\xi}(p), \mathrm{SUP}_t^+\widehat{\xi}(q)\}.$$

By the two cases, we obtain that $\widehat{\xi}$ is a $\sup_t^+$-HFUPs of $\mathcal{A}$.

Similarly, we can prove the other results.   □

The following example shows the converse of Proposition 3.1.

EXAMPLE 3.2. Let $\mathcal{A} = \{0, p, q, r\}$ be a UP-algebra defined in Example 3.1. We define a HFS $\widehat{\xi}$ on $\mathcal{A}$ as follows:

$$\widehat{\xi}(0) = [0.5, 0.6], \widehat{\xi}(p) = [0, 1], \widehat{\xi}(q) = [0.4, 0.5], \widehat{\xi}(r) = \{0.1, 0.2, 0.3, 0.4\}.$$

Then $\widehat{\xi}$ is a $\sup_t^+$-HFUPs (also, $\sup_t^+$-HFUPf, $\sup_t^+$-HFUPi, $\sup_t^+$-HFSUPi) of $\mathcal{A}$ for all $t \in [0.6, 1]$ but, $\widehat{\xi}$ is not an sup-HFUPs (also, sup-HFUPf, sup-HFUPi, sup-HFSUPi) of $\mathcal{A}$ because $\mathrm{SUP}\widehat{\xi}(0) \not\geq \mathrm{SUP}\widehat{\xi}(p)$.

PROPOSITION 3.2. *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$ and $s \in (0, 1]$. If $\widehat{\xi}$ is a $\sup_t^+$-HFUPs (resp., $\sup_t^+$-HFUPf, $\sup_t^+$-HFUPi, $\sup_t^+$-HFSUPi) of $\mathcal{A}$ for all $t \in (0, s]$, then $\widehat{\xi}$ is a sup-HFUPs (resp., sup-HFUPf, sup-HFUPi, sup-HFSUPi) of $\mathcal{A}$.*

P r o o f. Suppose that there exist $p, q \in \mathcal{A}$ such that
$$\mathrm{SUP}\widehat{\xi}(p * q) < \min\{\mathrm{SUP}\widehat{\xi}(p), \mathrm{SUP}\widehat{\xi}(q)\}.$$
Let
$$t = \min\{\frac{\min\{\mathrm{SUP}\widehat{\xi}(p), \mathrm{SUP}\widehat{\xi}(q)\} - \mathrm{SUP}\widehat{\xi}(p * q)}{2}, s\}.$$
Then $t \in (0, s]$ and $\mathrm{SUP}\widehat{\xi}(p * q) + t < 1$. Thus,
$$\begin{aligned}
\min\{\mathrm{SUP}\widehat{\xi}(p) + t, \mathrm{SUP}\widehat{\xi}(q) + t\} &= \min\{\mathrm{SUP}\widehat{\xi}(p), \mathrm{SUP}\widehat{\xi}(q)\} + t \\
&> \mathrm{SUP}\widehat{\xi}(p * q) + t \\
&= \min\{\mathrm{SUP}\widehat{\xi}(p * q) + t, 1\} \\
&= \mathrm{SUP}_t^+\widehat{\xi}(p * q).
\end{aligned}$$
Hence, $\mathrm{SUP}_t^+\widehat{\xi}(p * q) < \mathrm{SUP}_t^+\widehat{\xi}(p)$ and $\mathrm{SUP}_t^+\widehat{\xi}(p * q) < \mathrm{SUP}_t^+\widehat{\xi}(q)$. Since $\widehat{\xi}$ is a $\sup_t^+$-HFUPs of $\mathcal{A}$, we get
$$\mathrm{SUP}_t^+\widehat{\xi}(p * q) \geq \min\{\mathrm{SUP}_t^+\widehat{\xi}(p), \mathrm{SUP}_t^+\widehat{\xi}(q)\} > \mathrm{SUP}_t^+\widehat{\xi}(p * q),$$
which is a contradiction. Therefore, $\widehat{\xi}$ is a sup-HFUPs of $\mathcal{A}$.

Similarly, we can prove the other results.   □

Next, we introduce $\sup_t^-$-hesitant fuzzy UP-subalgebras, $\sup_t^-$-hesitant fuzzy UP-filters, $\sup_t^-$-hesitant fuzzy UP-ideals and $\sup_t^-$-hesitant fuzzy strong UP-ideals of UP-algebras.

DEFINITION 3.2. A HFS $\widehat{\xi}$ on $\mathcal{A}$ is called:
(1) a $\sup_t^-$-*hesitant fuzzy UP-subalgebra* ($\sup_t^-$-HFUPs) of $\mathcal{A}$ if
   $\mathrm{SUP}_t^-\widehat{\xi}(p * q) \geq \min\{\mathrm{SUP}_t^-\widehat{\xi}(p), \mathrm{SUP}_t^-\widehat{\xi}(q)\}$ for all $p, q \in \mathcal{A}$,
(2) a $\sup_t^-$-*hesitant fuzzy UP-filter* ($\sup_t^-$-HFUPf) of $\mathcal{A}$ if
   (i) $\mathrm{SUP}_t^-\widehat{\xi}(0) \geq \mathrm{SUP}_t^-\widehat{\xi}(p)$ for all $p \in \mathcal{A}$, and
   (ii) $\mathrm{SUP}_t^-\widehat{\xi}(q) \geq \min\{\mathrm{SUP}_t^-\widehat{\xi}(p * q), \mathrm{SUP}_t^-\widehat{\xi}(p)\}$ for all $p, q \in \mathcal{A}$,
(3) a $\sup_t^-$-*hesitant fuzzy UP-ideal* ($\sup_t^-$-HFUPi)of $\mathcal{A}$ if

(i) $\mathrm{SUP}_t^- \widehat{\xi}(0) \geq \mathrm{SUP}_t^- \widehat{\xi}(p)$ for all $p \in \mathcal{A}$, and
(ii) $\mathrm{SUP}_t^- \widehat{\xi}(p * r) \geq \min\{\mathrm{SUP}_t^- \widehat{\xi}(p * (q * r)), \mathrm{SUP}_t^- \widehat{\xi}(q)\}$ for all $p, q, r \in \mathcal{A}$,

(4) a $\sup_t^-$-*hesitant fuzzy strong UP-ideal* ($\sup_t^-$-HFSUPi) of $\mathcal{A}$ if
(i) $\mathrm{SUP}_t^- \widehat{\xi}(0) \geq \mathrm{SUP}_t^- \widehat{\xi}(p)$ for all $p \in \mathcal{A}$, and
(ii) $\mathrm{SUP}_t^- \widehat{\xi}(p) \geq \min\{\mathrm{SUP}_t^- \widehat{\xi}((r * q) * (r * p)), \mathrm{SUP}_t^- \widehat{\xi}(q)\}$ for all $p, q, r \in \mathcal{A}$.

If $\widehat{\xi}$ is a $\sup_t^-$-HFUPs of $\mathcal{A}$, then it follows from (2.1), we can obviously see that $\mathrm{SUP}_t^- \widehat{\xi}(0) \geq \mathrm{SUP}_t^- \widehat{\xi}(p)$ for all $p \in \mathcal{A}$.

EXAMPLE 3.3. Let $\mathcal{A} = \{0, p, q, r\}$ be a UP-algebra with a binary operation $*$ defined by the following Cayley table:

$*$	0	p	q	r
0	0	p	q	r
p	0	0	q	r
q	0	0	0	0
r	0	0	q	0

(1) Define a HFS $\widehat{\xi}_1$ on $\mathcal{A}$ by: $\widehat{\xi}_1(0) = [0.4, 0.8]$, $\widehat{\xi}_1(p) = (0.6, 0.7)$, $\widehat{\xi}_1(q) = \{0.5\}$, $\widehat{\xi}_1(r) = \emptyset$, we get that $\widehat{\xi}_1$ is a $\sup_t^-$-HFUPs of $\mathcal{A}$ for all $t \in [0, 1]$.
(2) Define a HFS $\widehat{\xi}_2$ on $\mathcal{A}$ by: $\widehat{\xi}_2(0) = (0, 1)$, $\widehat{\xi}_2(p) = (0.6, 0.9]$, $\widehat{\xi}_2(q) = [0, 0.5]$, $\widehat{\xi}_2(r) = \{0, 0.1\}$, we get that $\widehat{\xi}_2$ is a $\sup_t^-$-HFUPf of $\mathcal{A}$ for all $t \in [0.5, 1]$.
(3) Define a HFS $\widehat{\xi}_3$ on $\mathcal{A}$ by: $\widehat{\xi}_3(0) = \{0.4, 0.7\}$, $\widehat{\xi}_3(p) = [0.2, 0.6]$, $\widehat{\xi}_3(q) = (0.2, 0.4)$, $\widehat{\xi}_3(r) = \emptyset$, we get that $\widehat{\xi}_3$ is a $\sup_t^-$-HFUPi of $\mathcal{A}$ for all $t \in [0.4, 1]$.
(4) Define a HFS $\widehat{\xi}_4$ on $\mathcal{A}$ by: $\widehat{\xi}_4(0) = \{0.2, 0.4, 0.5\}$, $\widehat{\xi}_4(p) = [0.1, 0.6]$, $\widehat{\xi}_4(q) = 0$, $\widehat{\xi}_4(r) = \emptyset$, we get that $\widehat{\xi}_4$ is a $\sup_t^-$-HFSUPi of $\mathcal{A}$ for all $t \in [0.6, 1]$.

PROPOSITION 3.3. *Every sup-HFUPs (resp., sup-HFUPf, sup-HFUPi, sup-HFSUPi) of $\mathcal{A}$ is a $\sup_t^-$-HFUPs (resp., $\sup_t^-$-HFUPf, $\sup_t^-$-HFUPi, $\sup_t^-$-HFSUPi) of $\mathcal{A}$.*

P r o o f. Let $\widehat{\xi}$ be a sup-HFUPs of $\mathcal{A}$ and $p, q \in \mathcal{A}$. Then $\mathrm{SUP}\widehat{\xi}(p * q) \geq \mathrm{SUP}\widehat{\xi}(p)$ or $\mathrm{SUP}\widehat{\xi}(p * q) \geq \mathrm{SUP}\widehat{\xi}(q)$. Suppose that $\mathrm{SUP}\widehat{\xi}(p * q) \geq \mathrm{SUP}\widehat{\xi}(q)$,

then we have $\text{SUP}\widehat{\xi}(p * q) - t \geq \text{SUP}\widehat{\xi}(q) - t$ and so,

$$\text{SUP}_t^-\widehat{\xi}(p * q) = \max\{\text{SUP}\widehat{\xi}(p * q) - t, 0\}$$
$$\geq \max\{\text{SUP}\widehat{\xi}(q) - t, 0\}$$
$$= \text{SUP}_t^-\widehat{\xi}(q)$$
$$\geq \min\{\text{SUP}_t^-\widehat{\xi}(p), \text{SUP}_t^-\widehat{\xi}(q)\}.$$

Thus,

$$\text{SUP}_t^-\widehat{\xi}(p * q) \geq \min\{\text{SUP}_t^-\widehat{\xi}(p), \text{SUP}_t^-\widehat{\xi}(q)\}.$$

On the other hand, if $\text{SUP}\widehat{\xi}(p * q) \geq \text{SUP}\widehat{\xi}(p)$, then we can prove that

$$\text{SUP}_t^-\widehat{\xi}(p * q) \geq \min\{\text{SUP}_t^-\widehat{\xi}(p), \text{SUP}_t^-\widehat{\xi}(q)\}.$$

Therefore, we have that $\widehat{\xi}$ is a $\sup_t^-$-HFUPs of $\mathcal{A}$.

Similarly, we can prove the other results.   □

The following example shows the converse of Proposition 3.3.

EXAMPLE 3.4.   Let $\mathcal{A} = \{0, p, q, r\}$ be a UP-algebra defined in Example 3.3 and define a HFS $\widehat{\xi}$ on $\mathcal{A}$ by

$$\widehat{\xi}(0) = \emptyset, \widehat{\xi}(p) = \{0.5\}, \widehat{\xi}(q) = [0.4, 0.7), \widehat{\xi}(r) = \{0.1, 0.2, 0.3\}.$$

Then $\widehat{\xi}$ is a $\sup_t^-$-HFUPs (also, $\sup_t^-$-HFUPf, $\sup_t^-$-HFUPi, $\sup_t^-$-HFSUPi) of $\mathcal{A}$ for all $t \in [0.7, 1]$ but, $\widehat{\xi}$ is not a sup-HFUPs (also, sup-HFUPf, sup-HFUPi, sup-HFSUPi) of $\mathcal{A}$ because $\text{SUP}\widehat{\xi}(0) \not\geq \text{SUP}\widehat{\xi}(q)$.

By Proposition 3.3 and Example 3.4, we have that the concepts of a $\sup_t^-$-HFUPs (resp., $\sup_t^-$-HFUPf, $\sup_t^-$-HFUPi, $\sup_t^-$-HFSUPi) of a UP-algebra is a generalization of the concept of a sup-HFUPs (resp., sup-HFUPf, sup-HFUPi, sup-HFSUPi).

PROPOSITION 3.4.   Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$ and $s \in (0, 1]$. If $\widehat{\xi}$ is a $\sup_t^-$-HFUPs (resp., $\sup_t^-$-HFUPf, $\sup_t^-$-HFUPi, $\sup_t^-$-HFSUPi) of $\mathcal{A}$ for all $t \in (0, s]$, then $\widehat{\xi}$ is a sup-HFUPs (resp., sup-HFUPf, sup-HFUPi, sup-HFSUPi) of $\mathcal{A}$.

P r o o f. Suppose that $\text{SUP}\widehat{\xi}(p * q) < \min\{\text{SUP}\widehat{\xi}(p), \text{SUP}\widehat{\xi}(q)\}$ for some $p, q \in \mathcal{A}$. Choose $t \in (0, s]$ such that $\min\{\text{SUP}\widehat{\xi}(p), \text{SUP}\widehat{\xi}(q)\} - t > 0$. Then

$$\text{SUP}_t^-\widehat{\xi}(p) = \max\{\text{SUP}\widehat{\xi}(p) - t, 0\} = \text{SUP}\widehat{\xi}(p) - t$$

and
$$\mathrm{SUP}_t^-\widehat{\xi}(q) = \max\{\mathrm{SUP}\widehat{\xi}(q) - t, 0\} = \mathrm{SUP}\widehat{\xi}(q) - t.$$
Since $\widehat{\xi}$ is a $\sup_t^-$-HFUPs of $\mathcal{A}$, we get
$$\begin{aligned}\mathrm{SUP}_t^-\widehat{\xi}(p * q) &\geq \min\{\mathrm{SUP}_t^-\widehat{\xi}(p), \mathrm{SUP}_t^-\widehat{\xi}(q)\}\\ &= \min\{\mathrm{SUP}\widehat{\xi}(p) - t, \mathrm{SUP}\widehat{\xi}(q) - t\}\\ &= \min\{\mathrm{SUP}\widehat{\xi}(p), \mathrm{SUP}\widehat{\xi}(q)\} - t\\ &> 0.\end{aligned}$$
Thus,
$$\begin{aligned}\mathrm{SUP}_t^-\widehat{\xi}(p * q) &= \max\{\mathrm{SUP}\widehat{\xi}(p * q) - t, 0\}\\ &= \mathrm{SUP}\widehat{\xi}(p * q) - t\\ &< \min\{\mathrm{SUP}\widehat{\xi}(p), \mathrm{SUP}\widehat{\xi}(q)\} - t\\ &= \min\{\mathrm{SUP}_t^-\widehat{\xi}(p), \mathrm{SUP}_t^-\widehat{\xi}(q)\},\end{aligned}$$
this is a contradiction. Therefore, $\widehat{\xi}$ is a sup-HFUPs of $\mathcal{A}$.

Similarly, we can prove the other results.　　□

THEOREM 3.1.　*Every $\sup_t^\alpha$-HFUPf of $\mathcal{A}$ is a $\sup_t^\alpha$-HFUPs.*

P r o o f. Let $\widehat{\xi}$ be a $\sup_t^\alpha$-HFUPf of $\mathcal{A}$. Then
$$\begin{aligned}\mathrm{SUP}_t^\alpha\widehat{\xi}(p * q) &\geq \min\{\mathrm{SUP}_t^\alpha\widehat{\xi}(q * (p * q)), \mathrm{SUP}_t^\alpha\widehat{\xi}(q)\}\\ &= \min\{\mathrm{SUP}_t^\alpha\widehat{\xi}(0), \mathrm{SUP}_t^\alpha\widehat{\xi}(q)\} \qquad ((2.5))\\ &= \mathrm{SUP}_t^\alpha\widehat{\xi}(q)\\ &\geq \min\{\mathrm{SUP}_t^\alpha\widehat{\xi}(p), \mathrm{SUP}_t^\alpha\widehat{\xi}(q)\}\end{aligned}$$
for all $p, q \in \mathcal{A}$. Therefore, $\widehat{\xi}$ is a $\sup_t^\alpha$-HFUPs of $\mathcal{A}$.　　□

The following example shows the converse of Theorem 3.1.

EXAMPLE 3.5.　Let $\mathcal{A} = \{0, p, q, r\}$ be a UP-algebra with a binary operation $*$ defined by the following Cayley table:

$*$	0	p	q	r
0	0	p	q	r
p	0	0	p	r
q	0	0	0	r
r	0	0	0	0

We define a HFS $\widehat{\xi}$ on $\mathcal{A}$ as follows:

$\widehat{\xi}(0) = [0,1], \widehat{\xi}(p) = (0.1, 0.3), \widehat{\xi}(q) = (0.3, 0.5], \widehat{\xi}(r) = \{0.2, 0.4, 0.6, 0.7\}.$

Then, the following statements hold:

(1) $\widehat{\xi}$ is a $\sup_{0.2}^-$-HFUPs of $\mathcal{A}$ but not a $\sup_{0.2}^-$-HFUPf of $\mathcal{A}$ because

$$\mathrm{SUP}_{0.2}^- \widehat{\xi}(p) = 0.1 \not\geq 0.3 = \min\{\mathrm{SUP}_{0.2}^- \widehat{\xi}(q * p), \mathrm{SUP}_{0.2}^- \widehat{\xi}(q)\}.$$

(2) $\widehat{\xi}$ is a $\sup_{0.6}^+$-HFUPs of $\mathcal{A}$ but not a $\sup_{0.6}^+$-HFUPf of $\mathcal{A}$ because

$$\mathrm{SUP}_{0.6}^+ \widehat{\xi}(p) = 0.9 \not\geq 1 = \min\{\mathrm{SUP}_{0.6}^+ \widehat{\xi}(q * p), \mathrm{SUP}_{0.6}^+ \widehat{\xi}(q)\}.$$

By taking $t = 0$ in Theorem 3.1, we have next Corollary 3.1.

COROLLARY 3.1. [10] *Every sup-HFUPf of $\mathcal{A}$ is a sup-HFUPs.*

THEOREM 3.2. *Every $\sup_t^\alpha$-HFUPi of $\mathcal{A}$ is a $\sup_t^\alpha$-HFUPf.*

P r o o f. Let $\widehat{\xi}$ be a $\sup_t^\alpha$-HFUPi of $\mathcal{A}$ and $p, q \in \mathcal{A}$. Then $\mathrm{SUP}_t^\alpha \widehat{\xi}(0) \geq \mathrm{SUP}_t^\alpha \widehat{\xi}(p)$ and

$$\mathrm{SUP}_t^\alpha \widehat{\xi}(q) = \mathrm{SUP}_t^\alpha \widehat{\xi}(0 * q) \qquad\qquad ((\text{UP-2}))$$
$$\geq \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(0 * (p * q)), \mathrm{SUP}_t^\alpha \widehat{\xi}(p)\}$$
$$= \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(p * q), \mathrm{SUP}_t^\alpha \widehat{\xi}(p)\}. \qquad ((\text{UP-2}))$$

Hence, $\widehat{\xi}$ is a $\sup_t^\alpha$-HFUPf of $\mathcal{A}$. □

The following example shows the converse of Theorem 3.2.

EXAMPLE 3.6. Let $\mathcal{A} = \{0, p, q, r\}$ be a UP-algebra defined in Example 3.1. We define a HFS $\widehat{\xi}$ on $\mathcal{A}$ by:

$\widehat{\xi}(0) = (0.7, 0.8], \widehat{\xi}(p) = [0.5, 0.6], \widehat{\xi}(q) = \{0.4, 0.5\}, \widehat{\xi}(r) = (0.3, 0.5).$

Hence, the following statements are true:

(1) $\widehat{\xi}$ is a $\sup_{0.3}^-$-HFUPf of $\mathcal{A}$ and since $\mathrm{SUP}_{0.3}^- \widehat{\xi}(r * q) = 0.2 \not\geq 0.3 = \min\{\mathrm{SUP}_{0.3}^- \widehat{\xi}(r * (p * q)), \mathrm{SUP}_{0.3}^- \widehat{\xi}(p)\}$, we get that $\widehat{\xi}$ is not a $\sup_{0.3}^-$-HFUPi of $\mathcal{A}$.

(2) $\widehat{\xi}$ is a $\sup_{0.2}^+$-HFUPf of $\mathcal{A}$ and since $\mathrm{SUP}_{0.2}^+ \widehat{\xi}(r * q) = 0.7 \not\geq 0.8 = \min\{\mathrm{SUP}_{0.2}^+ \widehat{\xi}(r * (p * q)), \mathrm{SUP}_{0.2}^+ \widehat{\xi}(p)\}$, we have that $\widehat{\xi}$ is not a $\sup_{0.2}^+$-HFUPi of $\mathcal{A}$.

By taking $t = 0$ in Theorem 3.2, we have next Corollary 3.2.

COROLLARY 3.2. [10] *Every* sup-*HFUPi of* $\mathcal{A}$ *is a* sup-*HFUPf.*

THEOREM 3.3. *Every* $\sup_t^\alpha$-*HFSUPi of* $\mathcal{A}$ *is a* $\sup_t^\alpha$-*HFUPi.*

P r o o f. Let $\widehat{\xi}$ be a $\sup_t^\alpha$-HFSUPi of $\mathcal{A}$ and $p, q, r \in \mathcal{A}$. Then $\mathrm{SUP}_t^\alpha \widehat{\xi}(0) \geq \mathrm{SUP}_t^\alpha \widehat{\xi}(q)$ and

$$\mathrm{SUP}_t^\alpha \widehat{\xi}(p * r) \geq \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}((r * q) * (r * (p * r))), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\}$$
$$= \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}((r * q) * 0), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\} \qquad ((2.5))$$
$$= \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(0), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\} \qquad ((\text{UP-3}))$$
$$= \mathrm{SUP}_t^\alpha \widehat{\xi}(q)$$
$$\geq \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(p * (q * r)), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\}.$$

Thus, $\widehat{\xi}$ is a $\sup_t^\alpha$-HFUPi of $\mathcal{A}$. □

The following example shows the converse of Theorem 3.3.

EXAMPLE 3.7. Let $\mathcal{A} = \{0, p, q, r\}$ be a UP-algebra defined in Example 3.3. We define a HFS $\widehat{\xi}$ on $\mathcal{A}$ as follows:

$$\widehat{\xi}(0) = \{0.8, 0.9\}, \widehat{\xi}(p) = [0.2, 0.7], \widehat{\xi}(q) = \{0\}, \widehat{\xi}(r) = (0.4, 0.7).$$

Then, the following statements hold:

(1) $\widehat{\xi}$ is a $\sup_{0.7}^-$-HFUPi of $\mathcal{A}$ but not a $\sup_{0.7}^-$-HFSUPi of $\mathcal{A}$ because

$$\mathrm{SUP}_{0.7}^- \widehat{\xi}(p) = 0 \not\geq 0.2 = \min\{\mathrm{SUP}_{0.7}^- \widehat{\xi}((r * 0) * (r * p)), \mathrm{SUP}_{0.7}^- \widehat{\xi}(0)\}.$$

(2) $\widehat{\xi}$ is a $\sup_{0.1}^+$-HFUPi of $\mathcal{A}$ but not a $\sup_{0.1}^+$-HFSUPi of $\mathcal{A}$ because

$$\mathrm{SUP}_{0.1}^+ \widehat{\xi}(p) = 0.8 \not\geq 1 = \min\{\mathrm{SUP}_{0.1}^+ \widehat{\xi}((r * 0) * (r * p)), \mathrm{SUP}_{0.1}^+ \widehat{\xi}(0)\},$$

By taking $t = 0$ in Theorem 3.3, we have next Corollary 3.3.

COROLLARY 3.3. [10] *Every* sup-*HFSUPi of* $\mathcal{A}$ *is a* sup-*HFUPi.*

By Theorem 3.1, Theorem 3.2, Theorem 3.3, Example 3.5, Example 3.6, and Example 3.7, we have that the concept of a $\sup_t^\alpha$-HFUPs is a generalization of the concept of a $\sup_t^\alpha$-HFUPf, the concept of a $\sup_t^\alpha$-HFUPf is a generalization of the concept of a $\sup_t^\alpha$-HFUPi, and the concept of a $\sup_t^\alpha$-HFUPi is a generalization of the concept of a $\sup_t^\alpha$-HFSUPi.

THEOREM 3.4. *A HFS* $\widehat{\xi}$ *on* $\mathcal{A}$ *is a* $\sup_t^\alpha$-*HFSUPi of* $\mathcal{A}$ *if and only if* $\mathrm{SUP}_t^\alpha \widehat{\xi}(p) = \mathrm{SUP}_t^\alpha \widehat{\xi}(0)$ *for all* $p \in \mathcal{A}$.

P r o o f. Assume that $\widehat{\xi}$ is a $\sup_t^\alpha$-HFSUPi of $\mathcal{A}$. Then for all $p, q, r \in \mathcal{A}$, we have

$$\mathrm{SUP}_t^\alpha \widehat{\xi}(0) \geq \mathrm{SUP}_t^\alpha \widehat{\xi}(p) \text{ and}$$
$$\mathrm{SUP}_t^\alpha \widehat{\xi}(p) \geq \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}((r * q) * (r * p)), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\}.$$

Thus,

$$\begin{aligned}
\mathrm{SUP}_t^\alpha \widehat{\xi}(p) &\geq \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}((p * 0) * (p * p)), \mathrm{SUP}_t^\alpha \widehat{\xi}(0)\} \\
&= \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(0 * 0), \mathrm{SUP}_t^\alpha \widehat{\xi}(0)\} \\
&= \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(0), \mathrm{SUP}_t^\alpha \widehat{\xi}(0)\} \\
&= \mathrm{SUP}_t^\alpha \widehat{\xi}(0) \\
&\geq \mathrm{SUP}_t^\alpha \widehat{\xi}(p)
\end{aligned}$$

for all $p \in \mathcal{A}$. Hence, $\mathrm{SUP}_t^\alpha \widehat{\xi}(0) = \mathrm{SUP}_t^\alpha \widehat{\xi}(p)$ for all $p \in \mathcal{A}$.

Conversely, assume that $\mathrm{SUP}_t^\alpha \widehat{\xi}(0) = \mathrm{SUP}_t^\alpha \widehat{\xi}(p)$ for all $p \in \mathcal{A}$. Then for all $p, q, r \in \mathcal{A}$, we get $\mathrm{SUP}_t^\alpha \widehat{\xi}(0) \geq \mathrm{SUP}_t^\alpha \widehat{\xi}(p)$ and

$$\mathrm{SUP}_t^\alpha \widehat{\xi}(p) \geq \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}((r * q) * (r * p)), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\}.$$

Therefore, $\widehat{\xi}$ is a $\sup_t^\alpha$-HFSUPi of $\mathcal{A}$. $\square$

THEOREM 3.5.   A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a $\sup_t^-$-HFSUPi of $\mathcal{A}$ if and only if the supremum of all images of $\widehat{\xi}$ is equal or $\sup\{\mathrm{SUP}\widehat{\xi}(p) \mid p \in \mathcal{A}\} \leq t$.

P r o o f. ($\Rightarrow$) Assume that $\widehat{\xi}$ is a $\sup_t^-$-HFSUPi of $\mathcal{A}$. By Theorem 3.4, we obtain that $\mathrm{SUP}_t^- \widehat{\xi}(0) = \mathrm{SUP}_t^- \widehat{\xi}(p)$ for all $p \in \mathcal{A}$. Now, we consider the following two cases:

Case 1: Suppose that $\mathrm{SUP}\widehat{\xi}(0) \leq t$. Then $\mathrm{SUP}\widehat{\xi}(0) - t \leq 0$ and so $\mathrm{SUP}_t^- \widehat{\xi}(0) = \max\{\mathrm{SUP}\widehat{\xi}(0) - t, 0\} = 0$. Thus, for all $p \in \mathcal{A}$, we see that

$$\max\{\mathrm{SUP}\widehat{\xi}(p) - t, 0\} = \mathrm{SUP}_t^- \widehat{\xi}(p) = \mathrm{SUP}_t^- \widehat{\xi}(0) = 0.$$

Hence, $\mathrm{SUP}\widehat{\xi}(p) - t \leq 0$ for all $p \in \mathcal{A}$, which implies that $\mathrm{SUP}\widehat{\xi}(p) \leq t$ for all $p \in \mathcal{A}$. Therefore, we get $\sup\{\mathrm{SUP}\widehat{\xi}(p) \mid p \in \mathcal{A}\} \leq t$.

Case 2:   Suppose that $\mathrm{SUP}\widehat{\xi}(0) > t$. Then $\mathrm{SUP}\widehat{\xi}(0) - t > 0$ and so for all $p \in \mathcal{A}$, we get

$$
\begin{aligned}
\max\{\mathrm{SUP}\widehat{\xi}(p) - t, 0\} &= \mathrm{SUP}_t^{-}\widehat{\xi}(p) \\
&= \mathrm{SUP}_t^{-}\widehat{\xi}(0) \\
&= \max\{\mathrm{SUP}\widehat{\xi}(0) - t, 0\} \\
&= \mathrm{SUP}\widehat{\xi}(0) - t \\
&> 0.
\end{aligned}
$$

Thus, $\mathrm{SUP}\widehat{\xi}(0) - t = \mathrm{SUP}\widehat{\xi}(p) - t$ for all $p \in \mathcal{A}$, which signifies that $\mathrm{SUP}\widehat{\xi}(0) = \mathrm{SUP}\widehat{\xi}(p)$ for all $p \in \mathcal{A}$. Therefore, the supremum of all images of $\widehat{\xi}$ is equal.

($\Leftarrow$) Assume that $\sup\{\mathrm{SUP}\widehat{\xi}(p) | p \in \mathcal{A}\} \leq t$ or the supremum of all images of $\widehat{\xi}$ is equal. If the supremum of all images of $\widehat{\xi}$ is equal, then it is clear that $\widehat{\xi}$ is a $\sup_t^{-}$-HFSUPi of $\mathcal{A}$. On the other hand, suppose that $\sup\{\mathrm{SUP}\widehat{\xi}(p) | p \in \mathcal{A}\} \leq t$. Then $\mathrm{SUP}_t^{-}\widehat{\xi}(p) = \max\{\mathrm{SUP}\widehat{\xi}(p) - t, 0\} = 0$ for all $p \in \mathcal{A}$. Thus, for all $p, q, r \in \mathcal{A}$, we have $\mathrm{SUP}_t^{-}\widehat{\xi}(0) \geq \mathrm{SUP}_t^{-}\widehat{\xi}(p)$ and $\mathrm{SUP}_t^{-}\widehat{\xi}(p) \geq \min\{\mathrm{SUP}_t^{-}\widehat{\xi}((r * q) * (r * p)), \mathrm{SUP}_t^{-}\widehat{\xi}(q)\}$. Therefore, $\widehat{\xi}$ is a $\sup_t^{-}$-HFSUPi of $\mathcal{A}$.   □

THEOREM 3.6.   *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a $\sup_t^{+}$-HFSUPi of $\mathcal{A}$ if and only if the supremum of all images of $\widehat{\xi}$ is equal or $\inf\{\mathrm{SUP}\widehat{\xi}(p) \mid p \in \mathcal{A}\} + t \geq 1$.*

P r o o f. ($\Rightarrow$) Assume that $\widehat{\xi}$ is a $\sup_t^{+}$-HFSUPi of $\mathcal{A}$. By Theorem 3.4, we have $\mathrm{SUP}_t^{+}\widehat{\xi}(0) = \mathrm{SUP}_t^{+}\widehat{\xi}(p)$ for all $p \in \mathcal{A}$. Now, we consider the following two cases:

Case 1: Suppose that $\mathrm{SUP}\widehat{\xi}(0) + t < 1$. Let $p \in S$. Then

$$
\begin{aligned}
\mathrm{SUP}_t^{+}\widehat{\xi}(p) &= \mathrm{SUP}_t^{+}\widehat{\xi}(0) \\
&= \min\{\mathrm{SUP}\widehat{\xi}(0) + t, 1\} \\
&= \mathrm{SUP}\widehat{\xi}(0) + t \\
&< 1,
\end{aligned}
$$

and so $\mathrm{SUP}_t^{+}\widehat{\xi}(p) = \mathrm{SUP}\widehat{\xi}(p) + t$. Thus, $\mathrm{SUP}\widehat{\xi}(p) + t = \mathrm{SUP}\widehat{\xi}(0) + t$ which signifies that $\mathrm{SUP}\widehat{\xi}(p) = \mathrm{SUP}\widehat{\xi}(0)$. Hence, the supremum of all images of $\widehat{\xi}$ is equal.

Case 2: Suppose that $\mathrm{SUP}\widehat{\xi}(0) + t \geq 1$. Let $p \in \mathcal{A}$. Then
$$\mathrm{SUP}_t^+\widehat{\xi}(p) = \mathrm{SUP}_t^+\widehat{\xi}(0)$$
$$= \min\{\mathrm{SUP}\widehat{\xi}(0) + t, 1\}$$
$$= 1.$$

Thus, $\mathrm{SUP}\widehat{\xi}(p) + t \geq 1$ and so $\mathrm{SUP}\widehat{\xi}(p) \geq 1 - t$. Hence, $\inf\{\mathrm{SUP}\widehat{\xi}(p) \mid p \in \mathcal{A}\} \geq 1 - t$. Therefore, $\inf\{\mathrm{SUP}\widehat{\xi}(p) \mid p \in \mathcal{A}\} + t \geq 1$.

By the two cases, we can conclude that the supremum of all images of $\widehat{\xi}$ is equal or $\inf\{\mathrm{SUP}\widehat{\xi}(p) \mid p \in \mathcal{A}\} + t \geq 1$.

$(\Leftarrow)$ Assume that the supremum of all images of $\widehat{\xi}$ is equal or $\inf\{\mathrm{SUP}\widehat{\xi}(p) \mid p \in \mathcal{A}\} + t \geq 1$. We have that $\widehat{\xi}$ is a $\sup_t^-$-HFSUPi of $\mathcal{A}$ if the supremum of all images of $\widehat{\xi}$ is equal. On the other hand, suppose that $\inf\{\mathrm{SUP}\widehat{\xi}(p) \mid p \in \mathcal{A}\} + t \geq 1$. Then $\mathrm{SUP}_t^+\widehat{\xi}(p) = \min\{\mathrm{SUP}\widehat{\xi}(p) + t, 1\} = 1$ for all $p \in \mathcal{A}$. Thus, for all $p, q, r \in \mathcal{A}$, we get $\mathrm{SUP}_t^+\widehat{\xi}(0) \geq \mathrm{SUP}_t^+\widehat{\xi}(p)$ and $\mathrm{SUP}_t^+\widehat{\xi}(p) \geq \min\{\mathrm{SUP}_t^+\widehat{\xi}((r * q) * (r * p)), \mathrm{SUP}_t^+\widehat{\xi}(q)\}$. Hence, $\widehat{\xi}$ is a $\sup_t^+$-HFSUPi of $\mathcal{A}$. $\square$

By taking $t = 0$ in Theorem 3.5 (or Theorem 3.6), we have next Corollary 3.4.

COROLLARY 3.4.   [10] *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a sup-HFSUPi of $\mathcal{A}$ if and only if the supremum of all images of $\widehat{\xi}$ is equal.*

4. Level Subsets

We divide this section into two parts. We study $\sup_t^\alpha$-upper s-level subsets in the first part and $\sup_t^\alpha$-lower s-level subsets in the second part.

DEFINITION 4.1.   ([10, 8]) Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$ and $s \in [0, 1]$, the sets
$$U[\widehat{\xi}; s] = \{x \in \mathcal{A} \mid \mathrm{SUP}\widehat{\xi}(x) \geq s\}$$
and
$$L[\widehat{\xi}; s] = \{x \in \mathcal{A} \mid \mathrm{SUP}\widehat{\xi}(x) \leq s\}$$
are called a sup-*upper s-level subset* and a sup-*lower s-level subset* of $\widehat{\xi}$, respectively.

In 2018, Mosrijai et al. [10] gave characterizations of sup-type of HFSs in terms of sup-upper and sup-lower s-level subsets. In this work, we introduce general concepts of sup-upper and sup-lower s-level subsets (seen in Definition 4.2) and use the concepts to characterize $\sup_t^\alpha$-HFUPss, $\sup_t^\alpha$-HFUPfs, $\sup_t^\alpha$-HFUPis and $\sup_t^\alpha$-HFSUPis of UP-algebras.

DEFINITION 4.2.   Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$ and $s \in [0,1]$, the sets

$$U_t^\alpha[\widehat{\xi}; s] = \{x \in \mathcal{A} \mid \mathrm{SUP}_t^\alpha \widehat{\xi}(x) \geq s\}$$

and

$$L_t^\alpha[\widehat{\xi}; s] = \{x \in \mathcal{A} \mid \mathrm{SUP}_t^\alpha \widehat{\xi}(x) \leq s\}$$

are called a $\sup_t^\alpha$-*upper s-level subset* and a $\sup_t^\alpha$-*lower s-level subset* of $\widehat{\xi}$, respectively. In this case: $t = 0$, we have $U_0^\alpha[\widehat{\xi}; s] = U[\widehat{\xi}; s]$ and $L_0^\alpha[\widehat{\xi}; s] = L[\widehat{\xi}; s]$.

4.1. $\sup_t^\alpha$-Upper s-Level Subsets.

In this part, we discuss the relations between $\sup_t^\alpha$-HFUPss (resp., $\sup_t^\alpha$-HFUPfs, $\sup_t^\alpha$-HFUPis, $\sup_t^\alpha$-HFSUPis) and their $\sup_t^\alpha$-upper s-level subsets.

THEOREM 4.1.   *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a $\sup_t^\alpha$-HFUPs of $\mathcal{A}$ if and only if $U_t^\alpha[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-subalgebra of $\mathcal{A}$ for all $s \in [0,1]$ with $U_t^\alpha[\widehat{\xi}; s] \neq \emptyset$.*

P r o o f. ($\Rightarrow$) Let $s \in [0,1]$ and $p, q \in U_t^\alpha[\widehat{\xi}; s]$. Then $\mathrm{SUP}_t^\alpha \widehat{\xi}(p) \geq s$ and $\mathrm{SUP}_t^\alpha \widehat{\xi}(q) \geq s$. Since $\widehat{\xi}$ is a $\sup_t^\alpha$-HFUPs of $\mathcal{A}$, we get

$$\mathrm{SUP}_t^\alpha \widehat{\xi}(p * q) \geq \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(p), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\} \geq s.$$

Hence, $p * q \in U_t^\alpha[\widehat{\xi}; s]$. Therefore, $U_t^\alpha[\widehat{\xi}; s]$ is a UP-subalgebra of $\mathcal{A}$.

($\Leftarrow$) Let $p, q \in \mathcal{A}$. We choose $s = \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(p), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\}$ and so we have $\mathrm{SUP}_t^\alpha \widehat{\xi}(p) \geq s$ and $\mathrm{SUP}_t^\alpha \widehat{\xi}(q) \geq s$. Thus, $p, q \in U_t^\alpha[\widehat{\xi}; s]$. By assumption, $U_t^\alpha[\widehat{\xi}; s]$ is a UP-subalgebra of $\mathcal{A}$ and so $p * q \in U_t^\alpha[\widehat{\xi}; s]$. Hence,

$$\mathrm{SUP}_t^\alpha \widehat{\xi}(p * q) \geq s = \min\{\mathrm{SUP}_t^\alpha \widehat{\xi}(p), \mathrm{SUP}_t^\alpha \widehat{\xi}(q)\}.$$

Therefore, $\widehat{\xi}$ is a $\sup_t^\alpha$-HFUPs of $\mathcal{A}$.   □

The proofs of Theorem 4.2 , Theorem 4.3 and Theorem 4.4 can be established by a similar arguments to the proof of Theorem 4.1.

THEOREM 4.2.   *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a $\sup_t^\alpha$-HFUPf of $\mathcal{A}$ if and only if $U_t^\alpha[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-filter of $\mathcal{A}$ for all $s \in [0,1]$ with $U_t^\alpha[\widehat{\xi}; s] \neq \emptyset$.*

THEOREM 4.3.   *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a $\sup_t^\alpha$-HFUPi of $\mathcal{A}$ if and only if $U_t^\alpha[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-ideal of $\mathcal{A}$ for all $s \in [0,1]$ with $U_t^\alpha[\widehat{\xi}; s] \neq \emptyset$.*

THEOREM 4.4.   *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a $\sup_t^\alpha$-HFSUPi of $\mathcal{A}$ if and only if $U_t^\alpha[\widehat{\xi}; s]$ of $\mathcal{A}$ is a strong UP-ideal of $\mathcal{A}$ for all $s \in [0,1]$ with $U_t^\alpha[\widehat{\xi}; s] \neq \emptyset$.*

By taking $t = 0$ in Theorem 4.1 (resp., Theorem 4.2, Theorem 4.3, Theorem 4.4), we have next Corollary 4.1 (resp., Corollary 4.2, Corollary 4.3, Corollary 4.4).

COROLLARY 4.1. [10] *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a sup-HFUPs of $\mathcal{A}$ if and only if $U[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-subalgebra of $\mathcal{A}$ for all $s \in [0,1]$ with $U[\widehat{\xi}; s] \neq \emptyset$.*

COROLLARY 4.2. [10] *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a sup-HFUPf of $\mathcal{A}$ if and only if $U[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-filter of $\mathcal{A}$ for all $s \in [0,1]$ with $U[\widehat{\xi}; s] \neq \emptyset$.*

COROLLARY 4.3. [10] *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a sup-HFUPi of $\mathcal{A}$ if and only if $U[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-ideal of $\mathcal{A}$ for all $s \in [0,1]$ with $U[\widehat{\xi}; s] \neq \emptyset$.*

COROLLARY 4.4. [10] *A HFS $\widehat{\xi}$ on $\mathcal{A}$ is a sup-HFSUPi of $\mathcal{A}$ if and only if $U[\widehat{\xi}; s]$ of $\mathcal{A}$ is a strong UP-ideal of $\mathcal{A}$ for all $s \in [0,1]$ with $U[\widehat{\xi}; s] \neq \emptyset$.*

4.2. $\sup_t^\alpha$-Lower s-Level Subsets.

In this part, we discuss the relations between $\sup_t^\alpha$-HFUPss (resp., $\sup_t^\alpha$-HFUPfs, $\sup_t^\alpha$-HFUPis, $\sup_t^\alpha$-HFSUPis) and their $\sup_t^\alpha$-lower s-level subsets.

For a HFS $\widehat{\xi}$ on $\mathcal{A}$, the HFS $\widehat{\xi}^C$ defined by $\widehat{\xi}^C(p) = \{1 - \text{SUP}\widehat{\xi}(p)\}$ for all $p \in \mathcal{A}$ is said to be the *supremum complement* of $\widehat{\xi}$. Then $\text{SUP}\widehat{\xi}^C(p) = 1 - \text{SUP}\widehat{\xi}(p)$ for all $p \in \mathcal{A}$. We observe that $(\widehat{\xi}^C)^C(p) = \{\text{SUP}\widehat{\xi}(p)\}$ for all $p \in \mathcal{A}$, and then $\text{SUP}(\widehat{\xi}^C)^C(p) = \text{SUP}\widehat{\xi}(p)$ for all $p \in \mathcal{A}$.

THEOREM 4.5. *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$. Then $\widehat{\xi}^C$ is a $\sup_t^\alpha$-HFUPs of $\mathcal{A}$ if and only if $L_t^\alpha[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-subalgebra of $\mathcal{A}$ for all $s \in [0,1]$ with $L_t^\alpha[\widehat{\xi}; s] \neq \emptyset$.*

P r o o f. ($\Rightarrow$) Let $s \in [0,1]$ and $p, q \in L_t^\alpha[\widehat{\xi}; s]$. Then $\max\{\text{SUP}_t^\alpha\widehat{\xi}(p), \text{SUP}_t^\alpha\widehat{\xi}(q)\} \leq s$. Since $\widehat{\xi}^C$ is a $\sup_t^\alpha$-HFUPs of $\mathcal{A}$, we obtain that

$$1 - \text{SUP}_t^\alpha\widehat{\xi}(p * q) = \text{SUP}_t^\alpha\widehat{\xi}^C(p * q)$$
$$\geq \min\{\text{SUP}_t^\alpha\widehat{\xi}^C(p), \text{SUP}_t^\alpha\widehat{\xi}^C(q)\}$$
$$= \min\{1 - \text{SUP}_t^\alpha\widehat{\xi}(p), 1 - \text{SUP}_t^\alpha\widehat{\xi}(q)\}$$
$$= 1 - \max\{\text{SUP}_t^\alpha\widehat{\xi}(p), \text{SUP}_t^\alpha\widehat{\xi}(q)\}.$$

Thus, $\text{SUP}_t^\alpha\widehat{\xi}(p * q) \leq \max\{\text{SUP}_t^\alpha\widehat{\xi}(p), \text{SUP}_t^\alpha\widehat{\xi}(q)\} \leq s$. Hence, $p * q \in L_t^\alpha[\widehat{\xi}; s]$. Therefore, $L_t^\alpha[\widehat{\xi}; s]$ is a UP-subalgebra of $\mathcal{A}$.

($\Leftarrow$) Let $p, q \in \mathcal{A}$. Choose $s = \max\{\mathrm{SUP}_t^\alpha\widehat{\xi}(p), \mathrm{SUP}_t^\alpha\widehat{\xi}(q)\}$, then we have $\mathrm{SUP}_t^\alpha\widehat{\xi}(p) \leq s$ and $\mathrm{SUP}_t^\alpha\widehat{\xi}(q) \leq s$. Thus, $p, q \in L_t^\alpha[\widehat{\xi}; s]$. By assumption, we get $L_t^\alpha[\widehat{\xi}; s]$ is a UP-subalgebra of $\mathcal{A}$ and so $p * q \in L_t^\alpha[\widehat{\xi}; s]$. Thus, $\mathrm{SUP}_t^\alpha\widehat{\xi}(p * q) \leq s = \max\{\mathrm{SUP}_t^\alpha\widehat{\xi}(p), \mathrm{SUP}_t^\alpha\widehat{\xi}(q)\}$. Hence,

$$\begin{aligned}
\mathrm{SUP}_t^\alpha\widehat{\xi}^C(p * q) &= 1 - \mathrm{SUP}_t^\alpha\widehat{\xi}(p * q) \\
&\geq 1 - \max\{\mathrm{SUP}_t^\alpha\widehat{\xi}(p), \mathrm{SUP}_t^\alpha\widehat{\xi}(q)\} \\
&= \min\{1 - \mathrm{SUP}_t^\alpha\widehat{\xi}(p), 1 - \mathrm{SUP}_t^\alpha\widehat{\xi}(q)\} \\
&= \min\{\mathrm{SUP}_t^\alpha\widehat{\xi}^C(p), \mathrm{SUP}_t^\alpha\widehat{\xi}^C(q)\}.
\end{aligned}$$

Therefore, $\widehat{\xi}^C$ is a $\sup_t^\alpha$-HFUPs of $\mathcal{A}$.   □

The proofs of Theorem 4.6, Theorem 4.7 and Theorem 4.8 can be established by a similar argument to the proof of Theorem 4.5.

THEOREM 4.6. *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$. Then $\widehat{\xi}^C$ is a $\sup_t^\alpha$-HFUPf of $\mathcal{A}$ if and only if $L_t^\alpha[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-filter of $\mathcal{A}$ for all $s \in [0,1]$ with $L_t^\alpha[\widehat{\xi}; s] \neq \emptyset$.*

THEOREM 4.7. *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$. Then $\widehat{\xi}^C$ is a $\sup_t^\alpha$-HFUPi of $\mathcal{A}$ if and only if $L_t^\alpha[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-ideal of $\mathcal{A}$ for all $s \in [0,1]$ with $L_t^\alpha[\widehat{\xi}; s] \neq \emptyset$.*

THEOREM 4.8. *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$. Then $\widehat{\xi}^C$ is a $\sup_t^\alpha$-HFSUPi of $\mathcal{A}$ if and only if $L_t^\alpha[\widehat{\xi}; s]$ of $\mathcal{A}$ is a strong UP-ideal of $\mathcal{A}$ for all $s \in [0,1]$ with $L_t^\alpha[\widehat{\xi}; s] \neq \emptyset$.*

By taking $t = 0$ in Theorem 4.5 (resp., Theorem 4.6, Theorem 4.7, Theorem 4.8), we have next Corollary 4.5 (resp., Corollary 4.6, Corollary 4.7, Corollary 4.8).

COROLLARY 4.5. [10] *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$. Then $\widehat{\xi}^C$ is a sup-HFUPs of $\mathcal{A}$ if and only if $L[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-subalgebra of $\mathcal{A}$ for all $s \in [0,1]$ with $L[\widehat{\xi}; s] \neq \emptyset$.*

COROLLARY 4.6. [10] *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$. Then $\widehat{\xi}^C$ is a sup-HFUPf of $\mathcal{A}$ if and only if $L[\widehat{\xi}; s]$ of $\mathcal{A}$ is a UP-filter of $\mathcal{A}$ for all $s \in [0,1]$ with $L[\widehat{\xi}; s] \neq \emptyset$.*

COROLLARY 4.7. [10] *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$. Then $\widehat{\xi}^C$ is a sup-HFUPi of $\mathcal{A}$ if and only if $L[\widehat{\xi};s]$ of $\mathcal{A}$ is a UP-ideal of $\mathcal{A}$ for all $s \in [0,1]$ with $L[\widehat{\xi};s] \neq \emptyset$.*

COROLLARY 4.8. [10] *Let $\widehat{\xi}$ be a HFS on $\mathcal{A}$. Then $\widehat{\xi}^C$ is a sup-HFSUPi of $\mathcal{A}$ if and only if $L[\widehat{\xi};s]$ of $\mathcal{A}$ is a strong UP-ideal of $\mathcal{A}$ for all $s \in [0,1]$ with $L[\widehat{\xi};s] \neq \emptyset$.*

5. Conclusions and Future Works

In the present paper, we have introduced the concept of a $\sup_t^\alpha$-HFUPs (resp., $\sup_t^\alpha$-HFUPf, $\sup_t^\alpha$-HFUPi, $\sup_t^\alpha$-HFSUPi), which is a generalization of the concept of a sup-HFUPs (resp., sup-HFUPf, sup-HFUPi, sup-HFSUPi), of UP-algebras, and investigated some of its essential properties. Then, we have the generalization diagram of these HFSs below.

sup-HFUPs ⇄ sup-HFUPf ⇄ sup-HFUPi ⇄ sup-HFSUPi

$\sup_t^\alpha$-HFUPs ⇄ $\sup_t^\alpha$-HFUPf ⇄ $\sup_t^\alpha$-HFUPi ⇄ $\sup_t^\alpha$-HFSUPi
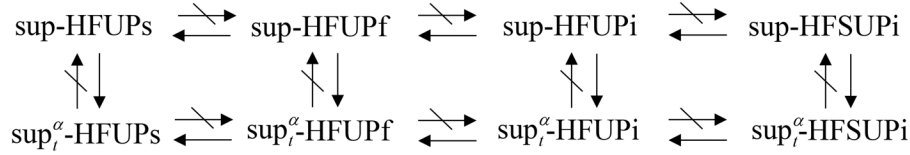
In our future study of UP-algebras, the following objectives are considered:

- to get more results of $\sup_t^\alpha$-hesitant fuzzy substructures,
- to define anti-type of $\sup_t^\alpha$-hesitant fuzzy substructures,
- to define neutrosophic sets by means of $\sup_t^\alpha$-hesitant fuzzy substructures.

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