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SOLITON SOLUTIONS FOR THE DISAGGREGATED MANHATTAN LATTICE

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Abstract

The article is devoted to the study of soliton solutions in a disaggregated model of traffic flow in the Manhattan lattice. Such a study is carried out within the framework of the dualism of the space of soliton solutions and solutions of an induced pointwise functional differential equation. A dual pair

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of "function-operator" arises, which allows the same function that sets the functional differential equation to match the most "simple" operator that generates a "simple" dynamic system. For the characteristics of soliton solutions from a given range, the entire set of soliton solutions is described, and their asymptotics in both space and time are indicated. All bounded soliton solutions are described for the considered disaggregated Manhattan lattice.

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1. **Introduction**

One of the important areas of research is related to the study of soliton solutions (traveling wave type solutions) and, in particular, both the existence of soliton solutions and the models in which they arise. A large number of works are devoted to this and presented both in the form of articles, monographs, and in the form of a review [16, 24, 22, 23]. Methods of studying soliton solutions are also diverse. One of the most frequently used methods of studying such systems is the direct construction of solutions using the explicit form of the right side, as well as the presence of various symmetries, or its possible periodicity, infinite differentiability, and analyticity. Further, using the methods of perturbation theory, soliton solutions are also established for the close right-hand sides. A review of works in this direction for infinite-dimensional (homogeneous) systems with Frenkel-Kontorova and Fermi-Pasta-Ulama potentials is given in [23]. However, this approach does not allow us to describe the space of all soliton solutions, as well as the asymptotics of their possible growth.

The presented work is devoted to the study of a disaggregated model of motion in the Manhattan lattice. In such a model, we assume that the number of nodes of a two-dimensional lattice is quite large. Therefore, we will consider the mentioned lattice to be complete in all directions and homogeneous. Another assumption, in contrast to the work [14, 1], concerns the detail of the observed flow parameters.

Steady-state flows in such a lattice are described by soliton solutions for an infinite-dimensional ordinary differential equation in the form of a finite-difference analog of a parabolic equation. A formalism is proposed below in which soliton solutions are in one-to-one correspondence with solutions of a parametric family of induced pointwise functional differential equations. The parameter of such a family of equations is the characteristic of the soliton solution. Within the framework of the proposed formalism, it is possible to describe the entire space of soliton solutions with characteristics in the selected ranges. The possible growth of such solutions is described, where the growth is also related to the magnitude of the characteristic of the soliton solution.

The right part (operator) of an infinite-dimensional differential equation and the right part of a family of induced functional differential equations form a dual pair of "function-operator". Such dualism turns out to be very useful. In some cases, it turns out to be possible to use the properties of finiteness of the phase space of solutions to a family of induced functional differential equations. In other cases, for soliton solutions, it is possible to use the spectral properties of the right-hand side of the infinite-dimensional differential equation. For a dual function-operator pair, a natural question arises about the existence of a dual pair with the same family of functions, but a simpler operator. For the model under consideration, a solution to the question is given. Among soliton solutions, an important class of bounded soliton solutions stands out as the most informative.

The theorem of the existence and uniqueness of a soliton solution is established, the ranges of characteristics for which the statements of the theorem are valid are indicated, and the asymptotics of the possible growth of such solutions are obtained. A complete family of bounded soliton solutions is constructed for the model under consideration.

The approach of studying soliton solutions with a given characteristic for an infinite-dimensional dynamical system based on the existence of a *one-to-one correspondence* of such solutions with solutions of a family of induced functional differential equations has been used in a number of works [3, 4, 5, 6, 7, 8, 20, 26, 25, 21].

Beyond the discussion, we leave such an important problem as the question of the solvability of functional differential equations. On this issue, you can refer to the works [18, 19, 15, 8, 2] and a number of other works.

1.1. **The disaggregated Manhattan lattice.**

The traffic pattern in the Manhattan lattice is such that from each point of the node of the two-dimensional lattice, movement is possible in three directions along the main highways (forward, left, right) and movement in the opposite direction through the stock. In the previous work [14], each node of the lattice was characterized by a scalar parameter in the form of the value of the total flow. In this paper, we will proceed to the description of a disaggregated model of flow motion in a lattice. In particular, one of the variants of the disaggregated model can be a model in which the values of flows in directions are observed at each node of the lattice. In such a model, each node of the lattice is characterized by a four-dimensional parameter space $x = (x_1, \dots, x_4)'$.

For the group $\Gamma = \mathbb{Z}^2$, through $\{\gamma_1, \gamma_2\}$ we denote the system of its generators, where $\gamma_1 = (1,0)$, $\gamma_2 = (0,1)$, and through e we denote the neutral element of the group $\mathbb{Z}^2$. Using $\mathcal{K}^n_{\mathbb{Z}^2}$, we denote the space of sequences with elements $\varkappa = \{x_\gamma\}_{\gamma \in \mathbb{Z}^2}$, $x_\gamma = (x_{\gamma 1}, \ldots, x_{\gamma n})' \in \mathbb{R}^n$, endowed with Tikhonov topology.

We consider the n-dimensional finite difference analog of the parabolic equation

$$\dot{x}_\gamma(t) = \sum_{l=1}^{2} A_{\gamma l}[x_{\gamma_l \gamma} - x_\gamma] + \sum_{l=1}^{2} B_{\gamma l}[x_{\gamma_l^{-1}\gamma} - x_\gamma] + \phi(x_\gamma), \qquad (1)$$
$$\forall \gamma \in \mathbb{Z}^2, \quad x_\gamma \in \mathbb{R}^n, \quad \text{for almost all } t \in \mathbb{R},$$

where $A_{\gamma_l}, B_{\gamma_l}$, $l = 1, 2$, are $n \times n$ matrices with non-negative elements, and the potential ϕ is given by a continuous function. On the right side of the equation (1) the first two linear terms define the diffusion component of the interaction in the system, and the nonlinear term defines the convective component of the interaction in the system.

The solution of such a system is called any vector function $\{x_\gamma(\cdot)\}_{\gamma \in \mathbb{Z}^2}$, coordinates $x_\gamma(\cdot)$, $\gamma \in \mathbb{Z}^2$ of which are absolutely continuous functions and satisfy the (1) system for almost all $t \in \mathbb{R}$.

For each $\gamma \in \mathbb{Z}^2$ (for each node of a two-dimensional lattice), the vector of observed quantities (parameters) is denoted by x_γ the flow of the transport stream in the node $\gamma \in \mathbb{Z}^2$. The potential $\phi(\cdot)$ has the form

$$\phi(x) = \begin{cases} 0, & x \notin \mathbb{R}^n_+, \\ \phi(x) > 0, & x \in \mathbb{R}^n_+, \quad \|x\|_{\mathbb{R}^n} < \Delta, \quad \Delta > 0, \\ \sigma[\|x\|_{\mathbb{R}^n} - \triangle], & \sigma < 0, \quad x \in \mathbb{R}^n_+, \quad \|x\|_{\mathbb{R}^n} \geq \Delta. \end{cases}$$

The value of the norm of the vector of flow characteristics equal to $\triangle$ is critical for the node and is determined by its technical capabilities. At values less than the critical value of $\triangle$, the stream flows from the stocks to the node γ. When the flow values are greater than the critical value of $\triangle$, the steram flows from the node γ to the stocks.

We rewrite the system (1) in a different way

$$\dot{x}_\gamma(t) = \sum_{l=1}^{2} A_{\gamma l} x_{\gamma_l \gamma} - \sum_{l=1}^{2} [A_{\gamma l} + B_{\gamma l}] x_\gamma + \sum_{l=1}^{2} B_{\gamma l} x_{\gamma_l^{-1}\gamma} \qquad (2)$$
$$+\phi(x_\gamma), \quad \forall \gamma \in \mathbb{Z}^2, \quad x_\gamma \in \mathbb{R}^n, \quad \text{for almost all } t \in \mathbb{R}.$$

For the system under consideration, we will study traveling wave type solutions (soliton solutions) as the most informative class of solutions for the steady-state flow structure

Definition 1.1. We say that the solution $\{x_\gamma(\cdot)\}_{\gamma \in \mathbb{Z}^2}$ of the system (1), defined for all $t \in \mathbb{R}$, has a traveling wave type (soliton solution) if there exists $\tau \geq 0$, independent of t and γ, that for all $\gamma \in \mathbb{Z}^2$ and $t \in \mathbb{R}$, the following equality is fulfilled

$$x_\gamma(t+\tau) = x_{\gamma_l\gamma}(t), \quad l = 1,2. \tag{3}$$

The constant τ will be called the *characteristic* of a traveling wave.

The task is to determine in which ranges of the characteristic τ the soliton solution exists, what is the asymptotic behavior of such solutions and their dependence on the system parameters. Among soliton solutions, an important class of bounded soliton solutions stands out as one of the most informative classes of solutions. Thus, we should study the solutions of the system (2)-(3), which is a boundary value problem with non-local boundary conditions (3).

2. **Dual pairs of “function-operator”**

We will present the boundary value problem under consideration in an operator form and construct a pointwise functional differential equation induced by such a boundary value problem. The operator defining the right-hand side of an infinite-dimensional differential equation and the function defining the right-hand side of an induced pointwise functional differential equation form a dual pair of “function-operator”. For such a dual pair, statements will be formulated about the correspondence between the solutions of the operator boundary value problem and the induced functional differential equation of the pointwise type.

2.1. **The operator form of the boundary value problem and the induced functional differential equation.** We are going to rewrite the boundary value problem (2)-(3) in operator form. In the sequence space $\mathcal{K}^n_{\mathbb{Z}^2}$ endowed with Tikhonov topology, we define the linear operator $\mathbb{A}$, the nonlinear operator $\mathbb{F}$ and the group of shift operators $\mathbb{T} = \{T_{\hat\gamma} : \hat\gamma \in \mathbb{Z}^2\}$, acting continuously from the space $\mathcal{K}^n_{\mathbb{Z}^2}$ into itself according to the following rule: for any $\gamma \in \mathbb{Z}^2$, $\varkappa \in \mathcal{K}^n_{\mathbb{Z}^2}$, $\varkappa = \{x_\gamma\}_{\gamma\in\mathbb{Z}^2}$

$$(\mathbb{A}\varkappa)_\gamma = \sum_{l=1}^{2} A_{\gamma_l} x_{\gamma_l\gamma} - \sum_{l=1}^{2}(A_{\gamma_l} + B_{\gamma_l})x_\gamma + \sum_{l=1}^{2} B_{\gamma_l} x_{\gamma_l^{-1}\gamma},$$
$$(\mathbb{F}(\varkappa))_\gamma = \phi(x_\gamma), \quad T_{\hat\gamma}\{x_\gamma\}_{\gamma\in\mathbb{Z}^2} = \{x_{\gamma\hat\gamma}\}_{\gamma\in\mathbb{Z}^2}.$$

We define the cyclic group $Q =< \check q >$, $\check q(t) = t+\tau$, as well as the epimorphism $\eta : \mathbb{Z}^2 \to Q$, where $\eta(\gamma_l) = \check q$, $l = 1,2$, for the generators γ_l, $l = 1,2$, of the group $\mathbb{Z}^2$. We introduce the notation $\mathbb{G}_{\mathbb{Z}^2} = \mathbb{A} + \mathbb{F}$. Then the boundary value problem (2)-(3) will have the following equivalent operator representation

$$\dot\varkappa(t) = \mathbb{G}_{\mathbb{Z}^2}(\varkappa), \qquad \text{for almost all } t \in \mathbb{R}, \tag{4}$$
$$\varkappa(\eta(\hat\gamma)(t)) = T_{\hat\gamma}\varkappa(t), \quad \forall\hat\gamma \in \{\gamma_1,\gamma_2\}, \quad \forall t \in \mathbb{R}. \tag{5}$$

On the left side of the equation (4) there is a Gateaux derivative. The *solution of an infinite-dimensional differential equation* (4) is every vector function $\varkappa(t) = \{x_\gamma\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$, each coordinate of which is given by an absolutely continuous function, and the vector function itself satisfies to this equation for almost all $t \in \mathbb{R}$. Since $\{\gamma_1, \gamma_2\}$ are the generators of the group $\mathbb{Z}^2$, the system (4)-(5) is equivalent to the following system

$$\dot{\varkappa}(t) = \mathbb{G}_{\mathbb{Z}^2}(\varkappa), \qquad \text{for almost all } t \in \mathbb{R}, \tag{6}$$

$$\varkappa(\eta(\hat{\gamma})(t)) = T_{\hat{\gamma}}\varkappa(t), \quad \forall\hat{\gamma} \in \mathbb{Z}^2, \quad \forall t \in \mathbb{R}. \tag{7}$$

The condition (7), which provides a traveling wave condition (soliton solution), means that *time shift is equal to space shift.* It is not difficult to notice that the operator $\mathbb{G}_{\mathbb{Z}^2}$ is permuted with the shift operator $T_{\hat{\gamma}}$, $\hat{\gamma} \in \mathbb{Z}^2$, i.e. for any $\hat{\gamma} \in \mathbb{Z}^2$ there is an equality $T_{\hat{\gamma}}\mathbb{G}_{\mathbb{Z}^2} = \mathbb{G}_{\mathbb{Z}^2}T_{\hat{\gamma}}$. This commutativity property is a consequence of the spatial homogeneity of the system and the reason for the presence of all canonical properties of soliton solutions.

We consider a *induced* pointwise functional differential equation

$$\begin{aligned}\dot{x}(t) = \sum_{l=1}^{2} A_{\gamma_l}x(t+\tau) - \sum_{l=1}^{2}(A_{\gamma_l} + B_{\gamma_l})x(t) + \sum_{l=1}^{2} B_{\gamma_l}x(t-\tau)\\ +\phi(x(t)), \qquad \text{for almost all } t \in \mathbb{R}.\end{aligned} \tag{8}$$

Such an equation is a representation of the coordinate of the operator equation (6), corresponding to the unit element e of the group $\mathbb{Z}^2$, using the traveling wave condition (7).

The *solution of a pointwise functional differential equation* (8) is every absolutely continuous function $x(t)$, $t \in \mathbb{R}$, satisfying this equation almost everywhere.

We transform a functional differential equation of pointwise type (8). To do this, we introduce the notation $A_{\gamma_1} + A_{\gamma_2} = A$, $B_{\gamma_1} + B_{\gamma_2} = B$. Then the equation will be rewritten as

$$\begin{aligned}&\dot{x}(t) = Ax(t+\tau) - (A+B)x(t) + Bx(t-\tau) + \phi(x(t)),\\ &\text{for almost all } t \in \mathbb{R}.\end{aligned} \tag{9}$$

The right side of such a pointwise functional differential equation is given by the mapping $g : \mathbb{R}^{3n} \to \mathbb{R}$ and has the form

$$g(z_1, z_0, z_{-1}) = Az_1 - (A+B)z_0 + Bz_{-1} + \phi(z_0), \quad z_1, z_0, z_{-1} \in \mathbb{R}^n.$$

The operator $\mathbb{G}_{\mathbb{Z}^2}$, as the right part of an infinite-dimensional ordinary differential equation (6), and the function g, as the right part of an induced functional differential equation of pointwise type (8), form a *dual pair* $(g|\mathbb{G}_{\mathbb{Z}^2})$. The following important statement takes place.

THEOREM 2.1. *Let* $(g|\mathbb{G}_{\mathbb{Z}^2})$ *be a dual pair. Each solution* $x(t)$, $t \in \mathbb{R}$, *of the pointwise functional differential equation* (8) *corresponds to the solution* $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$, *of the boundary value problem* (6)-(7) *(traveling wave type solution) and vice versa. Such solutions are related by ratios* $x_\gamma(t) = x_e(\eta(\gamma)(t))$, $x_e(t) = x(t)$, $\forall\gamma \in \mathbb{Z}^2$, $\forall t \in \mathbb{R}$.

Proof. Let $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$, is a solution of the boundary value problem (6)-(7). For such a solution in the infinite-dimensional differential equation (6), the coordinate $x_e(t)$, $t \in \mathbb{R}$, corresponding to the unit element $e = (0, 0)$ of the group $\mathbb{Z}^2$ satisfies the equation

$$\begin{aligned}\dot{x}_e(t) = {} & \sum_{l=1}^{2} A_{\gamma_l} x_{\gamma_l}(t) - \sum_{l=1}^{2}(A_{\gamma_l} + B_{\gamma_l})x_e(t) + \sum_{l=1}^{2} B_{\gamma_l} x_{\gamma_l^{-1}}(t) \\ & +\phi(x_e(t)), \quad \text{for almost all } t \in \mathbb{R}. \end{aligned} \tag{10}$$

From the boundary condition (7) it follows that for the solution under consideration, the system of equalities $x_\gamma(t) = x(\eta(\gamma)(t)$, $\gamma \in \mathbb{Z}^2$, is fulfilled and, in particular, the equalities $x_{\gamma_l}(t) = x_e(\eta(\gamma_l)(t)) = x_e(t + \tau)$, $x_{\gamma_l^{-1}}(t) = x_e(\eta(\gamma_l^{-1})(t)) = x_e(t - \tau)$, $l = 1, 2$, are satisfied. If we substitute these values $x_{\gamma_l}(t), x_{\gamma_l^{-1}}(t)$, $l = 1, 2$, into the equation (10) and reassign the variable x_e to the variable x, then such an equation will coincide with the functional differential equation (8).

In the opposite direction. Let $x(\cdot)$ be the solution of a functional differential equation of pointwise type (8). We define an infinite-dimensional vector function $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$, where $x_\gamma(t) = x(\eta(\gamma)(t))$, $\gamma \in \mathbb{Z}^2$. Then, by virtue of the functional differential equation (8), the system of equalities will be fulfilled

$$\begin{aligned}\dot{x}_\gamma(t) = {} & \sum_{l=1}^{2} A_{\gamma_l} x_{\gamma_l\gamma}(t) - \sum_{l=1}^{2}(A_{\gamma_l} + B_{\gamma_l})x_\gamma(t) + \sum_{l=1}^{2} B_{\gamma_l} x_{\gamma_l^{-1}\gamma}(t) \\ & +\phi(x_\gamma(t)), \quad \forall\gamma \in \mathbb{Z}^2, \quad \text{for almost all } t \in \mathbb{R}, \end{aligned}$$

and the vector function $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$, will be the solution of the infinite dimensional differential equation (6). On the other hand, thus defined the vector function $\varkappa(\cdot)$ satisfies the boundary condition (7). The theorem is proved. □

2.2. **Theorems of existence and uniqueness of the solution for the dual pair "function-operator".** We transform a functional differential equation of pointwise type (9) from the considered dual pair. We replace the time so that the deviations of the argument become integer, and the characteristic

τ is considered as a parameter

$$\begin{aligned}&\dot{\tilde{x}}(t)=\tau[A\tilde{x}(t+1)-(A+B)\tilde{x}(t)+B\tilde{x}(t-1)+\phi(\tilde{x}(t))],\\&\text{for almost all } t\in\mathbb{R}.\end{aligned}$$

Such an equation was investigated in the monograph [8] with minimal restrictions on the potential $\pi(\cdot)$ in the form of the Lipschitz condition (quasi-linear potentials).

Let the potential ϕ satisfy the Lipschitz condition with the constant L_ϕ. Consider a transcendental equation with respect to two variables $\tau\in(0,+\infty)$ and $\mu\in(0,1)$

$$2C_\phi\tau\left(\mu^{-1}+1\right)=\ln\mu^{-1}, \tag{11}$$

where

$$C_\phi=\max\left\{\|A\|+\|B\|;L_\phi\right\}.$$

The set of solutions to the equation (11) is described by the functions $\mu_1(\tau)$, $\mu_2(\tau)$, given in Figure 1.

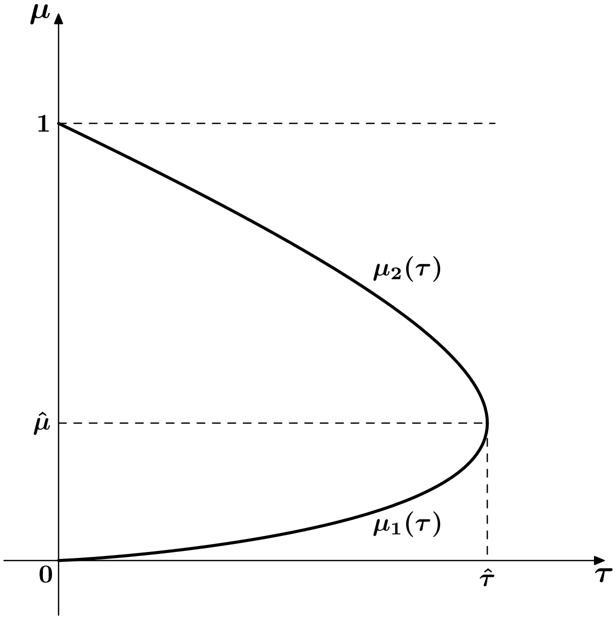

FIGURE 1. Graphs of functions $\mu_1(\tau),\mu_2(\tau)$.

To study the existence and uniqueness of solutions of pointwise functional differential equations, their localization in the spaces of functions dominated by functions of a given exponential growth with an exponential power as a parameter is proposed

$$\mathcal{L}^n_\mu C^{(0)}(\mathbb{R}) = \left\{ z(\cdot) : z(\cdot) \in C^{(0)}\left(\mathbb{R}, \mathbb{R}^n\right), \sup_{t\in\mathbb{R}} \|z(t)\mu^{|t|}\|_{R^n} < +\infty \right\},$$
$$\|z(\cdot)\|^{(0)}_\mu = \sup_{t\in\mathbb{R}} \|z(t)\mu^{|t|}\|_{R^n}, \quad \mu \in (0,1).$$

We formulate the theorem of the existence and uniqueness of a solution for the induced functional differential equation (8).

Theorem 2.2 ([8]). *Let the potential ϕ satisfy the Lipschitz condition with the constant L_ϕ. Then for any initial data $\bar{a} \in \mathbb{R}$, $\bar{t} \in \mathbb{R}$, and characteristics $\tau > 0$ satisfying the condition*

$$0 < \tau < \hat{\tau},$$

in the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, $\forall \mu$, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$, for the functional differential equation (8), *there exists and moreover a unique solution $x(t)$, $t \in \mathbb{R}$, such that it satisfies the initial condition $x(\bar{t}) = \bar{a}$. Such a solution, as an element of the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, continuously depends on the initial given $\bar{a} \in \mathbb{R}$, the characteristic τ and the potential $\phi(\cdot)$.*

Theorem 2.2 not only guarantees the existence of a solution, but also sets a limit on its possible growth in time t. Moreover, for each $0 < \tau < \hat{\tau}$, the spaces $\mathcal{L}^1_{(\sqrt[\tau]{\mu_2(\tau)} - \varepsilon)} C^{(0)}(\mathbb{R})$ for small $\varepsilon > 0$ is much narrower than the spaces $\mathcal{L}^1_{(\sqrt[\tau]{\mu_1(\tau)} + \varepsilon)} C^{(0)}(\mathbb{R})$. The theorem guarantees the existence of a solution in narrower spaces and uniqueness in wider spaces. This property of solutions, along with Theorem 2.1 on the one-to-one correspondence of soliton solutions to solutions of a family of induced functional differential equations, underlies the proof of the theorems of the existence and uniqueness of soliton solutions, which will be given below.

Theorem 2.2 allows reformulation in terms of traveling wave type solutions (soliton solutions) for the initial finite difference analogue of the parabolic equation (2). To do this, in the space $\mathcal{K}^n_{\mathbb{Z}^2}$, we define a family of Hilbert subspaces $\mathcal{K}^n_{\mathbb{Z}^2 2\mu}$, $\mu \in (0,1)$

$$\mathcal{K}^n_{\mathbb{Z}^2 2\mu} = \left\{ \varkappa : \varkappa \in \mathcal{K}^n_{\mathbb{Z}^2}; \quad \textstyle\sum_{\gamma\in\mathbb{Z}^2} \|x_\gamma\|^2_{\mathbb{R}^n} \mu^{2|\gamma|} < +\infty \right\},$$
$$|\gamma| = |(i,j)| = |i| + |j|$$

with the norm

$$\|\varkappa\|_{\mathbb{Z}^2 2\mu} = \Big[\sum_{\gamma\in\mathbb{Z}^2} \|x_\gamma\|^2_{\mathbb{R}^n} \mu^{2|\gamma|} \Big]^{\frac{1}{2}}.$$

THEOREM 2.3. *Let the potential ϕ satisfy the Lipschitz condition with the constant L_ϕ. Then for any initial data $\bar{\gamma} \in \mathbb{Z}^2$, $\bar{a} \in \mathbb{R}$, $\bar{\in}\mathbb{R}$, and characteristics $\tau > 0$ satisfying the condition*

$$0 < \tau < \hat{\tau},$$

for the initial infinite-dimensional system of differential equations (2) *there is a unique soliton solution $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$ (solution of the boundary value problem* (6)-(7)*), with the characteristic τ such that it satisfies the initial condition $x_{\bar{\gamma}}(\bar{t}) = \bar{a}$, and for any parameter μ, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$, values of the vector function*

$$\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$$

for any $t \in \mathbb{R}$ belong to the space $\mathcal{K}^n_{\mathbb{Z}^2 2\mu}$, and the function

$$\rho(t) = \|\varkappa(t)\|_{\mathbb{Z}^2 2\mu}$$

belongs to the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$. Such a solution continuously depends on the initial given $a \in \mathbb{R}$, the characteristic τ and the potential of $\phi(\cdot)$.

Proof. By virtue of Theorem 2.2, for any initial data $\bar{a} \in \mathbb{R}$, $\bar{t} \in \mathbb{R}$, in the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, $\forall\mu$, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$, for the functional differential equation (8) there exists and moreover a unique solution $x(t)$, $t \in \mathbb{R}$, such that it satisfies the initial condition $x(\bar{t}) = \bar{a}$. Then by Theorem 2.1 vector function $\varkappa(t) = \{x_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $x_\gamma(t) = x(\eta(\gamma)(t))$, is a solution to the boundary value problem (6)-(7) (a soliton solution). We set $\bar{a}, \bar{\gamma}, \bar{t}$. If we select $\tilde{t} = \eta(\bar{\gamma})(\bar{t})$, the initial condition $x_{\bar{\gamma}}(\bar{t}) = \bar{a}$ will be met. It remains to show that $\varkappa(t) \in \mathcal{K}^n_{\mathbb{Z}^2 2\mu}$, $\forall t \in \mathbb{R}$, and $\rho(\cdot) \in \mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, $\forall\mu$, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$. For an arbitrary $t \in \mathbb{R}$ and $\forall\mu$, $\forall\bar{\mu}$, $\mu^\tau, \bar{\mu}^\tau \in (\mu_1(\tau), \mu_2(\tau))$, $\mu < \bar{\mu}$, taking into account the estimate $|\eta(\gamma)(t)| \le |\gamma| + |t|$ we get

$$\begin{aligned}
\|\varkappa(t)\|_{\mathbb{Z}^2 2\mu} &= \Big[\sum_{\gamma\in\mathbb{Z}^2} \|x_\gamma(t)\|^2_{\mathbb{R}^n} \mu^{2|\gamma|}\Big]^{\frac{1}{2}} = \Big[\sum_{\gamma\in\mathbb{Z}^2} \|x(\eta(\gamma)(t))\|^2_{\mathbb{R}^n} \mu^{2|\gamma|}\Big]^{\frac{1}{2}} \\
&= \Big[\sum_{\gamma\in\mathbb{Z}^2} \|x(\eta(\gamma)(t))\|^2_{\mathbb{R}^n} \bar{\mu}^{2|\gamma|} \left(\frac{\mu}{\bar{\mu}}\right)^{2|\gamma|}\Big]^{\frac{1}{2}} \\
&\le \Big[\sum_{\gamma\in\mathbb{Z}^2} \|x(\eta(\gamma)(t))\|^2_{\mathbb{R}^n} \bar{\mu}^{2(|\gamma|+|t|)} \bar{\mu}^{-2|t|} \left(\frac{\mu}{\bar{\mu}}\right)^{2|\gamma|}\Big]^{\frac{1}{2}} \\
&\le \Big[\sum_{\gamma\in\mathbb{Z}^2} \|x(\eta(\gamma)(t))\|^2_{\mathbb{R}^n} \bar{\mu}^{2|\eta(\gamma)(t)|} \bar{\mu}^{-2|t|} \left(\frac{\mu}{\bar{\mu}}\right)^{2|\gamma|}\Big]^{\frac{1}{2}} \\
&\le \|x(\cdot)\|^{(0)}_{\bar{\mu}} \bar{\mu}^{-|t|} \Big[\sum_{\gamma\in\mathbb{Z}^2} \left(\frac{\mu}{\bar{\mu}}\right)^{2|\gamma|}\Big]^{\frac{1}{2}}.
\end{aligned}$$

Since the group $\mathbb{Z}^2$ is commutative (polynomial growth) and $\frac{\mu}{\bar{\mu}} < 1$, the sum is finite and equal to some A. We finally get the estimate

$$\|\varkappa(t)\|_{\mathbb{Z}^2 2\mu} \leq A\|x(\cdot)\|_{\bar{\mu}}^{(0)}\bar{\mu}^{-|t|}, \quad \forall t \in \mathbb{R}.$$

It follows from the obtained estimate that for $\forall\mu$, $\forall\bar{\mu}$, $\mu^\tau, \bar{\mu}^\tau \in (\mu_1(\tau), \mu_2(\tau))$, $\mu < \bar{\mu}$, for the vector function $\varkappa(\cdot)$, the conditions $\varkappa(t) \in \mathcal{K}^n_{\mathbb{Z}^2 2\mu}$ for any $t \in \mathbb{R}$ and $\rho(\cdot) \in \mathcal{L}^1_{\bar{\mu}}C^{(0)}(\mathbb{R})$ take place. It follows that $\forall\mu$, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$ the conditions $\varkappa(t) \in \mathcal{K}^n_{\mathbb{Z}^2 2\mu}$ are valid for any $t \in \mathbb{R}$, as well as $\rho(\cdot) \in \mathcal{L}^1_{\mu}C^{(0)}(\mathbb{R})$. The existence of a soliton solution has been proved. It remains to prove its uniqueness.

Let the vector function $\varkappa(\cdot)$ be a soliton solution (solution of the boundary value problem (6)-(7)) and satisfies the conditions: $x_{\bar{\gamma}}(\bar{t}) = \bar{a}$, $\varkappa(t) \in \mathcal{K}^n_{\mathbb{Z}^2 2\mu}$, $\forall t \in \mathbb{R}$, and $\rho(\cdot) \in \mathcal{L}^1_{\mu}C^{(0)}(\mathbb{R})$, $\forall\mu$, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$. Then the function $x(t) = x_e(t)$, $\forall t \in \mathbb{R}$, will be the solution of the induced functional differential equation (8) and satisfies the initial condition $x(\tilde{t}) = \bar{a}$, $\tilde{t} = \eta(\bar{\gamma})(\bar{t})$. Moreover, from the condition $\rho(\cdot) \in \mathcal{L}^1_{\mu}C^{(0)}(\mathbb{R})$, it follows that $x(\cdot) \in \mathcal{L}^1_{\mu}C^{(0)}(\mathbb{R})$. Then by Theorem 2.2 such a solution $x(\cdot)$ of the equations (8) is the only one, from which the uniqueness of the soliton solution follows. The theorem has been proved. $\square$

2.3. **Canonical dual pair "function-operator" as a dual pair with the simplest operator.** In the previous section, we studied the existence and uniqueness of a soliton solution, as well as for the dual pair $(g|\mathbb{G}_{\mathbb{Z}^2})$ the relationship of soliton solutions of an infinite-dimensional ordinary differential equation with the right side $G_{\mathbb{Z}^2}$ and solutions of a family of induced functional differential equations with the right side g. The question is to identify the dual pair with the simplest lattice and its corresponding operator among the dual pairs with the same function g.

For the group $\Gamma = \mathbb{Z}$, we denote the generator by γ_1, where $\gamma_1 = 1$. Consider the finite difference analogue of the parabolic equation

$$\begin{aligned} &\dot{\bar{x}}_\gamma(t) = A[\bar{x}_{\gamma_1\gamma} - \bar{x}_\gamma] + B[\bar{x}_{\gamma_1^{-1}\gamma} - \bar{x}_\gamma] + \phi(\bar{x}_\gamma), \\ &\bar{x}_\gamma \in \mathbb{R}, \quad \forall\gamma \in \mathbb{Z}, \qquad \text{for almost all } t \in \mathbb{R}, \end{aligned} \tag{12}$$

We rewrite the system (12) in a different way

$$\begin{aligned} &\dot{\bar{x}}_\gamma(t) = A\bar{x}_{\gamma_1\gamma} - (A+B)\bar{x}_\gamma + B\bar{x}_{\gamma_1^{-1}\gamma} + \phi(\bar{x}_\gamma), \\ &\bar{x}_\gamma \in \mathbb{R}, \quad \forall\gamma \in \mathbb{Z}, \qquad \text{for almost all } t \in \mathbb{R}, \end{aligned} \tag{13}$$

Definition 2.1. We say that the solution $\{\bar{x}_\gamma(\cdot)\}_{\gamma\in\mathbb{Z}}$ of the systems (12) defined for all $t \in \mathbb{R}$ has a traveling wave type (soliton solution) if there exists $\tau \geq 0$ independent of t and γ, that for all $\gamma \in \mathbb{Z}$ and $t \in \mathbb{R}$ the following equality is fulfilled

$$\bar{x}_\gamma(t+\tau) = \bar{x}_{\gamma_1\gamma}(t). \tag{14}$$

The constant τ will be called the *characteristic* of a traveling wave.

We rewrite the boundary value problem (13)-(14) in operator form. Using $\mathcal{K}^n_{\mathbb{Z}}$, we denote the space of sequences with elements $\bar{\varkappa} = \{\bar{x}_\gamma\}_{\gamma\in\mathbb{Z}}$, $\bar{x}_\gamma \in \mathbb{R}^n$, endowed with a Tikhonov topology. We define the linear operator $\bar{\mathbb{A}}$, the nonlinear operator $\bar{\mathbb{F}}$ and the group of shift operators $\mathbb{T} = \{T_{\hat{\gamma}} : \hat{\gamma} \in \mathbb{Z}$, acting continuously from the space $\mathcal{K}^n_{\mathbb{Z}}$ into itself according to the following rule: for any $\gamma, \hat{\gamma} \in \mathbb{Z}$, $\bar{\varkappa} \in \mathcal{K}^n_{\mathbb{Z}}$, $\bar{\varkappa} = \{\bar{x}_\gamma\}_{\gamma\in\mathbb{Z}}$

$$(\bar{\mathbb{A}}\bar{\varkappa})_\gamma = A\bar{x}_{\gamma_l\gamma} - (A+B)\bar{x}_\gamma + B\bar{x}_{{\gamma_l}^{-1}\gamma}, \quad (\bar{\mathbb{F}}(\bar{\varkappa}))_\gamma = \phi(\bar{x}_\gamma),$$
$$T_{\hat{\gamma}}\{\bar{x}_\gamma\}_{\gamma\in\mathbb{Z}} = \{\bar{x}_{\gamma\hat{\gamma}}\}_{\gamma\in\mathbb{Z}}.$$

We define the cyclic group $Q =< \check{q} >$, $\check{q}(t) = t+\tau$, as well as the epimorphism $\bar{\eta} : \mathbb{Z} \to Q$, where $\bar{\eta}(\gamma_1) = \check{q}$ for the generator γ_1 of the group $\mathbb{Z}$. We introduce the notation $\mathbb{G}_{\mathbb{Z}} = \bar{\mathbb{A}} + \bar{\mathbb{F}}$.

Boundary value problem (13)-(14) has the following equivalent operator representation

$$\dot{\bar{\varkappa}}(t) = \mathbb{G}_{\mathbb{Z}}(\bar{\varkappa}), \quad \text{for almost all } t \in \mathbb{R}. \tag{15}$$
$$\bar{\varkappa}(\eta(\gamma_1)(t)) = T_{\gamma_1}\bar{\varkappa}(t), \quad \forall t \in \mathbb{R}. \tag{16}$$

On the left side of the equation (15) there is also a Gateaux derivative. The *solution of an infinite-dimensional differential equation* (15) is every vector function $\bar{\varkappa}(t) = \{\bar{x}_\gamma(t)\}_{\gamma\in\mathbb{Z}^2}$, $t \in \mathbb{R}$, each coordinate of which is given by an absolutely continuous function, and the vector function itself satisfies this equation for almost all $t \in \mathbb{R}$. Since γ_1 is the generator of the group $\mathbb{Z}$, the system (15)-(16) is equivalent to the system

$$\dot{\bar{\varkappa}}(t) = \mathbb{G}_{\mathbb{Z}}(\bar{\varkappa}), \quad \text{for almost all } t \in \mathbb{R}, \tag{17}$$
$$\bar{\varkappa}(\eta(\hat{\gamma})(t)) = T_{\hat{\gamma}}\bar{\varkappa}(t), \quad \forall \hat{\gamma} \in \mathbb{Z}, \quad \forall t \in \mathbb{R}. \tag{18}$$

The condition (18), which provides a traveling wave condition (soliton solution), also means that *time shift is equal to space shift.* It is not difficult to notice that the operator $\mathbb{G}_{\mathbb{Z}}$ is permuted with the shift operator $T_{\hat{\gamma}}$, $\hat{\gamma} \in \mathbb{Z}$, i.e. for any $\hat{\gamma} \in \mathbb{Z}$ there is the equality $T_{\hat{\gamma}}\mathbb{G}_{\mathbb{Z}} = \mathbb{G}_{\mathbb{Z}}T_{\hat{\gamma}}$.

If for the system (17)-(18) we construct an induced pointwise functional differential equation using the procedure given after the equation (8), then we get the equation (9). The right-hand side of such an equation is also given by the function g. In this case, we will get a dual pair of "function-operator" of the form $(g|\mathbb{G}_{\mathbb{Z}})$. Such a dual pair is called *canonical*, is a dual pair with the same function g and the simplest operator $\mathbb{G}_{\mathbb{Z}}$.

For such a dual pair, the existence and uniqueness theorem for the induced functional differential equation coincides with Theorem 2.2. The theorem of existence and uniqueness of a soliton solution for the (13) system is similar to Theorem 2.3. To formulate it in the space $\mathcal{K}^n_{\mathbb{Z}}$, we define a family of Hilbert subspaces $\mathcal{K}^n_{\mathbb{Z}2\mu}$, $\mu \in (0,1)$

$$\mathcal{K}^n_{\mathbb{Z}2\mu} = \Big\{ \bar{\varkappa} : \bar{\varkappa} \in \mathcal{K}^n_{\mathbb{Z}}; \quad \textstyle\sum_{\gamma\in\mathbb{Z}} \|\bar{x}_\gamma\|^2_{\mathbb{R}^n} \mu^{2|\gamma|} < +\infty \Big\},$$
$$|\gamma| = |(i,j)| = |i| + |j|$$

with the norm

$$\|\varkappa\|_{\mathbb{Z}2\mu} = \big[\sum_{\gamma\in\mathbb{Z}} \|\bar{x}_\gamma\|^2_{\mathbb{R}^n} \mu^{2|\gamma|} \big]^{\frac{1}{2}}.$$

THEOREM 2.4. *Let the potential ϕ satisfy the Lipschitz condition with the constant L_ϕ. Then for any initial data $\bar{\gamma} \in \mathbb{Z}$, $\bar{a} \in \mathbb{R}$, $\bar{t} \in \mathbb{R}$, and characteristics $\tau > 0$ satisfying the condition*

$$0 < \tau < \hat{\tau},$$

for the initial infinite-dimensional system of differential equations (13), *there is a unique solution $\bar{\varkappa}(t) = \{\bar{x}_\gamma(t)\}_{\gamma\in\mathbb{Z}}$, $t \in \mathbb{R}$, of the traveling wave type (soliton solution) with characteristic τ such that it satisfies the initial condition $\bar{x}_{\bar{\gamma}}(\bar{t}) = \bar{a}$, and for any parameter μ, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$, values of the vector function*

$$\bar{\varkappa}(t) = \{\bar{x}_\gamma(t)\}_{\gamma\in\mathbb{Z}}$$

for any $t \in \mathbb{R}$ belong to the space $\mathcal{K}^n_{\mathbb{Z}2\mu}$, and the function

$$\rho(t) = \|\bar{\varkappa}(t)\|_{\mathbb{Z}2\mu}$$

belongs to the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$. Such a solution continuously depends on the initial value $\bar{a} \in \mathbb{R}$, the characteristic τ and the potential $\phi(\cdot)$.

Proof. The proof repeats verbatim the proof of Theorem 2.3. The only difference is that everywhere the group $\mathbb{Z}^2$ should be replaced by the group $\mathbb{Z}$. □

2.4. **Bounded soliton solutions.** Bounded soliton solutions occupy a special place among soliton solutions. By virtue of Theorems 2.1-2.4, bounded soliton solutions correspond to bounded solutions of the induced functional differential equation. Let us proceed to the study of bounded solutions of a family of induced pointwise functional differential equations (9)

$$\dot{x}(t) = Ax(t+\tau) - (A+B)x(t) + Bx(t-\tau) + \phi(x(t)), \qquad (19)$$
$$\text{for almost all } t \in \mathbb{R},$$

in which the parameter of the family is the characteristic τ of the soliton solution. To begin with, we should describe *stationary solutions.* Obviously, the state $\bar{a} \in \mathbb{R}^n$ is stationary if and only if the condition $\phi(\bar{a}) = 0$ holds. By virtue of the definition of the potential ϕ, the points of the set

$$\{a : a \notin \mathbb{R}^n_+\} \cup \{a : \|a\|_{\mathbb{R}^n} = \Delta\}$$

and only they set the stationary states. Then, by virtue of the theorem of existence and uniqueness of the solution for the functional differential equation (19) (Theorem 2.2) for any initial value of $\bar{a}$ from the set

$$\Omega = \{x : x \in (\mathbb{R}^n_+ \cap B_\Delta(0))\},$$

where $B_\Delta(0)$ is the interior of a ball in the space $\mathbb{R}^n$ of radius Δ, there is a solution from the space $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, $\forall \mu$, $\mu^\tau \in (\mu_1(\tau), \mu_2(\tau))$, with such an initial value and such a solution is the only one. Then such solutions will be bounded and their phase portraits belong to the set of Ω.

We consider an example where there are two observable characteristics at each node of the lattice. For example, at each point of the lattice, the observed values are the cumulative flows along the highway, i.e. the cumulative flows forward and reverse, as well as the cumulative flows left and right. Figure 2 shows a phase portrait of the entire family of bounded soliton solutions for the example under consideration. The range of characteristics τ, μ, under which the existence of soliton solutions is guaranteed, is determined by the inequality

$$2C_\phi \tau \left(\mu^{-1} + 1\right) < \ln \mu^{-1},$$

where

$$C_\phi = \max\left\{\|A\| + \|B\|; L_\phi\right\}$$

and it is shown in Figure 1. Since all soliton solutions with initial conditions from the domain Ω are bounded and belong to each of the spaces $\mathcal{L}^1_\mu C^{(0)}(\mathbb{R})$, $\mu \in (0, 1)$, then for such solutions the set of characteristics τ varies from 0 to $\hat{\tau}$.

3. Discussion

This work is a stage in the development of research on soliton solutions for complex dynamical systems. In particular, an aggregated traffic flow model on the Manhattan lattice was previously studied. The disaggregation of such a system means the expansion and description of important observable characteristics for a steady flow. An important assumption is that the lattice is very large, the functioning of the nodes is uniform. This means that there is a homogeneous environment. An important development of such a study is associated with the rejection of the condition of the environment uniformity, which leads to its complication. In such systems, the space of soliton solutions is either trivial or completely absent. The space of “quasi-soliton” solutions turns out to be important, which, with a decrease in the level of heterogeneity, become more and more similar to soliton solutions. Quasi-soliton solutions are implemented in real systems and their study is important.

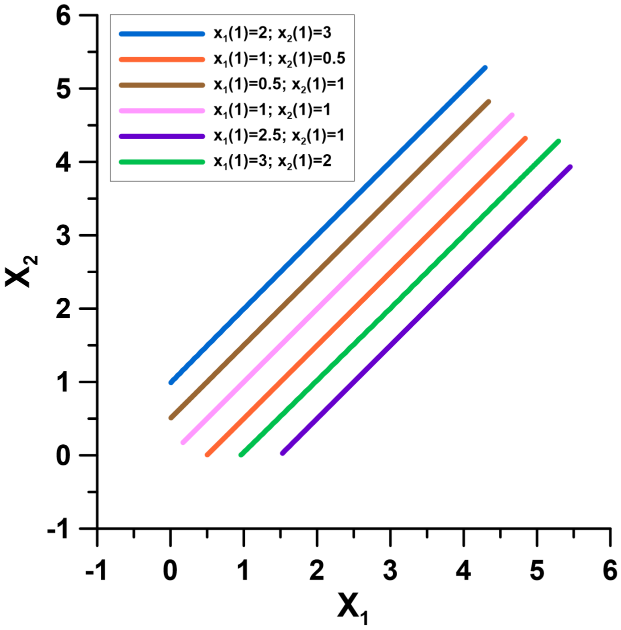

FIGURE 2. Phase curves for the equation (19) constructed using the OPTCON-F software package [17]. The parameter values are as follows: $\tau = 0.01$, $\Delta = 10$, $a_{11} = 1.0$, $a_{12} = 0.5$, $a_{21} = 3.0$, $a_{22} = 1.0$, $b_{11} = 2.0$, $b_{12} = 2.0$, $b_{21} = 5.0$, $b_{22} = 2.0$.

4. **Conclusions**

There were two important assumptions in the Manhattan lattice model under consideration.

First, the number of vertices in the lattice is considered to be quite large, and the procedures for regulating flows in the lattice are uniform throughout the lattice. Therefore, we can consider the lattice complete in all directions. The completeness of the lattice and the unity of flow control procedures determine the spatial uniformity of the system (the right side of the considered infinite-dimensional ordinary differential equation is permutable with a group of shift operators), and with it they guarantee the presence of a rich family of soliton solutions.

Secondly, the disaggregation of the observed flow parameters, which leads to a multidimensional nature.

When passing to the consideration of incomplete lattices (lattices with boundaries), the spatial uniformity of the system is lost (the right part of the considered infinite-dimensional ordinary differential equation is not permutable with a group of shift operators), which leads to a narrowing of the class of soliton solutions, or even their absence. In this case, it is necessary to expand the concept of a traveling wave type solution to quasi-traveling wave type solutions ("soliton solutions with a defect"). If in a homogeneous model solutions (absolutely continuous) of an induced family of pointwise functional differential equations corresponded to soliton solutions, then in an inhomogeneous model quasi-traveling waves ("soliton solutions with a defect") will correspond to impulse solutions of an induced family of functional differential equations. This approach was implemented for the finite difference analogue of the wave equation in one plastic deformation problem on longitudinal vibrations in an inhomogeneous infinite rod and the thermal conductivity equation in one problem on transportation [7, 9, 10, 11, 12, 13].

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