

SUM-CONNECTIVITY ENERGY ON GRAPH OPERATIONS

S. Sripriya¹, A. Anuradha^{2,§}

^{1,2} Department of Mathematics

Faculty of Engineering and Technology

SRM Institute of Science and Technology

Kattankulathur 603203, Tamil Nadu, INDIA

e-mails: ¹ss3832@srmist.edu.in, ^{2,§} anuradha@srmist.edu.in

Abstract

Consider a simple, finite, undirected, connected graph G on q vertices. Depending on the context, various matrix representations of G are available in literature. One such matrix representation is sum-connectivity matrix $SC(G)$ which is of order $q \times q$ with G vertices indexing its rows and columns. The $(ij)^{\text{th}}$ entry of $SC(G)$ is $\frac{1}{\sqrt{d_i+d_j}}$ if the corresponding vertex pair $(v_i, v_j) \in E$; otherwise it is 0. The summation of absolute eigenvalues of $SC(G)$ is called sum-connectivity energy and denoted as $E[SC(G)]$. In the article, we determine sum-connectivity energy for some generalized graph operations other than product graphs. Further, we establish the results in terms of the base graphs.

Math. Subject Classification: 05C05; 05C38; 05C50

Key Words and Phrases: sum-connectivity energy; t-splitting graph; t-shadow graph; duplication; subdivision

1. Introduction

Consider simple, undirected and connected graph G with $|V| = q$ and $|E| = s$. Two vertices v_i and v_j are said to be adjacent if there is an edge

connecting them and denoted as $v_i \sim v_j$. A vertex degree is termed as total amount of edges attached in it, and written as d_i . When an edge connects each vertex to every other vertex, then the graph is said to be complete. Adjacency matrix of graph G is a matrix of order q denoted $B(G) = [b_{ij}]$ such that $b_{ij} = 1$ for each adjacent pair v_i, v_j and 0, otherwise. A p -regular graph is one in which each of its vertex has same degree p . A bipartite graph is a graph whose vertices are partitioned into two disjoint vertex sets such that no edges connects vertices within the same set. Further in a bipartite graph, if every vertex of one set is attached to every other vertex of the partitioned set, then it becomes a complete bipartite graph.

Randic, in 1975, defined a molecular descriptor $R(G) = \sum_{v_i \sim v_j} \frac{1}{\sqrt{d_i d_j}}$, called the product-connectivity (also called as Randic) index. In analogous to this product-connectivity index, another molecular descriptor sum-connectivity index is defined as follows:

$$S(G) = \sum_{v_i \sim v_j} \frac{1}{\sqrt{d_i + d_j}}.$$

Todeschini and Consonni [9] compiled the application of topological indices in modelling the structural property relationships. Molecular descriptors is highly applicable in the study of physical and chemical properties of molecular compounds, biomedical fields, bio informatics etc. The prime factor for the popularity of these indices is its intensive affability in solving problems in connection to the theory of those disciplines. Some of the applications of sum-connectivity indices in modelling few molecular properties are given in [3].

The sum-connectivity matrix $SC(G) = (s_{ij})$ of graph G is constructed as

$$s_{ij} = \begin{cases} \frac{1}{\sqrt{d_i + d_j}}, & \text{if } (v_i, v_j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

As the entries of the matrix $SC(G)$ is symmetric and real values, its eigenvalues are also real values. Let $\phi(SC(G), \eta) = \det(\eta I - SC(G))$ represent characteristic polynomial of $SC(G)$. If $\eta_1, \eta_2, \dots, \eta_s$ are distinct eigenvalues of $SC(G)$, then the sum-connectivity spectrum along with multiplicities is represented by

$$\text{Spec}(SC(G)) = \begin{pmatrix} \eta_1 & \eta_2 & \dots & \eta_s \\ m_1 & m_2 & \dots & m_s \end{pmatrix},$$

where m_i is the multiplicity of the eigenvalue η_i .

The summation of absolute values of sum-connectivity matrix is called sum-connectivity energy and denoted as

$$E[SC(G)] = \sum_{i=1}^q |\eta_i|.$$

DEFINITION 1.1. For every vertex u of G , when a new vertex v is added in such a way v is connected to each vertex which is connected to u , the splitting graph $S^1(G)$ is formed. Similarly by adding t additional vertices $u_1, u_2, u_3, \dots, u_t$ to each vertex u of G , such that for $1 \leq i \leq t$, the t -splitting graph $S^t(G)$ is formed by joining u_i to all the vertices that is connected to u in G . Take G_1, G_2 as two copies of G and attach every vertex u of G_1 to the corresponding neighbouring vertices v of G_2 . This will result in the construction of the shadow graph $D^2(G)$. Similarly by making t copies G_1, G_2, \dots, G_t of G , the shadow graph $D^t(G)$ is formed by connecting every node/vertex u of G_i to neighbours of adjacent node/vertex v of G_j for $1 \leq i, j \leq t$.

DEFINITION 1.2. The Kronecker product of P , a matrix of order $a \times b$ and Q , a matrix of order $c \times d$ is calculated by

$$P \otimes Q = \begin{pmatrix} d_{11}Q & \dots & d_{1b}Q \\ \vdots & \ddots & \vdots \\ d_{a1}Q & \dots & d_{ab}Q \end{pmatrix}.$$

LEMMA 1.1. Let X, Y be the eigenvectors of the matrix P, Q of order a and b with respect to the eigenvalues η_i and η_j . Then, XY is the eigenvector of the Kronecker product $P \otimes Q$ corresponding to the eigenvalue η_{ij} .

LEMMA 1.2. Let $A_{11}, A_{12}, A_{13}, A_{14}$ be matrices of order q . If A_{14} is invertible and $S = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix}$. Then, $\det S = \det A_{14} \cdot \det[A_{11} - A_{12}A_{14}^{-1}A_{13}]$.

B. Zhou and N. Trinajstic [11] have found lower and upper bounds for sum-connectivity energy. K.N. Prakasha and S.K. Reddy [2] computed sum-connectivity energy for some specific types of graphs. Puttaswamy et al. [4, 5] computed Randic and sum-connectivity energies for certain graph families. M.J. Sridevi et al. [8] obtained inverse sum-connectivity energy and its bounds for certain graphs.

In this study, we first determine sum-connectivity energy for t -splitting and t -shadow graphs. Also we investigate sum-connectivity energy for duplication, subdivision for p -regular graphs and establish a relation of sum-connectivity spectrum in terms of adjacency matrix spectrum of the graphs taken into consideration.

2. Sum-connectivity energy of t -splitting and t -shadow graphs

In this section, computations of sum-connectivity energy of t -splitting and t -shadow graphs of regular graphs have been done.

THEOREM 2.1. *Let G be p -regular graph of order q . The sum-connectivity energy of t -splitting graph of G is given by*

$$E[SC(S^t(G))] = \frac{\sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{(t + 1)(t + 2)} E[SC(G)].$$

P r o o f. Let v_1, v_2, \dots, v_q be vertices of graph G . Let $S^t(G)$ be graph obtained on adding vertices $v_1^1, v_2^1, \dots, v_q^1, v_1^2, v_2^2, \dots, v_q^2, \dots, v_1^t, v_2^t, \dots, v_q^t$ to $v_1, v_2, v_3, \dots, v_q$. For each vertex v_i in G , $v_i^j, 1 \leq j \leq t$, is adjacent to all neighbours v_i of G . Hence, sum-connectivity matrix for $S^t(G)$ denoted as $SC[S^t(G)]$ can be expressed by block matrix of order $(t + 1) \times (t + 1)$ as

$$\begin{pmatrix} \frac{1}{\sqrt{t+1}}SC(G) & \sqrt{\frac{2}{t+2}}SC(G) & \dots & \sqrt{\frac{2}{t+2}}SC(G) \\ \sqrt{\frac{2}{t+2}}SC(G) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{2}{t+2}}SC(G) & 0 & \dots & 0 \end{pmatrix} = A \otimes SC(G),$$

where the matrix A is given by

$$A = \begin{pmatrix} \frac{1}{\sqrt{t+1}} & \sqrt{\frac{2}{t+2}} & \sqrt{\frac{2}{t+2}} & \dots & \sqrt{\frac{2}{t+2}} \\ \sqrt{\frac{2}{t+2}} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{2}{t+2}} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Therefore characteristic polynomial of A is given by the polynomial $\phi(A - \mu I) = \mu^{t-1} \left(\mu^2 - \frac{1}{\sqrt{t+1}}\mu - t \left(\frac{2}{t+2} \right) \right)$. The corresponding spectrum is

$$\text{Spec}(A) = \begin{pmatrix} 0 & \frac{(t+2)\sqrt{t+1} \pm \sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{2(t+1)(t+2)} \\ t - 1 & 1 \end{pmatrix}.$$

Hence the spectrum of $SC[S^t(G)]$ is

$$\text{Spec}(SC[S^t(G)]) = \begin{pmatrix} 0\eta_i & \left(\frac{(t+2)\sqrt{t+1} \pm \sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{2(t+1)(t+2)} \right) \eta_i \\ t - 1 & 1 \end{pmatrix},$$

where η_i are eigenvalues of $SC(G)$. The computation of the sum-connectivity energy is then as follows:

$$\begin{aligned}
 E[SC(S^t(G))] &= \sum_{i=1}^q \left[\left| \left(\frac{(t+2)\sqrt{t+1} + \sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{2(t+1)(t+2)} \right) \eta_i \right| \right. \\
 &\quad \left. + \left| \left(\frac{(t+2)\sqrt{t+1} - \sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{2(t+1)(t+2)} \right) \eta_i \right| \right] \\
 &= \sum_{i=1}^q |\eta_i| \left[\frac{(t+2)\sqrt{t+1} + \sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{2(t+1)(t+2)} \right. \\
 &\quad \left. + \frac{\sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4} - (t+2)\sqrt{t+1}}{2(t+1)(t+2)} \right] \\
 &= \sum_{i=1}^q |\eta_i| \frac{2\sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{2(t+1)(t+2)} \\
 &= \frac{\sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{(t+1)(t+2)} E[SC(G)].
 \end{aligned}$$

□

The following corollaries are obtained by substituting the suitable regularity parameters in the above result.

COROLLARY 2.1. *The sum-connectivity energy of t -splitting graph of cycle graph C_q is given by*

$$E[SC(S^t(C_q))] = \frac{\sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{(t+1)(t+2)} \sum_{k=0}^{q-1} \left| \cos \frac{2\pi k}{q} \right|.$$

COROLLARY 2.2. *The sum-connectivity energy of the t -splitting graph of complete graph K_q is*

$$E[SC(S^t(K_q))] = \sqrt{2q-2} \frac{\sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{(t+1)(t+2)}.$$

COROLLARY 2.3. *The sum-connectivity energy of t -splitting graph of complete bipartite graph $K_{q,q}$ is given by*

$$E[SC(S^t(K_{q,q}))] = \sqrt{2q} \frac{\sqrt{8t^4 + 33t^3 + 45t^2 + 24t + 4}}{(t+1)(t+2)}.$$

THEOREM 2.2. *The sum-connectivity energy of t -shadow graph of $D^t(G)$ is*

$$E[SC(D^t(G))] = \sqrt{t} E[SC(G)].$$

P r o o f. The sum-connectivity matrix $SC[D^t(G)]$ of $D^t(G)$ can be expressed as $\frac{1}{\sqrt{t}}J \otimes SC(G)$ where J is the all-ones (i.e., all the elements are exactly one) matrix of order $t \times t$. The spectrum of $\frac{1}{\sqrt{t}}J = \begin{pmatrix} \frac{t}{\sqrt{t}} & 0 \\ 1 & t-1 \end{pmatrix}$. Therefore, sum-connectivity spectrum of $D^t(G)$ is $\begin{pmatrix} \sqrt{t}\eta_i & 0\eta_i \\ 1 & t-1 \end{pmatrix}$ where $\eta_i, i = 1, 2, \dots, q$ are eigenvalues of $SC(G)$. Hence, $E[SC[D^t(G)]] = \sum_{i=1}^q |\sqrt{t}\eta_i| = \sqrt{t}E[SC(G)]$. \square

The following corollaries are obtained by substituting the suitable regularity parameters in the above result.

COROLLARY 2.4. $E[SC(D^t(C_q))] = \sqrt{t} \sum_{k=0}^{q-1} \left| \cos \frac{2\pi k}{q} \right|$.

COROLLARY 2.5. $E[SC(D^t(K_q))] = \sqrt{t(2q-2)}$.

COROLLARY 2.6. $E[SC(D^t(K_{q,q}))] = \sqrt{2qt}$.

3. Sum-connectivity energy for graph duplication

DEFINITION 3.1. *Duplication of a vertex v by an edge* is obtained when we add a new pair of adjacent vertices v' and v'' (the adjacent edge is labeled as $e_v = (v'v'')$) such that v is made adjacent to both v' and v'' . When this duplication is done to all the vertices of G , the result is a new graph G^* with $3|V(G)|$ vertices and $|E(G)| + 3|V(G)|$ edges.

THEOREM 3.1. *Consider a p -regular graph G of order q . Let $B(G)$ be the adjacency matrix with $\eta_1, \eta_2, \dots, \eta_q$ being its eigenvalues. Let G^* be new graph formed on duplicating each vertex of G by an edge. Then*

$$E[SC(G^*)] = \frac{q}{2} + \sum_{i=1}^q \left| \sqrt{\frac{\eta_i^2}{2p+4} - \frac{\eta_i}{\sqrt{2p+4}} + \frac{p+36}{4(p+4)}} \right|.$$

P r o o f. Let v_1, v_2, \dots, v_q be vertices of G . Let we duplicate vertices v_1, v_2, \dots, v_q all together by edges e_1, e_2, \dots, e_q respectively such that $e_1 = v'_1v''_1, e_2 = v'_2v''_2, \dots, e_q = v'_qv''_q$ to obtain new graph G^* . Then $SC(G^*)$ can be

written in block matrix as

$$SC(G^*) = \begin{pmatrix} \frac{1}{\sqrt{2p+4}}B(G) & C \\ C^T & I_q \otimes \frac{1}{2}B(K_2) \end{pmatrix},$$

where

$$C = \begin{pmatrix} \frac{1}{\sqrt{p+4}} & \frac{1}{\sqrt{p+4}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{p+4}} & \frac{1}{\sqrt{p+4}} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{\sqrt{p+4}} & \frac{1}{\sqrt{p+4}} \end{pmatrix}.$$

The characteristic polynomial for $SC(G^*)$ is

$$\begin{aligned} \phi(SC(G^* : \mu)) &= \left| \begin{array}{cc} \mu I_q - \frac{1}{\sqrt{2p+4}}B(G) & C \\ C^T & I_q \otimes (\mu I_2 - \frac{1}{2}B(K_2)) \end{array} \right| \\ &= (\mu^2 - \frac{1}{4})^q \left| \mu I_q - \frac{1}{\sqrt{2p+4}}B(G) - C (\mu I_2 - \frac{1}{2}B(K_2))^{-1} \otimes I_q^{-1} C^T \right| \\ &= (\mu^2 - \frac{1}{4})^q \left| \mu I_q - \frac{1}{\sqrt{2p+4}}B(G) - C \left(\frac{1}{\mu^2 - \frac{1}{4}}(\mu I_2 + B(K_2) \otimes I_q) \right) C^T \right| \\ &= \left| (\mu^2 - \frac{1}{4}) (\mu I_q - B(G)) - C((\mu I_2 + B(K_2)) \otimes I_q) C^T \right|. \end{aligned}$$

Now,

$$\begin{aligned} &C((\mu I_2 + B(K_2)) \otimes I_q) C^T \\ &= \begin{pmatrix} \frac{1}{\sqrt{p+4}} & \frac{1}{\sqrt{p+4}} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{p+4}} & \frac{1}{\sqrt{p+4}} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{\sqrt{p+4}} & \frac{1}{\sqrt{p+4}} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
& \begin{pmatrix} \mu & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2} & \mu & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mu & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & \mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{p+4}} & 0 & \dots & 0 \\ \frac{1}{\sqrt{p+4}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{p+4}} & \dots & 0 \\ 0 & \frac{1}{\sqrt{p+4}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sqrt{p+4}} \\ 0 & 0 & \dots & \frac{1}{\sqrt{p+4}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{2\mu+1}{p+4} & 0 & 0 & \dots & 0 \\ 0 & \frac{2\mu+1}{p+4} & 0 & \dots & 0 \\ 0 & 0 & \frac{2\mu+1}{p+4} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{2\mu+1}{p+4} \end{pmatrix} = \frac{2\mu+1}{p+4} I_q.
\end{aligned}$$

Therefore, $\phi(SC(G^* : \mu)) = \left| (\mu^2 - \frac{1}{4})(\mu I_q - B(G)) - \frac{2}{p+4}(\mu + \frac{1}{2}) I_q \right|$. We know that the product of eigenvalues is equal to the determinant of a matrix. Hence we can derive the following:

$$\begin{aligned}
\phi(SC(G^* : \mu)) &= \prod_{i=1}^q (\mu - \frac{1}{2})(\mu + \frac{1}{2}) \left(\mu - \frac{\eta_i}{\sqrt{2p+4}} \right) - \frac{2}{p+4} (\mu + \frac{1}{2}) \\
&= (\mu + \frac{1}{2})^q \prod_{i=1}^q \left[(\mu - \frac{1}{2}) \left(\mu - \frac{\eta_i}{\sqrt{2p+4}} \right) - \frac{2}{p+4} \right] \\
&= (\mu + \frac{1}{2})^q \prod_{i=1}^q \left[\left(\mu^2 - \frac{\mu\eta_i}{\sqrt{2p+4}} \right) - \frac{\mu}{2} + \frac{\eta_i}{2\sqrt{2p+4}} - \frac{2}{p+4} \right] \\
&= (\mu + \frac{1}{2})^q \prod_{i=1}^q \left[\left(\mu^2 - \mu \left(\frac{\eta_i}{\sqrt{2p+4}} + \frac{1}{2} \right) + \left(\frac{\eta_i}{2\sqrt{2p+4}} - \frac{2}{p+4} \right) \right) \right].
\end{aligned}$$

Therefore the spectrum of $SC(G^*)$ is given as

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \left[\left(\frac{\eta_i}{\sqrt{2p+4}} + \frac{1}{2} \right) \pm \sqrt{\frac{\eta_i^2}{2p+4} - \frac{\eta_i}{\sqrt{2p+4}} + \frac{p+36}{4(p+4)}} \right] \\ q & 1 \end{pmatrix},$$

where $\eta_i, i = 1, 2, \dots, q$ are eigenvalues of G . Adding the absolute values from the spectrum we get

$$E[SC(G^*)] = \frac{q}{2} + \sum_{i=1}^q \left| \sqrt{\frac{\eta_i^2}{2p+4} - \frac{\eta_i}{\sqrt{2p+4}} + \frac{p+36}{4(p+4)}} \right|.$$

□

DEFINITION 3.2. The process *duplication of an edge* $e = u \sim v$ by a vertex w is such that $N(w) = \{u, v\}$. When this procedure is done for all the edges in a graph G , the resulting graph G^\ddagger has $|V(G)| + 2|E(G)$ vertices and $3|E(G)|$ edges.

THEOREM 3.2. Let G^\ddagger be the graph obtained on duplicating every edge of p -regular graph G by new vertex. Then the sum-connectivity energy of G^\ddagger given by

$$E[SC(G^\ddagger)] = \sum_{i=1}^q \left| \sqrt{\frac{\eta_i^2}{4p} + \frac{2(\eta_i+p)}{p+1}} \right|.$$

P r o o f. Let the vertices of G be labeled as v_1, v_2, \dots, v_q and edges as e_1, e_2, \dots, e_s where $s = |E(G)|$. Let $B(G)$ be its adjacency matrix with eigenvalues $\eta_1, \eta_2, \dots, \eta_q$ and let $F(G)$ be the incidence matrix of G . Let us duplicate edges e_1, e_2, \dots, e_s by vertices e'_1, e'_2, \dots, e'_s to obtain new graph G^\ddagger . Then sum-connectivity matrix $SC[G^\ddagger]$ is expressed as

$$SC(G^\ddagger) = \begin{pmatrix} \frac{1}{\sqrt{4p}}B(G) & \frac{1}{\sqrt{2p+2}}F(G) \\ \frac{1}{\sqrt{2p+2}}F(G)^T & 0_s \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \phi[SC(G^\ddagger : \mu)] &= \left| \begin{array}{cc} \mu I_q - \frac{1}{\sqrt{4p}}B(G) & \frac{1}{\sqrt{2p+2}}F(G) \\ \frac{1}{\sqrt{2p+2}}F(G)^T & \mu I_s \end{array} \right| \\ &= |\mu I_s| \left| \left(\mu I_q - \frac{1}{\sqrt{4p}}B(G) \right) - F(\mu I_s^{-1})F^T \right| \\ &= \mu^s \left| \left(\mu I_q - \frac{1}{\sqrt{4p}}B(G) \right) - \frac{1}{\mu}FF^T \right| \\ &= \mu^{s-q} \left| \left(\mu^2 I_q - \frac{\mu}{\sqrt{4p}}B(G) \right) - \frac{1}{2p+2}(B + pI_q) \right|. \end{aligned}$$

Since $\eta_i, i = 1, 2, \dots, q$ are eigenvalues of $B(G)$ and by applying the property of eigenvalues, we get

$$\phi[SC(G^\ddagger : \mu)] = \mu^{s-q} \prod_{i=1}^q \left(\mu^2 - \mu \frac{\eta_i}{\sqrt{4p}} - \frac{1}{2p+2}(\eta_i + p) \right).$$

Therefore the sum-connectivity spectrum of G^\ddagger is given as

$$\begin{pmatrix} 0 & \frac{1}{2} \left[\left(\frac{\eta_i}{\sqrt{4p}} \pm \sqrt{\frac{\eta_i^2}{4p} + \frac{2(\eta_i+p)}{(p+1)}} \right) \right] \\ s-q & 1 \end{pmatrix}.$$

Hence,

$$E[SC(G^\dagger)] = \sum_{i=1}^q \left| \sqrt{\frac{\eta_i^2}{4p} + \frac{2(\eta_i+p)}{p+1}} \right|.$$

□

4. Sum-connectivity energy for graph subdivision

THEOREM 4.1. *Let G be p -regular graph on q vertices v_1, v_2, \dots, v_q and s edges e_1, e_2, \dots, e_s . Let G^s be subdivision graph of G . Then $E[SC(G^s)] = 2 \sum_{i=1}^q \left| \sqrt{\frac{(\eta_i+p)}{p+2}} \right|$.*

P r o o f. Let $B(G)$ and $F(G)$ respectively represent the adjacency and incidence matrices of G . Then sum-connectivity matrix of G^s can be written as

$$SC(G^s) = \begin{bmatrix} 0_q & \frac{1}{\sqrt{p+2}}F(G) \\ \frac{1}{\sqrt{p+2}}F(G)^T & 0_s \end{bmatrix}.$$

The characteristic polynomial of sum-connectivity matrix of (G^s) is

$$\begin{aligned} \phi(SC(G^s) : \mu) &= \left| \begin{array}{cc} \mu I_q & \frac{1}{\sqrt{p+2}}F(G) \\ \frac{1}{\sqrt{p+2}}F(G)^T & \mu I_s \end{array} \right| \\ &= |\mu I_s| \left| \left(\mu I_q - \frac{1}{(p+2)}F(\mu I_s^{-1})F^T \right) \right| \\ &= \mu^s \left| \left(\mu I_q - \left(\frac{1}{p+2} \right) \frac{1}{\mu}F F^T \right) \right| \\ &= \mu^{s-q} \left| \left(\mu^2 I_q - \frac{1}{p+2}(B + pI_q) \right) \right|. \end{aligned}$$

Since $\eta_1, \eta_2, \dots, \eta_q$ are eigenvalues of $B(G)$, we have

$$\phi(SC(G^s) : \mu) = \mu^{s-q} \prod_{i=1}^q \left(\mu^2 - \frac{1}{p+2}(\eta_i + p) \right).$$

Hence

$$\text{Spec}(SC(G^s)) = \begin{pmatrix} 0 & \pm\sqrt{\frac{\eta_1+p}{p+2}} & \pm\sqrt{\frac{\eta_2+p}{p+2}} & \dots & \pm\sqrt{\frac{\eta_q+p}{p+2}} \\ s-q & 1 & 1 & \dots & 1 \end{pmatrix}.$$

Therefore,

$$E[SC(G^s)] = 2 \sum_{i=1}^q \left| \sqrt{\frac{(\eta_i+p)}{p+2}} \right|.$$

□

5. Conclusion

In this article, we have investigated sum-connectivity energy of t -splitting graph and t -shadow graph that can be obtained from sum-connectivity energy of G itself. Also we have obtained sum-connectivity spectrum and energy parameters for duplication, subdivision in relation of spectrum of adjacency matrix of graph taken under consideration.

Conflict of interest. The authors declare that they have no conflicts of interest.

Acknowledgments. For the corresponding author (A. Anuradha), this research was supported by SRM Institute of Science and Technology, Kattankulathur, Tamil Nadu, India under Selective Excellence Research Initiative-2021 grant.

References

- [1] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sect. Forschungsz. Graz* **103** (1978), 1–22.
- [2] K. N. Prakasha, P. S. Reddy, I. N. Cangul, Sum connectivity energy of graphs, *Advances in Mathematical Sciences and Applications* **28**, No 1 (2019), 85–98.
- [3] B. Lučić, S. Nikolić, N. Trinajstić, B. Zhou, S. Ivaniš Turk, Sum-connectivity index, in: I. Gutman, B. Furtula (Eds.), *Novel Molecular Structure Descriptors: Theory and Applications I*, Univ. Kragujevac (2010), 101–136.
- [4] Puttaswamy and C.A. Bhavya, On quotient of Randić and sum-connectivity energy of graphs, *International J. Math. Combin.* **2** (2020), 88–100.
- [5] Puttaswamy and R. Poojitha, On Laplacain of product of of Randić and sum-connectivity energy of graphs, *International J. Math. Combin.* **3** (2021), 35–48.
- [6] S. K. Vaidya and K. M. Popat, Spectra of graphs obtained by duplication of graph elements, *Mathematics Today* **35** 2019, 50–60.
- [7] S. K. Vaidya and K. M. Popat, Energy of m -splitting and m -shadow graphs, *For East Journal of Mathematics Sciences* **102** (2017), 1571–1578.
- [8] M. J. Sridevi and K. R. Rajanna, Inverse sum-connectivity Energy of a graph, *South East Asian J. of Math. & Math. Sci.* **14**, No 3 (2018), 31–48.
- [9] R. Todeschini, V. Consonni, *Handbook of Molecular Descriptors*, Wiley-VCH, Weinheim (2000), 84–90.

- [10] B. Zhou and N. Trinajstić, On a novel connectivity index, *J. Math. Chem.* **46** (2009), 1252–1270.
- [11] B. Zhou, N. Trinajstić, On sum-connectivity matrix and sum-connectivity energy of (molecular) graphs, *Acta Chim. Slov.* **57** (2010), 518–523.