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CHARACTERIZATION OF THE BASIC REPRODUCTION NUMBER R0 OF PDE EPIDEMIC MODELS

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Abstract

The basic reproduction number R0 has been used in epidemic models as an important threshold for controlling the spread of infections. In this paper we give a theoretical characterization of R0 for compartmental models that are based on partial differential equations. For any of such models we show that R0 is the spectral radius of the basic operator of the translation semigroup of operators that is solution of the model. The stability of steady states and the asymptotic behavior of the solutions are also established even if the important irreducibility property is satisfied only for a projection of the basic operator. For illustration, an age structured Susceptible-Infected-Recovered epidemic model is considered and R0 is explicitly computed in terms of the model's parameters using the established characterization.

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1. **Introduction**

Our main objective in this work is to give a theoretical mathematical characterization of the basic reproduction number R0, of the asynchronous exponential property and of the stability of steady states of age structured compartmental epidemic models that use partial differential equations (PDEs) in their modeling of the variation with respect to time and age of the various groups of a population exposed to an epidemic. This characterization is given in terms of the models core operators which are associated with their solution translation semigroups of operators.

During human history many diseases of infectious origin, also called epidemics, have affected many people either in specific regions or throughout the world. One of the key factors that contribute to the spreading of such epidemics is the contact between people that promotes the possibility of disease transmission. Humanity has indeed known various infective diseases during its history each with its proper intensity and rate of spreading. In particular, the last decades have been struck by severe epidemics like SARS (severe acute respiratory syndrome), AIDS (acquired immunodeficiency syndrome) caused by HIV (human immunodeficiency virus), the flu pandemic related to H1N1 virus and the more recent corona epidemic caused by the SARS-CoV-2 virus which was responsible for severe health problems and a high number of deaths during the last years because of its high transmissibility.

To control the outbreak of the aforementioned epidemics, the famous reproduction number $R0$ has been used, and it has been proven to be an important parameter in the study of the dynamics of epidemics. Indeed, values of $R0$ less than 1 ensure that the number of infected individuals will not increase, while values of $R0$ greater than 1 are a sign for a continuous spreading of the infection. Because of its importance, computations of $R0$ have been considered in the study of a multitude of mathematical epidemic models that use either integral equations (IEs), ordinary differential equations (ODEs), fractional differential equations (FDEs), partial differential equations (PDEs), delay differential equations (DDEs) or stochastic equations (SEs) in their modeling ways. A nice discussion about the use of $R0$ and about its various definitions in such research works can be found in [8]. In particular numerous recent works dealing with $R0$ have been published for the modeling and monitoring of the dynamics of corona epidemic. Many of known classical mathematical modeling and analysis approaches used for the modeling and the analysis in epidemiology while considering the $R0$ parameter and using the technique of dividing the population in compartments (susceptible, recovered, infected, ...) have also been applied in such corona epidemic analysis research works. Examples in this context are the works [14] and [13] for the case of an ODE model, [11] for the case of a DDE model, [16] for the case of a FDE model, and [15] for the case of a SE model. In these works, the mathematical analysis objective is mainly to relate the stability of steady states to a condition on the reproduction number by directly determining steady states and separately showing for each model considered that these steady states are stable if and only if $R0 < 1$. Mathematically speaking, the first characterization of $R0$ was given in [9] for a particular case of an IE epidemic model by looking for solutions that are multiples a time exponential factor. A second characterization of $R0$ as the radius of the next generation matrix has been after that given for ODE models in [17]. This characterization has been used in the study of multiple ODE epidemic models (e.g. [10], [13], [14], [12]).

In terms of core operators of translation semigroup, $R0$ was characterized by the first author Larbi Alaoui in 1998 for the case of the famous Sharpe-Lotka PDE model [2], and in 2001 for the case of the IE epidemic linear model considered in [9], and also for a multi states version of the Sharpe-Lotka model [1]. The translation semigroups in question are those solution semigroups of renewal equations of the form

$$m(t) = \phi(m_t), \quad t \geq 0. \tag{1}$$

In Alaoui's works [2] and [1], R0 is theoretically characterized as the spectral radius of an operator associated with the core operators of the models translation semigroups. Our main purpose in this work is to give a generalization of the Alaoui's works for the case of linear PDE models by establishing results on the characterization of R0, on the stability of steady states and on the asynchronous exponential growth (AEG) property under less restrictive conditions on the core operators. To this end, we consider the qualitative study of a general multi states PDE age structured population model that models the dynamics of a population with multiple groups, that is a population whose entities may undergo various states or phases during their life cycle. Denoting by u_i the population density at state i, the PDE multi states model reads as follows:

$$\begin{cases} \mathcal{D}_{t,a} u_i(t,a) = \left[-\mu_i(a) - \sum_{j\in\{1,\cdots,n\}, j\neq i} \tau_{i,j}(a) \right] u_i(t,a), \\ \qquad\qquad + \sum_{j\in\{1,\cdots,n\}, j\neq i} \tau_{j,i}(a) u_j(t,a) \\ u_i(t,0) = \int_0^{\tilde{a}} \sum_{j\in\{1,\cdots,n\}} \beta_{i,j}(a) u_j(t,a)\, da, \quad 1 \leq i \leq n, \end{cases} \tag{2}$$

where $\mathcal{D}_{t,a} := \partial_t + \partial_a$.

For application, we consider a PDE age structured epidemiological model which describes the dynamics of a population exposed to an infectious disease, and which is given by

$$\begin{cases} \mathcal{D}_{t,a}S(t,a) = -\zeta(a)S(t,a) - \mu(a)S(t,a) \\ \mathcal{D}_{t,a}I(t,a) = \zeta(a)S(t,a) - \gamma(a)I(t,a) - \mu(a)I(t,a) \\ \mathcal{D}_{t,a}R(t,a) = \gamma(a)I(t,a) - \mu(a)R(t,a), \end{cases} \tag{3}$$

and by the following initial conditions

$$\begin{pmatrix} S \\ I \\ R \end{pmatrix}(t,0) = \begin{cases} \displaystyle\int_0^{\tilde{a}} \beta(a)\left[S(t,a) + (1-q)I(t,a) + R(t,a)\right]da \\ \displaystyle q\int_0^{\tilde{a}} \beta(a)I(t,a)da \\ 0 \end{cases}. \tag{4}$$

In this model S, I and R are the densities with respect to time and age of susceptibles, infectives and recovered, respectively. The age functions β, μ, ζ and γ denote the birth, mortality, infection and recovery rates, respectively.

We explicitly give details relative to the conditions yielding to properties of the solution translation semigroup of the general model (2) such as existence, positivity and compactness that make it possible to conclude the aforementioned qualitative results, in order to show how translation semigroups of operators provide a strong theoretical framework for the mathematical study of the dynamics for the model that makes it possible to derive such results in automatic way. We show that the framework makes it possible to establish necessary and sufficient conditions that yield in an automatic and theoretically well founded way to the AEG property, to the characterization and stability of steady states and of the associated reproduction number only in terms of the core operator of the solution translation semigroup.

The content of the following sections is as follows. In Section 2, we relate the study of (2) to the study of an equation of the form (1). More precisely, we determine the core operator of (2) and present the automatic way allowing the establishment of its qualitative results regarding its AEG property and the stability of its steady states based on the derivation of associated spectral and irreducibility properties of its core operator. The particular case of the model (35)-(36) is treated in Section 3. Section 4 concludes this work.

2. Case of the general multi-states PDE model

In this section we consider the mathematical analysis of the age structured populations PDE model (2) under the following assumption

$$(H_{\beta,\tau,\sigma}) \qquad \left\{\beta_{i,j}, \mu_i, \tau_{i,j} \in L^\infty((0,\tilde{a}), R^+), \quad \beta \not\equiv 0\right\}.$$

We show how translation semigroups can be used for the analysis of the associated dynamics to prove our conjuncture that the class of semigroups does provide a solid theoretical basis for the automatic derivation of various properties of the solutions and of steady states of the model. The theoretical results on translation semigroups that are relevant for the analysis of the model are given in parallel whenever they are needed.

Before starting with the analysis of the model (2) let us first briefly present the main features considered in the mathematical modeling leading to (2).

In the model (2) we are considering a population composed of n groups and we denote by $u := (u_1, \cdots, u_n)$ the associated vector of population densities. The constant $\tilde{a}$ is the maximum age of members of the population. and $\mu_i(a)$ is the rate of mortality, that is the rate at which an individual in state i dies at age a, $\tau_{i,j}(a)$ is the rate at which an individual transit from state i to state j at age a, and $\beta_{i,j}(a)$ is the fertility rate. The total subpopulation of group i with age between a_1 and a_2 is $\int_{a_1}^{a_2} u_i(t,a)dt$.

Notice that structuring the population with respect to age is very important in the modeling of the dynamics of the population since individuals with different ages may exhibit different behavior while interacting with each other. For transitions between the groups we therefore assume that associated rates also depend on age a and we denote the transition rate from group i to group j by $\tau_{i,j}$. We also suppose that new borns of a group j are not necessarily in the same group and could therefore belong to another group i. So, denoting by $\beta_{i,j}$ the age specific fertility rate or birth rate in group j yielding new borns in group i we get that the number of new borns belonging to group i at time t is $\int_0^{\tilde{a}} \sum_{j=1}^{n} \beta_{i,j}(a)u_j(t,a)da$. Likewise if we denote by $\mu_i(t,a)$ the death rate at age a of individuals of group i, then the number of deaths in group i at time t is $\int_0^{\tilde{a}} \mu_i(t,a)u_i(t,a)da$. With all these assumptions, we get the equations in (4) for the mathematical modeling of the dynamics of the considered population.

It is to also to be noticed that the study of (2) was considered by the first author in [4] using translation semigroups for the particular case where $\tau_{i,j} = 0$ for $i \neq j$, that is in the case where there are no transitions between the population groups. For this particular case, the conditions yielding the AEG property were mainly based on the property of irreducibility of the operators $\tilde{\phi}_\lambda$ which does not however hold for the general model (2) and in particular for the SIR model we are considering in the next section. We show how the AEG property can still be obtained using similar arguments by considering an essential dimensionally restricted operator ψ which is extracted from the main operator ϕ and for which irreducibility of associated operators $\tilde{\psi}_\lambda$ is satisfied. The main conditions on the spectral properties that yield to the AEG property are then proved using those of ψ.

The study of the AEG property for the considered models is the subject of the following sections where we also give results on the stability of steady states of the models and a characterization of associated basic reproduction number.

2.1. **Solution semigroup and core operator.** Before giving results on the existence of the solution semigroup of (2) and its associated core operator let us first give some definitions on semigroups.

If $(X, \|\cdot\|_X)$ is a Banach space, then a family of bounded linear operators $(V(t) : X \to X)_{t\geq 0}$ is called a semigroup on X if $V(0) = I_X$ ($I_X; X \to X$ is the identity operator) and $V(t+s) = V(t)V(s)$, for $t, s \geq 0$. It is called strongly continuous (or equivalently a C_0-semigroup), if $\lim_{t\to 0^+} \|V(t)x - x\|_X = 0$, for $x \in X$. The generator of a semigroup $(V(t))_{t\geq 0}$ on X is the operator A with domain $D(A) = \{x \in X, \lim_{t\to 0^+} t^{-1}(V(t)x - x) \text{ exists in X}\}$ and such that $Ax = \lim_{t\to 0^+} t^{-1}(V(t)x - x)$ for $x \in D(A)$. The growth bound of $(V(t))_{t\geq 0}$ is denoted by $\omega(V(t))$ or by $\omega(A)$ and is defined as the constant

$$\omega(A) := \inf\{\alpha \in R,\ \exists C \geq 1,\quad \|V(t)u\|_X \leq Ce^{\alpha t}\|u\|_X,\ u \in X, t \geq 0\}.$$

$(V(t))_{t\geq 0}$ on X is called eventually compact if there exists $t_0 \geq 0$ such that $V(t)$ is compact for every $t > t_0$.

For the study of the models we also use the notion of equivalent semigroups.

Given $T_1 := (T_1(t))_{t\geq 0}$ and $T_2 := (T_2(t))_{t\geq 0}$ semigroups on the Banach spaces X_1 and X_2, respectively, L an operator from X_1 to X_2 and $H_1 : D(H_1) \subset X_1 \to X_1$ and $H_2 : D(H_2) \subset X_2 \to X_2$ such that $L[D(H_1)] \subset D(H_2)$, we have (see [2]):

- H_1 is called L-similar to H_2 if L^{-1} exists and for $x \in D(H_1)$ it holds $H_1 x = L^{-1} H_2 L x$;
- T_1 is called L-similar to T_2 if for each $t \geq 0$, $T_1(t)$ is L-similar to $T_2(t)$.
- If furthermore the generator of T_1 is L-similar to the generator of T_2, then T_1 is said to be L-equivalent to T_2.

Now coming back to the model (2), the space on which we do the analysis of the model (2) is the Banach space E with its norm $||\cdot||_E$

$$E := L^1((0, \tilde{a}), R^n), \quad ||f||_E = \int_0^{\tilde{a}} ||f(s)||_{R^n} ds, \quad f \in E.$$

Setting

$$J := \{1, \cdots n\},$$

$$M_{i,j}(s) := \begin{cases} \mu_i(s) + \displaystyle\sum_{k\in J, k\neq i} \tau_{i,k}(s)) & \text{if } i = j, \\ -\tau_{j,i}(s) & \text{if } i \neq j \end{cases},$$

$$M(s) := (M_{i,j}(s))_{i,j\in J}, \tag{5}$$

$$B(s) := (\beta_{i,j}(s))_{i,j\in J} \tag{6}$$

and

$$U := (u_i)_{i\in J}, \tag{7}$$

the system (2) reads as follows

$$\begin{cases} \mathcal{D}_{t,a}U(t,a) = -M(a)U(t,a), \\ \qquad U(t,0) = \displaystyle\int_0^{\tilde{a}} B(s)U(t,s)ds. \end{cases} \tag{8}$$

To relate the model to an equation of the type (1), we follow the same approach in [4, 2, 7] by transforming this last system to the equivalent one

$$\begin{cases} \mathcal{D}_{t,a}V(t,a) = 0 \\ V(t,0) = \displaystyle\int_0^{\tilde{a}} B(s)\big([\Upsilon^{-1}V(t,.)](s)\big)ds, \end{cases} \tag{9}$$

using the transformation

$$V(t,.) = \Upsilon(U(t,\cdot)) \tag{10}$$

with

$$(\Upsilon f)(s) := Z(s)f(s), \qquad s \in (0,\tilde{a}), \tag{11}$$

$U(t,\cdot)$ being the function $s \mapsto U(t,s)$, and Z is the unique solution of

$$\big\{Z'(s) = Z(s)M(s), \qquad Z(0) = I_{n\times n},\big\} \tag{12}$$

where $I_{n\times n}$ denotes the real identity matrix of order n.

Since the determinant $\det(Z(\tau)) = \exp\big(\int_0^\tau \sum_{i=1}^n m_{i,i}(s)ds\big)$ is $\neq 0$, we may conclude that the inverse Z^{-1} of Z exists and Z^{-1} is solution of

$$\big\{X'(s) = -M(s)X(s), \qquad X(0) = I_{n\times n}\big\}. \tag{13}$$

We set

$$Z(s) := (z_{i,j}(s))_{i,j\in J} \quad \text{and} \quad Z^{-1}(s) = (\tilde{z}_{i,j}(s))_{i,j\in J}.$$

Taking into account the Cauchy problem (9) with its initial value condition, we get the core operator of (2) as follows

$$\phi : E := L^1((0,\tilde{a}),R^n) \to R^n, \quad \phi f = \int_0^{\tilde{a}} \Gamma(s)f(s)ds, \tag{14}$$

where

$$\Gamma(s) = B(s)Z^{-1}(s), \quad s \in (0,\tilde{a}). \tag{15}$$

The boundedness of ϕ yields the following result.

Lemma 2.1. *Let $(H_{\beta,\tau,\sigma})$ be satisfied. Then the first derivative operator $A_\phi : f \mapsto A_\phi f = -f'$ with its domain $D(A_\phi) := \{f \in W^{1,1}((0,\bar{a}),\mathbf{R}^n), f(0) = \phi f\}$ is the generator of a C_0-semigroup $\mathcal{T}_\phi$ on E*

$$[T_\phi(t)f](a) = \begin{cases} \phi(T_\phi(t-a)f) & \text{if } t-a \geq 0, \\ f(a-t) & \text{if } t-a < 0, \end{cases} \tag{16}$$

for $a \in (0, \bar{a})$*. Furthermore,* $\omega(\mathcal{T}_\phi) \leq \|\phi\|$ *and for* $f \in E$*, the solution of the problem*

$$\begin{cases} m(t) = \phi(m_t),\, t \geq 0 \quad \textit{and} \quad m_0(s) = f(-s), s \in (-\bar{a}, 0), \\ m \in L^1_{loc}((-\bar{a}, \infty), \mathbf{R}^n) \cap C(]0, \infty), \mathbf{R}^n), \end{cases}$$

(with $m_t(s) := m(t-s)$*) is given by*

$$m(t) = \begin{cases} f(-t) & \textit{if} \quad t \in (-\bar{a}, 0), \\ \phi(T_\phi(t)f) & \textit{if} \quad t \geq 0. \end{cases}$$

In the case where Z^{-1} *is nonnegative* $\big($*i.e.,* $\tilde{z}_{i,j}(s) \geq 0$ *for* $i, j \in J$ *and* $s \in (0, \tilde{a})\big)$*, the semigroup* T_ϕ *is also nonnegative.*

P r o o f. The result follows from [5] which states that taking F a Banach space, $0 < r < \infty$ and $\Phi : X := L^1((-r, 0), F) \to F$, the semigroup $\mathcal{H}_\Phi := (H_\Phi(t))_{t \geq 0}$ solution of $m(t) := \Phi(m_t)$ with $m_t(\tau) := m(t + \tau)$ if $\tau \in (-r; 0)$ is such that

$$[H_\Phi(t)f](\tau) = \begin{cases} \Phi(H_\Phi(t+\tau)f) & \text{if } t + \tau \geq 0, \\ f(\tau + t) & \text{if } t + \tau < 0. \end{cases}$$

Furthermore the generator $A_{\mathcal{H}_\Phi}$ of $\mathcal{H}_\Phi$ satisfies

$$A_{\mathcal{H}_\Phi} = \big\{ g \in W^{1,1}((-r, 0), F), g(0) = \Phi g \big\}$$
$$\text{and} \quad A_{\mathcal{H}_\Phi} g = g'.$$

If $X := L^1((0, r), F)$ and $m_t(s) := m(t-s)$, $s \in (0, r)$, then for $f \in X$ it suffices to take $(T_\Phi(t)f)(s) := (H_\Phi(t)g_f)(-s)$ with $g_f(s) := f(-s)$, $s \in (-r, 0)$, in order to get that the solution semigroup $\mathcal{T}_\Phi := (T_\Phi(t))_{t \geq 0}$ of $m(t) := \Phi(m_t)$ on X satisfies (16) and that the generator $A_{\mathcal{T}_\Phi}$ of $\mathcal{T}_\Phi$ is with domain $A_{\mathcal{T}_\Phi} = \{ f \in W^{1,1}((0, r), F), f(0) = \Phi f \}$ and satisfies $A_{\mathcal{T}_\Phi} f = -f'$. The nonnegativity of $\mathcal{T}_\Phi$ follows from the fact that Φ is nonnegative since $\Gamma(s)$ is nonnegative. □

The non-negativity of the core operator ϕ also plays an important role with regards to the spectral properties that yield the asymptotic behavior of the solution semigroup as we will see in the following subsections. In particular this positivity is guaranteed if all components of the matrix $Z^{-1}(s)$ are nonnegative. This last condition is in particular satisfied if for example $M(s)$ is a lower matrix as it is stated in the following lemma.

LEMMA 2.2. *Let* $(H_{\beta,\tau,\sigma})$ *be satisfied and assume that* M *is a lower matrix (i.e.* $M_{i,j} = 0$ *for* $i < j$*) and that* $M_{i,j} \leq 0$ *if* $i > j$*. Then* $Z^{-1}(a)$ *is also a* *lower matrix and the $\tilde{z}_{i,j}(s)$, $i,j \in J$, are nonnegative and satisfy*

$$\tilde{z}_{i,j}(a) = \begin{cases} 0 & \text{if } i < j, \\ e^{-\int_0^a M_{i,i}(\tau)d\tau} & \text{if } i = j, \\ -e^{-\int_0^a M_{i,i}(\tau)d\tau} \int_0^a \sum_{j\le k<i} M_{i,k}(s)\tilde{z}_{k,j}(s) e^{\int_0^s M_{i,i}(\tau)d\tau} ds & \\ & \text{if } i > j. \end{cases} \tag{17}$$

P r o o f. Using the fact that $[Z^{-1}]' = -MZ^{-1}$ the $\tilde{z}_{i,j}(s)$ satisfy the differential equations

$$\begin{cases} \tilde{z}'_{1,j}(a) + m_{1,1}(a)\tilde{z}_{1,j}(a) = 0, \\ \tilde{z}'_{i,j}(a) + m_{i,i}(a)\tilde{z}_{i,j}(a) = -\sum_{1\le k<i} m_{i,k}(a)\tilde{z}_{k,j}(a). \end{cases}$$

The result of the lemma follows by recursively solving these equations while considering the initial condition $Z^{-1}(0) = I_{n\times n}$. □

THEOREM 2.1. *Let $(H_{\beta,\tau,\sigma})$ be satisfied and Z^{-1} be nonnegative. Then the semigroup $\mathcal{G}_\phi := (G(t))_{t\ge 0}$ on E solution of (2) is given by*

$$G(t)f = \Upsilon^{-1}T_\phi(t)\Upsilon f, \qquad f \in E. \tag{18}$$

Both $\mathcal{T}_\phi := (T_\phi(t))_{t\ge0}$ and $\mathcal{G}_\phi$ are nonnegative and for $g \in E$, $g \ge 0$, the solution U of (2) such that $U(0,\cdot) = g$ satisfies

$$\begin{aligned} U(t,a) &= G(t)g(a) \\ &= \begin{cases} Z^{-1}(a)\int_0^{\tilde{a}} B(s)U(t-a,s)ds & \text{if } t-a \ge 0, \\ Z^{-1}(a)Z(t-a)g(t-a) & \text{if } t-a < 0. \end{cases} \end{aligned} \tag{19}$$

Furthermore the generator A^G of $\mathcal{G}_\phi$ satisfies

$$\begin{aligned} D(A^G) &= \{g \in W^{1,1}((0,\tilde{a}),\mathbf{R}^n), g(0) = \int_0^{\tilde{a}} B(a)g(a)da\}, \\ A^G g(a) &= -g'(a) - M(a)g(a), \qquad g \in D(A^G). \end{aligned}$$

(Next the operator ϕ is called the core operator of both semigroups $(G(t))_{t\ge0}$ and $(T_\phi(t))_{t\ge0}$).

P r o o f. Lemma 2.1 guaranties the existence of the translation C_0-semigroup $\mathcal{T}_\phi$ as solution semigroup (in a generalized sense) of the Cauchy problem (9) which satisfies (16). To conclude the proof of the theorem we simply use the fact that $\mathcal{G}_\phi$ given by (18) is Υ-equivalent to $\mathcal{T}_\phi$ since Υ and Υ^{-1} are bounded matrices.

The solution semigroup is given in a generalized sense to express that the considered semigroup is indeed the strong solution semigroup for initial values f in the domain of its generator, using the fact that if X is a Banach space, then the domain of a C_0-semigroup on X is dense in X. □

2.2. **Compactness and spectral properties.** Next we will show that the equivalence of the solution semigroup $\mathcal{G}_\phi$ of (2) to the translation semigroup $\mathcal{T}_\phi$ which is associated with the core operator ϕ given by (14)-(15) allows us to deduce in an automatic way the associated qualitative properties based on properties of ϕ. It is for this reason that we called ϕ a core operator to express the fact that properties of associated translation semigroup or any equivalent semigroup are an automatic consequence of those of ϕ.

We first give the definition of irreducibility and of compactness type [5]. Taking an ordered Banach space X, F a Banach space and $0 < A \leq \infty$, we have

- For an operator P on X, $\sigma(P)$, $r(P)$ and $s(P)$ denote as usual the spectrum, the spectral radius and the spectral bound of P. That is $r(P) := \sup\{|\alpha|, \alpha \in \sigma(P)\}$, $s(P) := \sup\{\Re(\alpha), \alpha \in \sigma(P)\}$ $\big(\Re(\lambda)$ is the real part of $\alpha\big)$. If P is nonnegative and X^* is the dual space of X, then P is called irreducible on X if for arbitrary positive elements $U \in X$ and $U^* \in X^*$, there exists $n \in \mathbf{N}$ such that $(U^*, P^n U) > 0$. $\big($Here a vector U is called nonnegative (resp. positive) if $U \geq 0$ (resp. $U \geq 0$ and $U \neq 0\big)$.
- If $X := L^1((0,A),F) \to F$, then an operator $\Phi : X \to F$ is called of compact type if for each $\alpha \in \mathbf{C}$ there exists $n \geq 1$ such that $\tilde{\Phi}_\alpha^n$ is compact where $\tilde{\Phi}_\alpha : F \to F$ is given by

$$\tilde{\Phi}_\alpha V := \Phi\big(e^{-\alpha\cdot} \otimes V\big) \quad \text{with} \quad (e^{-\alpha\cdot} \otimes V)(s) := e^{-\alpha s} V, \quad V \in F. \tag{20}$$

$\big($In the case $X := L^1((-A,0),F)$, we take $\tilde{\Phi}_\alpha V := \phi(e^{\alpha\cdot} \otimes V)$.$\big)$

Also we call a spectral value $\lambda_0 \in \sigma(P)$ dominant if

$$\Re(\lambda_0) > \sup\{\Re(\lambda), \lambda \in \sigma(P) \text{ and } \lambda \neq \lambda_0\}.$$

For the core operator ϕ of the model (2), we therefore have

$$\tilde{\phi}_\lambda x := \int_0^{\tilde{a}} e^{-\lambda a} \Gamma(a) x da, \quad x \in \mathbf{R}^n, \lambda \in \mathbf{C}. \tag{21}$$

Remark 2.1. The operators $\tilde{\phi}_\lambda$ are in fact obtained by solving the resolvent equation of $D(A_\phi)$. For $g \in E$, the resolvent equation

$$(\lambda I_{E_\phi} - A_\psi) f = g$$

with unknown $f \in D(A_\phi)$ reads $\lambda f + f' = g$ and yields $f(a) = e^{-\lambda a} f(0) + \int_0^a e^{\lambda(s-a)} g(s)ds$. It then follows that the condition $f(0) = \phi f$ is equivalent to $f(0) = \phi\big(a \mapsto e^{-\lambda a} f(0) + \int_0^a e^{\lambda(s-a)} g(s)ds\big)$, that is

$$f(0) - \tilde{\phi}_\lambda\big(f(0)\big) = \phi\Big(a \mapsto \int_0^a e^{\lambda(s-a)} g(s)ds\Big).$$

Compactness and irreducibility of such operators $\tilde{\phi}_\lambda$ play an essential role in the analysis of spectral properties of translation semigroups for the deduction of associated asymptotic behavior.

For the case of the generalized model (2), irreducibility of $\tilde{\phi}_\lambda$ operators is however not always satisfied as it is the case for the particular case of the SIR model (35)-(36).

The following result gives a sufficient and necessary condition to get the irreducibility of the operators $\tilde{\phi}_\lambda$ of the model.

LEMMA 2.3. *Let $(H_{\beta,\tau,\sigma})$ be satisfied and Z^{-1} be nonnegative. Then for $\lambda \in \mathbf{C}$, a sufficient condition for the operator $\tilde{\phi}_\lambda$ to be irreducible is*

$$(H_{\beta,Z}) \quad \left(\begin{array}{l}\text{For } i,j \in J \text{ there exists } k \in J \\ \text{such that } \beta_{i,k}\tilde{z}_{k,j} \not\equiv 0\end{array}\right). \tag{22}$$

P r o o f. The proof comes from the fact that for positive vectors $u, u^* \in \mathbf{R}^n$, the duality product

$$(u^*, \tilde{\phi}_\alpha(u)) = \int_0^{\tilde{a}} e^{-\alpha a} \sum_{i=1}^n u_i^* \Big(\sum_{k=1}^n \beta_{i,k}(a) \big(\sum_{j=1}^n \tilde{z}_{k,j}(a) u_j\big)\Big) da$$

is positive. □

Notice that the condition $(H_{\beta,Z})$ in (22) simply yields that all components of the matrix operator $\tilde{\phi}_\lambda$ are positive. Furthermore the negation of the condition (22) holds when we have

$$\Big(\text{There exists } i,j \in J \text{ for all } k \in J \text{ it holds } \beta_{i,k}\tilde{z}_{k,j} \equiv 0 \text{ on } (0,\tilde{a})\Big).$$

The extreme case for which this last condition is satisfied is when one or more lines of the matrix function β or one or more columns of the matrix function Z are identically zero. In this case, the operators $\tilde{\phi}_\lambda$ are of course not irreducible. Next we consider this latter case since it is also the case of the epidemic model (35)-(36).

We set

$$\begin{aligned} J' &:= \big\{i \in J \text{ such that} \beta_{i,j} \equiv 0 \text{ for all } j \in J\big\}, \\ J'' &:= \big\{j \in J \text{ such that } \tilde{z}_{i,j} \equiv 0 \text{ for all } i \in J\big\}. \end{aligned} \tag{23}$$

With a change of the order of the components of vectors of $\mathbf{R}^n$ and correspondingly of functions of E, we also adopt next the following setting

$$x = \begin{pmatrix} x_{J-L} \\ x_L \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} f_{J-L} \\ f_L \end{pmatrix} \text{ for } x \in \mathbf{R}^n, f \in E \text{ and } L \subset J.$$

Lemma 2.4. *Let $(H_{\beta,\tau,\sigma})$ and Z^{-1} be nonnegative. If $J' \cup J'' \neq \emptyset$, then for each $\lambda \in \mathbf{C}$, $\tilde{\phi}_\lambda$ is not irreducible.*

P r o o f. In the case $J'' \neq \emptyset$, taking $x \in \mathbf{R}^n$ and $x^* \in \mathbf{R}^n$ such that ($x_i = x_i^* = 1$ if $i \in J''$ and $x_i = x_i^* = 0$ if $i \notin J''$) we have $(x, x^*) > 0$ and $(x^*, \tilde{\phi}_\lambda x) = 0$ since $\tilde{\phi}_\lambda x = 0_{\mathbf{R}^n}$. In the case $J'' \neq \emptyset$, taking $x \in \mathbf{R}^n$ and $x^* \in \mathbf{R}^n$ such that ($x_i = x_i^* = 1$ if $i \in J'$ and $x_i = x_i^* = 0$ if $i \notin J'$) we also have $(x, x^*) > 0$ and $(x^*, \tilde{\phi}_\lambda x) = 0$ since $(\tilde{\phi}_\lambda x)_i = 0$ if $i \in J'$. □

We show next that the spectrum of the generator A_ϕ is characterized by the core operator given by

$$\psi : \ f \in E_\psi := L^1((0, \tilde{a}), \mathbf{R}^p) \mapsto \psi f := \int_0^{\tilde{a}} \tilde{\Gamma}(s) f(s) ds \in \mathbf{R}^p, \tag{24}$$

with

$$p := n - |J' \cup J''|, \quad \text{and} \quad \tilde{\Gamma}(s) := \tilde{B}(s)\tilde{W}(s),$$

where $|S|$ denotes the number of elements of a set S,

$$\tilde{B}(s) := (b_{i,j})_{i,j \in J,\, i \notin J' \cup J''} \quad \text{and} \quad \tilde{W}(s) := (\tilde{z}_{i,j})_{i,j \in J,\, j \notin J' \cup J''},$$

i.e., $\tilde{B}(s)$ (resp. $\tilde{W}(s)$) is the matrix extracted from the matrix $B(s)$ (resp. $Z^{-1}(s)$) by removing in $B(s)$ the lines with index i in $J' \cup J''$ (resp. the columns with index j in $J' \cup J''$). So we simply have $\tilde{\Gamma}(s) := (\Gamma_{i,j})_{i,j \notin J' \cup J''}$. The operators $\tilde{\psi}_\lambda$ associated with ψ satisfy

$$\tilde{\psi}_\lambda : \ x \in \mathbf{R}^p \mapsto \ \tilde{\psi}_\lambda x := \psi(e^{-\lambda \cdot} \otimes x) = \int_0^{\tilde{a}} e^{-\lambda s} \tilde{B}(s) \tilde{W}(s) x ds. \tag{25}$$

We show next that these operators are characterizing the spectral properties of the model semigroup.

Lemma 2.5. *Assume that Z^{-1} is nonnegative and that $(H_{\beta,\tau,\sigma})$ holds. Then we have:*

(i)

$$\det\left(\xi I_{n\times n} - \tilde{\phi}_\lambda\right) = \xi^p \det\left(\xi I_{p\times p} - \tilde{\psi}_\lambda\right). \tag{26}$$

(ii) *A sufficient condition for the operator $\tilde{\psi}_\lambda$ to be irreducible is*

$$(H_{\beta,Z})' \begin{pmatrix} \text{For } i,j \in J-(J'\cup J'') \text{ there exists } k \in J \\ \text{such that } \beta_{i,k}\tilde{z}_{k,j} \not\equiv 0 \text{ on } (0,\tilde{a}) \end{pmatrix}. \tag{27}$$

P r o o f. Since the entries of the matrix $(\xi I_{n\times n} - \tilde{\phi}_\lambda)$ that are in the lines with index in J' or that are in the columns with index in J'' are all equal to 0 except for the diagonal entries which are equal to λ, an iterative development of the determinant according to these lines followed by an iterative development with respect to these columns yields (26). To get the result of ii), it suffices to notice that with $(H_{\beta,Z})'$ we have $(u^*, \tilde{\psi}_\lambda(u)) > 0$ for $u, u^* \in \mathbf{R}^p$ such that u and u^* are positive vectors. □

To the operator ψ, and because of its compactness type and the irreducibility of its associated operators $\tilde{\psi}_\lambda$, we can apply the theory on translation semigroup developed in [5] to get the sufficient spectral properties that yield to the AEG property of the associated semigroup $(T_\psi(t))_{t\geq 0}$. The following result gathers these properties for the operator ψ.

LEMMA 2.6. *Assume that Z^{-1} is nonnegative and that $(H_{\beta,\tau,\sigma})$ holds. Then A_ψ such that $A_\psi f := -f'$ for $f \in D(A_\psi) := \{f \in W^{1,1}((0,\bar{a}),\mathbf{R}^p), f(0) = \psi f\}$ is the generator of a C_0-semigroup of translation $\mathcal{T}_\psi := (T_\psi(t))_{t\geq 0}$ on E_ψ and we have the following assertions.*
A1. *$\sigma(A_\psi) = \sigma_p(A_\psi) = \{\lambda \in \mathbf{C}, 1 \in \sigma(\tilde{\psi}_\lambda)\}$, $dim\big[Kernel(\lambda I_{E_\psi} - A_\psi)\big] = dim\big[Kernel(I_F - \tilde{\psi}_\lambda)\big]$ and*

$$Kernel(\lambda I_{E_\psi} - A_\psi) = \big\{e^{-\lambda\cdot} \otimes x, x \in Kernel(I_{\mathbf{R}^p} - \tilde{\psi}_\lambda)\big\}.$$

A2. *$r(\tilde{\psi}_\lambda) > 0$ and $r(\tilde{\psi}_\lambda)$ is an eigenvalue of $\tilde{\psi}_\lambda$ with a geometric multiplicity equal to one.*
A3. *$\tilde{\psi}_\lambda$ is irreducible for each $\lambda \in \mathbf{C}$, the mapping $\alpha \in \mathbf{R} \mapsto r(\tilde{\psi}_\alpha)$ is continuous and decreasing. Furthermore, $\lim_{\alpha\to-\infty} r(\tilde{\psi}_\alpha) = \infty$ and $\lim_{\alpha\to\infty} r(\tilde{\psi}_\alpha) = 0$.*
A4. *$s(A_\psi)$ is a simple pole of A_ψ and is dominant in $\sigma(A_\psi)$. Furthermore $\lambda_0 := s(A_\psi)$ is the unique real value α, which exists, that satisfies $r(\tilde{\psi}_\alpha) = 1$. Furthermore $Kernel\big(\lambda_0 I_{E_\psi} - A_\psi\big)$ and $Range\big(\lambda_0 I_{E_\psi} - A_\psi\big)$ are closed in X and are closed under $(T_\psi(t))_{t\geq 0}$ and we have $E_\psi = Kernel\big(\lambda I_{E_\psi} - A_\psi\big) \oplus Range\big(\lambda I_{E_\psi} - A_\psi\big)$.*

P r o o f. In this proof we give the conditions allowing to conclude each assertion following the work in [5]. The result of each assertion is also true for every operator $\Phi : L^1((-r, 0), F) \to F$ with F being a Banach if of course Φ satisfies the associated conditions.

Assertion A1 holds because ψ of bounded and is of compact type. Assertion A2 holds since $\tilde{\psi}_\lambda$ is irreducible and with compact iterate. Since ψ is nonnegative and of compact type and because of the irreducibility of $\tilde{\psi}_\lambda$ for at least one λ (yielding the irreducibility for all λ), assertion A3 holds. Assertion A4 holds since ψ is of compact type, $\tilde{\psi}_\lambda$ is irreducible for one λ and and the fact that ψ is of finite rank yielding the eventually compactnes of $(T_\psi(t))_{t\geq 0}$ (see Proposition of [5]. □

As already mentioned, for the case of the operator ϕ of the model (2) we do not have the irreducibility of $\tilde{\phi}_\lambda$ in general. However, and as it is shown next, similar results of assertions $A1 - A4$ of Lemma 2.6 can also be obtained for ϕ.

It is in fact because of the property (26) on the characteristic polynomial of $\tilde{\phi}_\lambda$ that the analysis of spectral properties of the generator of the solution semigroup $\mathcal{T}_\phi$ can be restricted to associated spectral properties of ψ as we will see next. Indeed, with this fact, we only have to consider the matrix $\tilde{\Gamma}(s)$ obtained from the matrix $\Gamma(s)$ by removing each line with index in $J' \cup J''$, and each column with index in $J' \cup J''$ instead of considering $\Gamma(s)$. Indeed, as a consequence of Lemma 2.3 and of the equivalence between the semigroups $\mathcal{T}_\phi$ and $\mathcal{G}_\phi$ we get the following result on the compactness of the solution semigroup and on the simplicity and dominance of the spectral bound of the its associated generator.

Theorem 2.2. *Assume that Z^{-1} is nonnegative and that $(H_{\beta,\tau,\sigma})$ holds. Then it holds*:

(i) *The translation semigroup $\mathcal{T}_\phi$ on E is compact for $t \geq \tilde{a}$ and is nonnegative. Furthermore the spectrum $\sigma(A_\phi)$ of A_ϕ (resp. $\sigma(A_\psi)$ of A_ψ with A_ψ considered on E_ψ) is equal to the point spectrum $\sigma_p(A_\phi)$ (resp. $\sigma_p(A_\psi)$) and satisfies*

$$\sigma(A_\phi) = \sigma(A_\psi) = \{\lambda \in \mathbf{C}, 1 \in \sigma(\tilde{\psi}_\lambda)\} = \Big\{\lambda \in \mathbf{C},$$
$$det\Big(I_{p\times p} - \Big[\int_0^{\tilde{a}} e^{-\lambda a} \sum_{k\in J} \beta_{i,k}(a)\tilde{z}_{k,j}(a)da)\Big]_{i,j\in J-(J'\cup J'')}\Big) = 0\Big\};$$

(ii) *The spectrum $\sigma(A^G)$ of A^G is equal to its point spectrum $\sigma_p(A^G)$ and satisfies $\sigma(A^G) = \sigma(A_\phi)$;*
(iii) *If ψ_λ is irreducible for a $\lambda \in \mathbf{R}$ (In particular if $(H_{\beta,Z})'$ is satisfied) then $s(A_\psi)$ is a dominant eigenvalue of A^G (and of A_ϕ and A_ψ) and is the unique real value λ_0 such that $r(\tilde{\psi}_\lambda) = 1$ and a simple pole and a dominant eigenvalue of A_ψ (and of A_ϕ and A^G).*

P r o o f. Using [5], Proposition 4, the compactness of $\mathcal{T}_\phi$ comes from the fact that ϕ is of compact type since ϕ which is in $\mathcal{L}(E, \mathbf{R}^n)$ is of finite rank. The positivity of $(T_\phi(t))_{t\geq 0}$ also follows from the positivity of ϕ (see [5]-Proposition 1). The assertion (ii) follows from Lemma 2.6-A1 and Lemma 2.5. Finally (iii) follows from assertion (ii) of Lemma 2.5 and assertion A4 of Lemma 2.6. □

THEOREM 2.3. *Let $(H_{\beta,\tau,\sigma})$ and $(H_{\beta,Z})'$ be satisfied and assume that Z^{-1} is nonnegative and $J'' \subset J'$. Then*
(i) *$\lambda_0 := s(A_\phi)$ is a simple pole of A_ϕ and*

$$Kernel(\lambda_0 I_E - A_\phi) = \left\{ \begin{pmatrix} f_{J-J'} \\ f_{J'} = 0 \end{pmatrix}, f_{J-J'} \in Kernel(\lambda_0 I_{E_\psi} - A_\psi) \right\}$$
$$= \left\{ Ce^{-\lambda_0 \cdot} \begin{pmatrix} x_{\psi,\lambda_0} \\ 0_{\mathbf{R}^p} \end{pmatrix}, C \in \mathbf{R} \right\},$$

where $x_{\psi,\lambda_0} \in \mathbf{R}^{n-p}$ denotes the unique positive eigenvector with norm 1 of ψ_{λ_0} which is associated with the eigenvalue $1 = r(\tilde{\psi}_{\lambda_0})$. **(ii)** *$Range(\lambda_0 I_E - A_\phi)$ is closed in E and is closed under $(T_\phi(t))_{t\geq 0}$ and we have*

$$E = Kernel\big(\lambda_0 I_E - A_\phi\big) \oplus Range\big(\lambda_0 I_E - A_\phi\big).$$

P r o o f. From Remark 1 we have $f \in Kernel(\lambda_0 I_E - A_\phi)$ if and only if there exist $x \in \mathbf{R}^n$ such that $\tilde{\phi}_\lambda x = x$ and $f = e^{-\lambda_0 \cdot} \otimes x$. Using the fact that $\phi f = f(0)$ we get $f_{J''}(0) = x_{J'} = 0$ since the lines of matrix kernel Γ with index in J' are zero. Now since $J'' \subset J'$ it follows that $x_{J'}$ is an eigenvector of $\tilde{\psi}_\lambda$ which is associated with the eigenvalue 1 and the proof of (i) may be concluded using Lemma 2.6. To prove (ii) we take $g \in D(A_\phi)$ such that $(\lambda_0 I - A_\phi)^2 g = 0$. This gives that $(\lambda_0 I - A_\phi)g \in Kernel(\lambda_0 I - A_\phi)$ and there exists $C \in \mathbf{R}$ such that $(\lambda_0 I - A_\phi)g = Ce^{-\lambda_0 \cdot} \otimes x$ with $x_{J'} = 0$. So we get $(\lambda_0 I - A_\psi)^2 g_{J-J'} = 0$ and $C = 0$ since λ_0 is a simple pole of A_ψ. Finally we have $(\lambda_0 I - A_\phi)g = 0$ and λ_0 is a therefore simple pole of A_ϕ. □

2.3. **AEG property.** Next we show the AEG property of the solution semigroup of the general model (2). For a translation semigroup associated with a core operator Φ one main condition used in [5] to get its AEG property is the irreducibility of a $\tilde{\Phi}_\lambda$.

DEFINITION 2.1. For $d \in \mathbf{N}$, a semigroup $\mathcal{T} := (T(t))_{t\geq 0}$ on a Banach space X is said to have a λ_0-d-P-AEG property if P is a d dimensional projection on X and there exist $C, \delta > 0$ are constants such that

$$\|e^{-\lambda_0 t} T_\phi(t) - P\| \leq Ce^{-\delta t}, \quad t \geq 0.$$

λ_0 is called the Malthusian parameter of $\mathcal{T}$.

The AEG result of [5] cannot be applied to the semigroup of (2) since this condition is not satisfied in general as already stated in the previous section. However it can be applied to the case of the operator ψ to get the following result.

LEMMA 2.7. *Let $(H_{\beta,\tau,\sigma})$ be satisfied and Z^{-1} be nonnegative. Then denoting by P_ψ be the projection onto the eigenspace of A_ψ that is associated with the unique real value λ_0 such that $r(\tilde{\psi}_\lambda) = 1$, we have $\lambda_0 = s(A_\psi) = \omega(A_\psi)$, P_ψ is nonnegative. Furthermore $(T_\psi(t))_{t\geq 0}$ has the λ_0-1-P_ψ-AEG property.*

P r o o f. This result is a direct consequence of the work in [5] using the fact that ψ is nonnegative and of finite rank and the operator $\tilde{\psi}_\lambda$ are irreducible. □

Though the irreducibility condition is not satisfied for the model (2) the following result shows that its solution semigroup has the AEG property.

THEOREM 2.4. *Let $(H_{\beta,\tau,\sigma})$ and $(H_{\beta,Z})'$ be satisfied and let Z^{-1} be nonnegative and $J'' \subset J'$. Then denoting by P_ϕ the projection on $Kernel(\lambda_0 I_E - A_\phi)$ we have*
(i) *For each $f \in E$ there exists $C_f \in \mathbf{R}$ such that*

$$P_\phi f = C_f\Big(e^{-\lambda_0 \cdot} \otimes \begin{pmatrix} x_{\psi,\lambda_0} \\ 0_{\mathbf{R}^p} \end{pmatrix}\Big),$$

where $x_{\psi,\lambda_0} \in \mathbf{R}^{n-p}$ denotes the unique positive eigenvector with norm 1 of ψ_{λ_0} which is associated with the eigenvalue $1 = r(\tilde{\psi}_{\lambda_0})$.
(ii) *$\mathcal{T}_\phi$ has the λ_0-1-P_ϕ-AEG property and taking $\tilde{P}_\phi := \Upsilon^{-1} \circ P_\phi \circ \Upsilon$, $\mathcal{G}_\phi$ has the λ_0-1-$\tilde{P}_\phi$-AEG property.*
The solution U of (2) such that $u(0,\cdot) = f$, $f \in E$, shows the asymptotic behavior

$$\|U(t,\cdot) - C(\Upsilon f) e^{\lambda_0(t-\cdot)} Z^{-1}(\cdot) x_{\lambda_0}\| = o(e^{\lambda_0 t}) \quad (t \to \infty).$$

P r o o f. Assertion (i) is a direct consequence of Theorem 2.3. Since $R := Range(\lambda_0 I_E - A_\Phi)$ is closed under the semigroup $\mathcal{T}_\phi$ the restriction $\mathcal{T}_{\phi,R}$ of $\mathcal{T}_\phi$ to R defines a C_0-semigroup that is eventually compact with the generator $A_{\phi,R}$ equal to the restriction of A_ϕ to R. Since $s(A_\phi)$ is dominant it then follows that $\omega(\mathcal{T}_{\phi,R}) = s(A_{\phi,R}) < s(A_\phi) = \omega(\mathcal{T}_\phi)$ and the AEG property of $\mathcal{T}_\phi$ in (ii) follows from the fact that $X = Kernel(\lambda_0 I_E - A_\phi) \oplus R$. The AEG property of the semigroup $\mathcal{G}_\phi$ in (ii) and the asymptotic behavior of the solution u in (iii) follow, respectively, from the fact that $\mathcal{G}_\phi$ is Υ-similar to $\mathcal{T}_\phi$ and the fact that $u(t,\cdot) = G(t)f$. □

2.4. $R0$ **and stability.** Next X denotes a Banach space and $\mathcal{T} := (T(t))_{t\geq 0}$ is a C_0-semigroup on X. A steady state of $\mathcal{T}$ is any stationary point of $\mathcal{T}$ (i.e., $T(t)\tilde{x} = \tilde{x}$ for every $t \geq 0$). An steady state $\tilde{x}$ of $\mathcal{T}$ is called locally exponentially stable if there exist V a neighborhood of $\tilde{x}$ and constants $C, \delta > 0$ such that $\|T(t)x - \bar{x}\| \leq Ce^{-\delta t}$ for every $t \geq 0$ and $x \in V$ (i.e., $T(t)x$ converges uniformly and exponentially to $\bar{x}$ on V). If in addition $V = X$ then $\bar{x}$ is called (globally) exponentially stable.

The following lemma gives a characterization of exponential stability of steady states of semigroups having the AEG property.

LEMMA 2.8. *Assume that $\mathcal{T}$ has the λ_0-d-P-AEG property and that $\bar{x}$ is a steady state of $\mathcal{T}$. Then*

$$\big(\bar{x} \text{ is exponentially stable}\big) \Longleftrightarrow \big(\Re(\lambda_0) < 0\big).$$

P r o o f. It is sufficient to use the fact that for each $x_1, x_2 \in X$ we have $(T(t)x_1 - T(t)x_2)$ asymptotically behaves as $e^{\lambda_0 t}(Px_1 - Px_2)$. □

For similar semigroups, the following lemma gives an equivalence between the exponential stability of these steady states.

LEMMA 2.9. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two semigroups on X. Assume that $\mathcal{T}_1$ is L-similar to $\mathcal{T}_2$ (or equivalently that $\mathcal{T}_2$ is L-similar to $\mathcal{T}_1$) where L is an operator on X such that L and L^{-1} are bounded on X. Then we have*
(i) *$\big(x \in X$ is a steady state of $\mathcal{T}_1\big)$ if and only if $\big(Lx$ is a steady state of $\mathcal{T}_2\big)$.*
(ii) *If $x \in X$ is a steady state of $\mathcal{T}_1$, then $\big(x$ is exponentially stable for $\mathcal{T}_1\big)$ if and only if $\big(Lx$ is exponentially stable for $\mathcal{T}_2\big)$.*

P r o o f. (i) follows directly from $T_1(t) = L^{-1}T_2(t)L$, $t \geq 0$. For (ii), if $\bar{x}$ is an exponentially stable steady state of $\mathcal{T}_1$, then from

$$\|T_2(t)Lx - L\bar{x}| = \|LL^{-1}T_2(t)Lx - LL^{-1}L\bar{x}\| \leq \|L\|\|T_1(t)x - \bar{x}\|,$$

it follows that $L\bar{x}$ is also an exponentially stable steady state of $\mathcal{T}_2$. The inverse implication of (ii) follows in a symetric way. □

Coming back to the semigroup $\mathcal{T}_\phi$ of the generalized model (2), we have the following characterization of its steady states.

LEMMA 2.10. *Let $(H_{\beta,\tau,\sigma})$ be satisfied. Then $\tilde{f} \in E$ is a steady state of $\mathcal{T}_\phi$ if and only if there exists $\tilde{x} \in F$ such that $\big(\tilde{f} := \mathbf{1} \otimes \tilde{x}$ and $\phi(\tilde{f}) = \tilde{x}\big)$ where $\mathbf{1} \otimes \bar{x}$ is the function $s \mapsto \mathbf{1}(s)\bar{x} = \bar{x}$. In this case, the components of $\tilde{x}_{J'}$ are all equal to 0.*

P r o o f. The first statement comes from (16). The second statement follows directly from the fact that $(\phi f)_{J'}$ is the zero vector of $\mathbf{R}^{|J'|}$ and $\phi\tilde{f} = \tilde{f}$. □

With all these results and the established properties of $\mathcal{T}_\phi$ and of $\mathcal{G}$ in Sections 2.1-2.3, we may get a characterization of steady states of (2) and their stability.

THEOREM 2.5. *Let $(H_{\beta,\tau,\sigma})$ and $(H_{\beta,Z})'$ be satisfied and let Z^{-1} be nonnegative and $J'' \subset J'$. Then each steady state of the solution semigroup $\mathcal{G}_\phi$ of the model* (2) *is of the form $\Upsilon^{-1}(1 \otimes \bar{x}) : a \mapsto Z^{-1}(a)\bar{x}$ with $\bar{x} \in \mathbf{R}^n$ such that $\tilde{x}_i = 0$ for $i \in J'$ and it holds*

$$\left[\Upsilon^{-1}(\mathbf{1} \otimes \bar{x}) \text{ is exponentially stable}\right] \Longleftrightarrow \quad \left[r(\tilde{\phi}_0) < 1\right].$$

P r o o f. Since λ_0 is the unique root of $r(\tilde{\phi}_\lambda) = 1$ and since the mapping $\lambda \in \mathbf{R} \mapsto r(\tilde{\phi}_\lambda)$ is strictly decreasing, we get that $\Re(\lambda_0) < 0$ if and only if $r(\tilde{\phi}_0) < 1$. Using this, the proof may be deduced from Lemma 2.9 and Lemma 2.10 since $\mathcal{G}_\phi$ is Υ-similar to $\mathcal{T}_\phi$. □

Because of the importance of the spectral radius of $\tilde{\phi}_0$ we introduce the following definition.

DEFINITION 2.2. **(i)** The basic number R^ϕ of the core operator ϕ as well as of the semigroup $\mathcal{T}_\phi$ (and of any to $\mathcal{T}_\phi$ similar semigroup $\mathcal{G}_\phi$) is defined as $R^\phi := r(\tilde{\phi}_0)$. Also we call $\tilde{\phi}_0$ the basic operator of $\mathcal{T}_\phi$ (and also of $\mathcal{G}_\phi$ and of any to $\mathcal{T}_\phi$ similar semigroup $\mathcal{G}_\phi$).
(ii) The basic reproduction number $R0$ of a semigroup of operators (and also of any mathematical population model) as the threshold parameter $R0$ characterizing the asymptotic stability of any associated steady state $\tilde{u}$ in the sense that

$$(\tilde{u} \text{ is asymptotically stable}) \text{ if and only if } (R0 < 1).$$

As already mentioned, the characterization of $R0$ in terms of core operators of translation semigroups was first established in the works of the first author Larbi Alaoui in 1998 for the case of the Sharpe-Lotka PDE model ([2], Proposition 5.10), and in 2001 for the case of of a multi states version of the Sharpe-Lotka model [1]. In these works, $R0$ was characterized as the spectral radius of $\tilde{\phi}_0$ but under conditions on the parameters of each model yielding the irreducibility of the associated operators $\tilde{\phi}_\lambda$. The Sharpe-Lotka model is described by the following equations

$$\begin{cases} \mathcal{D}_{t,a}u(t,a) = -\mu(a)u(t,a), \\ u(t,0) = \displaystyle\int_0^{\tilde{a}} \beta(a)u(t,a)da. \end{cases} \tag{28}$$

The multi-dimensional version of this model, which was considered in [1] for the characterization of $R0$, reads as follows

$$\begin{cases} \mathcal{D}_{t,a}u_i(t,a) = -\mu_i(a)u_i(t,a) \\ u_i(t,0) = \displaystyle\int_0^{\tilde{a}} \sum_{j\in\{1,\cdots,n\}} \beta_{i,j}(a)u_j(t,a)da, \quad 1 \le i \le n \end{cases}. \tag{29}$$

The core operator of (29) satisfies

$$\phi : E := L^1((0,\tilde{a}), R^n) \to R^n, \quad \phi f = \int_0^{\tilde{a}} \Gamma(s)f(s)ds, \tag{30}$$

with

$$\Gamma(s) := B(s)Z^{-1}(s), \tag{31}$$

where $B(s)$ is the matrix

$$B(s) := (\beta_{i,j}(s))_{1\le i,j\le n} \tag{32}$$

and $Z^{-1}(s) := (\tilde{z}_{i,j}(s))_{1\le i,j\le n}$ is the matrix such that

$$\tilde{z}_{i,j}(s) := \begin{cases} e^{-\int_0^s \mu_{i,i}(\tau)d\tau} & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases} \tag{33}$$

Applying the previous results, we may now conclude the following result where, contrary to the result in [1] on the characterization of $R0$, we do not need to have the irreducibility of the whole $\tilde{\phi}_0$, but only the irreducibility of $\tilde{\psi}_0$.

Corollary 2.1. *Let ϕ is given by* (30)-(33) *and let $J := \{1, \cdots n\}$ and $J' := \{i \in J$ such that $\beta_{i,j} \equiv 0$ for all $j \in J\}$. Assume that the following condition is satisfied*

$$(H) \begin{pmatrix} \beta_{i,j}, \mu_i \in L^\infty((0,\tilde{a}), R^+), 1 \le i,j \le n \\ \text{For } i,j \in J - J' \text{ it holds } \beta_{i,j} \not\equiv 0 \text{ on } (0,\tilde{a}) \end{pmatrix}.$$

Then the basic number $R0$ of the n-dimensional model (29) *coincides with R^ϕ and R^ψ (i.e., $R0 = r(\tilde{\phi}_0) = r(\tilde{\psi}_0)$), where ψ is the operator*

$$\psi : \ g \in L^1((0,\tilde{a}), \mathbf{R}^{n-\|J'\|} \mapsto \psi f := \psi g = \int_0^{\tilde{a}} \tilde{B}(s)\tilde{W}(s)g(s)ds, \tag{34}$$

where $\tilde{B}(s)$ and $\tilde{W}(s)$ are the matrices

$$\tilde{B}(s) := (b_{i,j})_{i,j\in J, i\notin J'} \quad \text{and} \quad \tilde{W}(s) := (\tilde{z}_{i,j})_{i,j\in J, j\notin J'}.$$

P r o o f. For the model (29) we have $J'' = \emptyset$, and we have

$$\tilde{\phi}_\lambda x = \int_0^{\tilde{a}} e^{\lambda s}\tilde{B}(s)\tilde{W}(s)xds, \quad x \in \mathbf{R}^n, \lambda \in \mathbf{C}.$$

Under assumption (H), we have that $\tilde{B}(s)\tilde{W}(s) \not\equiv 0$ on $(0, \tilde{a})$, and that $\tilde{\phi}_\lambda$ is irreducible. So, the result of the corollary can be deduced using Theorem 2.5. □

REMARK 2.2. For the particular case of the Sharpe-Lotka model (28), because of the dimension 1, we have that the irreducibility of the whole associated operator ϕ_0 is needed, and that ψ coincides with ψ and is given by

$$\phi : E := L^1((0,\tilde{a}), \mathbf{R}) \to \mathbf{R}, \quad \phi f = \int_0^{\tilde{a}} \beta(s)e^{-\int_0^s \mu(\tau)d\tau} f(s)ds$$

and each $\tilde{\phi}_\lambda$, $\lambda \in \mathbf{C}$, satisfies

$$\tilde{\phi}_\lambda x = \int_0^{\tilde{a}} e^{-\lambda s}\beta(s)e^{-\int_0^s \mu(\tau)d\tau} xds, \quad x \in \mathbf{R}.$$

So, under the assumpltion

$$(H_{\mu,\beta})\left(\beta, \mu \in L^\infty((0,\tilde{a}), \mathbf{R}^+), \beta \not\equiv 0 \text{ on } (0,\tilde{a})\right),$$

$\tilde{\phi}_\lambda$ is irreducible for all $\lambda \in \mathbf{C}$, and the basic number of (28) is $R0 = r(\tilde{\psi}_0) = r(\tilde{\phi}_0) = \int_0^{\tilde{a}} \beta(s)e^{-\int_0^s \mu(\tau)d\tau}ds$.

3. Case of a PDE SIR epidemic model

In this section we consider the application of the results of the previous section to the case of the SIR model (35)-(36). We can write this model as follows

$$\mathcal{D}_{t,a}\begin{pmatrix} S \\ I \\ R \end{pmatrix}(t,a) = -M(a)\begin{pmatrix} S(t,a) \\ I(t,a) \\ R(t,a) \end{pmatrix} \tag{35}$$

and

$$\begin{pmatrix} S \\ I \\ R \end{pmatrix}(t,0) = \int_0^{\tilde{a}} B(a)\begin{pmatrix} S(t,a) \\ I(t,a) \\ R(t,a) \end{pmatrix} da, \tag{36}$$

where

$$M(a) := \begin{pmatrix} \zeta(a)+\mu(a) & 0 & 0 \\ -\zeta(a) & \gamma(a)+\mu(a) & 0 \\ 0 & -\gamma(a) & \mu(a) \end{pmatrix}$$

and

$$B(a) := \begin{pmatrix} \beta(a) & (1-q)\beta(a) & \beta(a) \\ 0 & q\beta(a) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this model S, I and R are the densities, with respect to time and age, of susceptibles, infectives and recovered, respectively. The age functions β, μ, ζ and γ denote the birth, mortality, infection and recovery rates, respectively.

The condition we assume next on the model parameter functions is as follows

$$(H_{\beta,\xi,\mu,\gamma}) \qquad \big\{\beta,\xi,\mu,\gamma \in L^\infty((0,\tilde{a}),\mathbf{R}^+), \quad \beta \not\equiv 0\big\}.$$

By setting

$$U := \big(S, I, R\big)^\tau,$$

and applying the previous result, the core operator of (35)-(36) is

$$\Phi : E := L^1((0,\tilde{a}), F := \mathbf{R}^3) \to F, \quad f \mapsto \phi f = \int_0^{\tilde{a}} \Gamma(s) f(s) ds, \tag{37}$$

where the kernel Γ is given by

$$\Gamma(s) = B(s) Z^{-1}(s),$$

with $Z^{-1} := (\tilde{z}_{i,j})_{1\le i,j\le 3}$ being the inverse matrix of matrix Z which the unique solution of

$$X' = XM, \quad X(0) = I_{3\times 3}.$$

The matrix Z^{-1}, which exists, is therefore the unique solution of

$$Y' = -MY, \quad Y(0) = I_{3\times 3}.$$

Resolving this equation, we get

$$Z^{-1}(a) = \begin{pmatrix} e^{-\int_0^a (\zeta(s)+\mu(s))ds} & 0 & 0 \\ \tilde{z}_{21}(a) & e^{-\int_0^a (\gamma(s)+\mu(s))ds} & 0 \\ \tilde{z}_{31}(a) & \tilde{z}_{32}(a) & e^{-\int_0^a \mu(s)ds} \end{pmatrix} \tag{38}$$

with

$$\begin{cases} \tilde{z}_{21}(a) = e^{-\int_0^a (\gamma(s)+\mu(s))ds} \displaystyle\int_0^a \zeta(s) e^{-\int_0^s (\zeta(\tau)-\gamma(\tau))d\tau}, \\ \tilde{z}_{31}(a) = e^{-\int_0^a \mu(s)ds} \displaystyle\int_0^a \gamma(s) e^{-\int_0^s \gamma(\tau)d\tau} \int_0^s \zeta(\tau) e^{-\int_0^\tau (\zeta(\sigma)-\gamma(\sigma))d\sigma} d\tau ds, \\ \tilde{z}_{32}(a) = e^{-\int_0^a \mu(s)ds} \displaystyle\int_0^a \gamma(s) e^{-\int_0^s \gamma(\tau)d\tau} ds. \end{cases} \tag{39}$$

Therefore, the kernel $\Gamma = (k_{ij})_{1\le i,j\le 3}$ satisfies

$$\Gamma(a) = \beta(a)\begin{pmatrix} \big(\tilde{z}_{11} + \tilde{q}\tilde{z}_{21} + \tilde{z}_{31}\big)(a) & \big(\tilde{q}\tilde{z}_{22} + \tilde{z}_{32}\big)(a) & \tilde{z}_{33}(a) \\ q\tilde{z}_{21}(a) & q\tilde{z}_{22}(a) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{40}$$

where $\tilde{q} := 1 - q$. So following the results of the previous section, we have

$$\sigma(\tilde{\phi}_\lambda) = \{0\} \cup \sigma(\tilde{\psi}_\lambda)$$

and ψ is given by

$$\psi : L^1((0,\bar{a}); \mathbf{R}^p) \to \mathbf{R}^p, \quad \psi g = \int_0^{\bar{a}} \tilde{\Gamma}(s) g(s) ds,$$

where

$$\tilde{\Gamma}(a) := \tilde{B}(a)\tilde{W}(a),$$

with

$$\tilde{B}(a) = \begin{cases} (B_{i,j})_{i,j\in\{1,2\}} & \text{if } q \neq 0, \\ (B_{i,j})_{i=j=1} & \text{if } q = 0, \end{cases}$$

and

$$\tilde{W}(a) = \begin{cases} (\tilde{z}_{i,j})_{i,j\in\{1,2\}} & \text{if } q \neq 0, \\ (\tilde{z}_{i,j})_{i=j=1} & \text{if } q = 0. \end{cases}$$

So, for the case $q = 0$ we get that the operators ψ and $\tilde{\psi}_\lambda$, $\lambda \in \mathbf{C}$, satisfy

$$\psi : E_{\psi,q=0} := L^1((0,\bar{a}); \mathbf{R}) \to \mathbf{R}, \quad \psi f = \int_0^{\bar{a}} \tilde{\Gamma}_{q=0}(s) f(s) ds,$$

$$\tilde{\psi}_\lambda : \mathbf{R} \to \mathbf{R}, \quad \tilde{\psi}_\lambda x = \int_0^{\bar{a}} e^{-\lambda s}\tilde{\Gamma}_{q=0}(s) x ds,$$

with

$$\tilde{\Gamma}_{q=0}(a) := (k_{i,j})_{1\le i,j\le 1} = \beta(a)\big(\tilde{z}_{1,1}(a) + \tilde{z}_{2,1}(a) + \tilde{z}_{3,1}(a)\big),$$

and for the case $q \neq 0$, we have

$$\psi : E_{\psi,q\neq 0} := L^1((0,\bar{a}); \mathbf{R}^2) \to \mathbf{R}^2, \quad \psi f = \int_0^{\bar{a}} \tilde{\Gamma}_{q\neq 0}(s) f(s) ds,$$

$$\text{and} \quad \tilde{\psi}_\lambda : \mathbf{R}^2 \to \mathbf{R}, \quad \tilde{\psi}_\lambda x = \int_0^{\bar{a}} e^{-\lambda s}\tilde{\Gamma}_{q\neq 0}(s) x ds,$$

where

$$\begin{aligned} \tilde{\Gamma}_{q\neq 0}(a) &:= \big(k_{i,j}\big)_{1\le i,j\le 2} \\ &= \beta(a)\begin{pmatrix} \tilde{z}_{11}(a) + \tilde{q}\tilde{z}_{21}(a) + \tilde{z}_{31}(a) & \tilde{q}\tilde{z}_{22}(a) + \tilde{z}_{32}(a) \\ q\tilde{z}_{21}(a) & q\tilde{z}_{22}(a) \end{pmatrix}. \end{aligned}$$

For the case $q = 0$, the spectrum of $\tilde{\psi}_\lambda$ is

$$\sigma(\tilde{\psi}_\lambda) = \Big\{ \int_0^{\bar{a}} e^{-\lambda s} k_{1,1}(s) \Big\},$$

and for the case $q \neq 0$, we have

$$\begin{aligned}\sigma(\tilde{\psi}_\lambda) = \Big\{ & \frac{1}{2}\left(\int_0^{\bar{a}} e^{-\lambda s}(k_{1,1}(a) + k_{2,2}(a))da - D_\lambda \right), \\ & \frac{1}{2}\left(\int_0^{\bar{a}} e^{-\lambda s}(k_{22}(a) + k_{1,1}(a))da + D_\lambda \right) \Big\},\end{aligned}$$

with

$$\begin{aligned}D_\lambda := \Big\{ \Big(\int_0^{\bar{a}} e^{-\lambda s} \big(k_{1,1}(a) + k_{2,2}(a)\big) da \Big)^2 \\ + 4 \int_0^{\bar{a}} e^{-\lambda s} k_{1,2}(a)da \int_0^{\bar{a}} e^{-\lambda s} k_{2,1}(a)da \Big\}^{1/2}.\end{aligned}$$

Using the fact that Φ is bounded under $H_{\beta,\xi,\mu,\gamma}$, the results of the previous section yield that the operator A_Φ with

$$D(A_\Phi) = \Big\{ g \in W^{1,1}((0,\bar{a}), \mathbf{R}^3), g(0) = \Phi g \Big\}$$

and

$$A_\Phi g = -g' \text{ for } g \in D(A_\Phi)$$

is the infinitesimal generator of a C_0-semigroup $\mathcal{T}_\phi := (T_\phi(t))_{t \geq 0}$ on E which is eventually compact.

Furthermore, the semigroup on E which is solution of (35)-(36) is $\mathcal{G}_\Phi := (G(t))_{t \geq 0}$ such that

$$G(t)f = \Upsilon^{-1} T_\Phi(t) \Upsilon f, \quad f \in E,$$

with

$$(\Upsilon f)(s) := Z(s)f(s), \qquad s \in (0, \tilde{a}).$$

Also the semigroup $\mathcal{G}_\Phi$ is eventually compact, nonnegative and with the generator A^G of $\mathcal{G}_\Phi$ satisfying

$$\begin{aligned}&D(A^G) = \{f \in W^{1,1}((0,\tilde{a}), \mathbf{R}^3), f(0) = \int_0^{\bar{a}} B(s)f(s)ds\}, \\ &A^G f(s) = -f'(s) - M(s)f(s), \qquad f \in D(A^G),\end{aligned}$$

i.e.,

$$A^G \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}(s) = \begin{pmatrix} -f_1'(s) \\ -f_2'(s) \\ -f_3'(s) \end{pmatrix} + \begin{pmatrix} -(\zeta(s) + \mu(s))f_1(s) \\ \zeta(s)f_1(s) - (\gamma(s) + \mu(s))f_2(s) \\ \gamma(s)f_2(s) - \mu(s)f_3(s) \end{pmatrix}.$$

The spectrum of A^G is given by

$$\sigma(A_\Phi^G) = \sigma_p(A_\Phi) = \big\{\lambda \in \mathbf{C}, 1 \in \sigma(\tilde{\psi}_\lambda)\big\}.$$

So, in the case $q = 0$,

$$\sigma(A_\Phi^G) = \Big\{\lambda \in \mathbf{C}, 1 = \int_0^{\tilde{a}} e^{-\lambda a}\beta(a)\tilde{z}_{1,1}(a)da\Big\},$$

and in the case $q \neq 0$,

$$\sigma(A_\Phi^G) = \Big\{\lambda \in \mathbf{C}, 1 = \int_0^{\tilde{a}} e^{-\lambda a}\beta(a)\big(\tilde{q}\tilde{z}_{2,1}(a) + \tilde{z}_{2,2}(a)\big)da \\ + q\int_0^{\tilde{a}} e^{-\lambda a}\beta(a)\tilde{z}_{2,1}(a)da \int_0^{\tilde{a}} e^{-\lambda a}\beta(a)\tilde{z}_{3,2}(a)da\Big\}.$$

Also we have that the spectral bound of A_Φ satisfies $s(A_\Phi) = \omega(A_\Phi)$, $s(A_\Phi)$ is dominant eigenvalue in $\sigma(A_\Phi)$ and it is the unique real value λ_0 such that $r(\tilde{\Phi}_{\lambda_0}) = 1$.

Applying Theorem 2.3, we also get

$$Kernel\big(\lambda_0 I_E - A_\Phi\big) = \Big\{Ce^{-\lambda_0 \cdot}\big(x_{\psi,\lambda_0,1}, x_{\psi,\lambda_0,2}, 0\big)^\tau, C \in \mathbf{R}\Big\},$$

with $\big(x_{\psi,\lambda_0,1} = 1$ and $x_{\psi,\lambda_0,2} = 0\big)$ if $q = 0$ and $\big(x_{\psi,\lambda_0,1}, x_{\psi,\lambda_0,2}\big) \in \mathbf{R}^2$ being the unique eigenvector of $\tilde{\psi}_{\lambda_0}$ with norm 1 that is associated with the eigenvalue 1 if $q \neq 0$.

Using Theorem 2.4 and Theorem 2.5, we may now easily conclude the following result on the AEG property of (35)-(36) and on the characterization of its basic number $R0$.

Theorem 3.1. *Let $(H_{\beta,\xi,\mu,\gamma})$ be satisfied. Then for each $f \in E$, $f \geq 0$, the solution U of* (35)-(36) *such that $U(0,\cdot) = f$ is given by $U(t,a) = G(t)f(a)$ and satisfies* (19). *The semigroup $\mathcal{G}_\Phi$ has the AEG property with the Malthusian parameter $\lambda_0 = s(A^G)$ that is the unique real number such that*

$$\int_0^{\bar{a}} e^{-\lambda s}k_{1,1}(s) = 1 \quad \textit{if} \quad q = 0,$$

or such that

$$\frac{1}{2}\left(\int_0^{\bar{a}} e^{-\lambda s}(k_{22}(a) + k_{1,1}(a))da + D_\lambda\right) = 1 \quad \textit{if} \quad q \neq 0.$$

For each nonnegative $f \in E$ the solution $u := u(t,a) := \big(S, I, R\big)^{\tau}$ such that $u(t,\cdot) = f$, there exist a nonnegative constant $C \in \mathbf{R}$ such that

$$\lim_{t\to\infty} e^{-\lambda_0 t}\begin{pmatrix} S(t,a)\\ I(t,a)\\ R(t,a)\end{pmatrix} = Ce^{-\lambda_0 a}\begin{pmatrix} \tilde z_{1,1}(a)x_{\psi,\lambda_0,1}\\ \tilde z_{2,1}(a)x_{\psi,\lambda_0,1} + \tilde z_{2,2}(a)x_{\psi,\lambda_0,2}\\ \tilde z_{3,1}(a)x_{\psi,\lambda_0,1} + \tilde z_{3,2}(a)x_{\psi,\lambda_0,1} + \tilde z_{3,3}(a)x_{\psi,\lambda_0,1}\end{pmatrix}.$$

Furthermore the basic reproduction number $R0$ of (35)-(36) *coincides with the basic number $R^{\phi} := r(\tilde\Phi_0)$ of ϕ and satisfies*

$$R0 : r(\tilde\psi_0) = \begin{cases} \displaystyle\int_0^{\bar a} k_{1,1}(a)da & \textit{if } q = 0,\\ \displaystyle\frac{1}{2}\left(\int_0^{\bar a}\big(k_{2,2}(a) + k_{1,1}(a)\big)da + D_0\right) & \textit{if } q \neq 0,\end{cases}$$

with

$$D_0 := \sqrt{\left(\int_0^{\bar a}\big(k_{1,1}(a) + k_{2,2}(a)\big)da\right)^2 + 4\int_0^{\bar a} k_{1,2}(a)da\int_0^{\bar a} k_{2,1}(a)da}.$$

Finally, each steady state is of the form $\bar U : a \mapsto Z^{-1}(a)\bar x$ with $\bar x := (x_1, 0, 0)^{\tau} \in \mathbf{R}^3$ if $q = 0$ and $\bar x := (x_1, x_2, 0)^{\tau} \in \mathbf{R}^3$ if $q \neq 0$ and

$$\big(\bar U \textit{ is exponentially stable }\big) \Longleftrightarrow \big(R0 < 1\big).$$

4. Conclusion

We considered a linear PDE SIR epidemic model describing the dynamics of a population exposed to an infectious disease. We analyzed this model by considering it as a particular case of a multi-dimensional linear PDE age structured population model in which the population is subdivided into n groups with possible transitions from one group to another.

Contrary to traditional approaches, the mathematical analysis we proposed for the models was done in an automatic way using the theoretical properties of translation semigroups. More precisely, properties on the associated core operator have been used to easily conclude properties of the models solutions. Beyond compactness and spectral properties this theory allowed us to directly get characterizations of the AEG property of the solutions of the models, of their basic reproduction numbers and of the stability of their steady states which are also explicitly derived. So we get a theoretical well founded characterization of the Malthusian parameters and of the basic reproduction numbers of the models only in terms of conditions on their core operators. This characterization also allowed us to automatically compute such values as a function of the parameters of the models. This has the advantage to better control the changes of these values while choosing parameter values that ensure the needed properties for a better control of the epidemic variations.

In summary, this work shows that translation semigroups provide a strong approach in the context of a well-defined framework for dealing with linear epidemic models that are described using transport PDE equations. We also believe that many other population models can be efficiently analyzed within this framework. In this sense we are also investigating the application of our approach to the study of the dynamics of SIR models that take into account other aspects and involve nonlinear terms in the modeling. In such an application, we are also planning to consider an associated numerical study with consideration of specific values of the models parameters. Furthermore, we are also investigating relationships with other existing approaches such as those using fractional derivatives. This can lead to important insights for the mathematical handling of epidemic models.

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