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GENERALIZED ERDÉLYI-KOBER FRACTIONAL INTEGRALS WITH I-FUNCTION AS KERNEL

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Abstract

The Erdélyi-Kober fractional integrals and derivatives are variants of the Riemann-Liuoville with very wide applications due to the freedom to chose their three arbitrary parameters. In this paper we introduce and study a generalization of the Erdélyi-Kober operators of fractional integration where the elementary kernel-function is replaced by a suitably chosen $I_{1,1}^{1,0}$-function. The I-functions are generalized hypergeometric functions, introduced by Rathie in 1997 as extensions of the Fox H-functions and Meijer G-functions, but have been not popular because of their rather complicated structure and multi-valued behaviour. However, it happens that they are useful not only in statistical physics, but include also important special functions in mathematics. In our previous works we related the I-functions to some specific special functions of fractional calculus, among which analogues of the Mittag-Leffler and Le Roy type functions. Here, we propose a theory of an I-function generalization of the Erdélyi-Kober fractional integrals, that will serve further as a base for further extension of the generalized fractional calculus with operators of fractional multi-order.

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1. **Introduction**

Here we provide a brief information on the special functions that are involved, in some particular cases, in this study. These are the generalized hypergeometric functions known as Fox H-functions, Meijer G-functions and Rathie I-functions. The definitions below are stated as in the original cited works.

Definition 1.1. (Ch. Fox [5], 1961, see books as [29], [12], [11], [10], etc.) The *Fox H-function* is a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral

$$H_{p,q}^{m,n}\left[z\left|\begin{array}{c}(a_i,A_i)_1^p\\(b_j,B_j)_1^q\end{array}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\mathcal{H}_{p,q}^{m,n}(s)\,z^{-s}ds,$$

$$\text{with}\quad \mathcal{H}_{p,q}^{m,n}(s)=\frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\prod\limits_{i=1}^{n}\Gamma(1-a_i-A_is)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod\limits_{i=n+1}^{p}\Gamma(a_i+A_is)},\tag{1.1}$$

for complex variable $z\neq 0$ and a contour $\mathcal{L}$ in the complex domain; the orders (m,n,p,q) are non negative integers so that $0\leq m\leq q$, $0\leq n\leq p$, the parameters $A_i>0, B_j>0$ are positive, and a_i, b_j, $i=1,...,p; j=1,...,q$ are arbitrary complex such that $A_i(b_j+l)\neq B_j(a_i-l'-1)$, $l,l'=0,1,2,...; i=1,...,n; j=1,...,m$ and so, the poles of the Gamma-functions in numerator do not coincide. The single valued branch of z^{-s} is chosen as $z^{-s}=\exp[-s\{\log|z|+i\arg z\}]$, $z\neq 0$. Note that $\mathcal{L}$ is an infinite contour which separates all the poles of $\Gamma(b_j+B_js)$ to the left and all the poles of $\Gamma(1-a_i-A_is)$ to the right of $\mathcal{L}$, and has one of the 3 following forms, as described in the book Kilbas-Saigo [10]:

(i) $\mathcal{L}=\mathcal{L}_{-\infty}$ is a left loop situated in the horizontal strip starting at the point $-\infty+i\varphi_1$ and terminating at the point $-\infty+i\varphi_2$ with $-\infty<\varphi_1<\varphi_2<+\infty$.

(ii) $\mathcal{L}=\mathcal{L}_{+\infty}$ is a right loop situated in the horizontal strip starting at the point $+\infty+i\varphi_1$ and terminating at the point $+\infty+i\varphi_2$ with $-\infty<\varphi_1<\varphi_2<+\infty$.

(iii) $\mathcal{L}=\mathcal{L}_{ic\infty}$ is a contour starting at the point $c-i\infty$ and terminating at the point $c+i\infty$, with suitable $c\in(-\infty,+\infty)$. This is also often denoted as $\mathcal{L}=(c-i\infty,c+i\infty)$.

Note that when the Mellin transform of the H-function exists (see Th.2.2 in [10]), it is equal to the integrand $\mathcal{H}_{p,q}^{m,n}(s)$. Also, in some books and other sources, the term z^{-s} in (1.1) is taken to be z^s, then the orientation of the above mentioned contours is to be changed.

The behaviour and properties of the H-function are described in terms of the following *characteristic parameters*:

$$R = \prod_{i=1}^{p} A_i^{-A_i} \prod_{j=1}^{q} B_j^{B_j} \;\; ; \;\; \Delta = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i; \;\; \delta = m + n - \tfrac{p+q}{2},$$
$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \tfrac{p-q}{2} \;\; ; \;\; a^* = \sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{1.2}$$

Depending on the values in (1.2), the H-function is a function analytic of z in disks $|z| < R$ or outside them, or in some sectors, or in the whole complex plane. More details on the properties of the Fox H-function can be found in many contemporary handbooks on special functions, as the above mentioned.

For studies on the behavior of the H-function around the singular points, and especially around the 3rd singular point $z = R$, one can see also the work of Karp [9], commenting and revisiting the old basic results of Braaksma [1].

When all $A_i = B_j = 1$, $i = 1, ..., p$; $j = 1, ..., q$, the H-function reduces to the *Meijer G-function* (C.S. Meijer [23], 1936-1941, etc.), see details in [4, Vol.1] and in the above mentioned books:

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i, 1)_1^p \\ (b_j, 1)_1^q \end{matrix} \right] = G_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i)_1^p \\ (b_j)_1^q \end{matrix} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{G}_{p,q}^{m,n}(s)\, z^s ds$$
$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{i=1}^{n} \Gamma(1 - a_i + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{i=n+1}^{p} \Gamma(a_i - s)} z^s ds, \quad z \neq 0. \tag{1.3}$$

Here we prefer to use the denotations by z^s instead of z^{-s}, as more conventional, that only yields a change of orientation of contours (described for the H-function). For the G-function, the behavior depends on conditions (1.2) with $R = 1$, $\Delta = q - p$, $\delta = m + n - \frac{p+q}{2}$. Although simpler than (1.1), the G-function is yet enough general as it incorporates most of the *Classical Special Functions* (known also as *Named SF*, *SF of Mathematical Physics*), the orthogonal polynomials, and many elementary functions. See lists of examples e.g. in [4, Vol.1], [12, Appendix C], etc.

The basic SF of FC that are Fox H-functions but *do not* reduce to Meijer G-functions in the general case (of *irrational* A_j, B_k) are the Wright generalized hypergeometric functions ${}_p\Psi_q$, extending the more popular ${}_pF_q$-functions, and the simplest but important example is by the Mittag-Leffler function, known as the Queen function of the fractional calculus, see details in [16], [19], [25], [26], [27], [28], etc.

However, it happens that many other SF, that have not been mentioned by now under the scheme of "SF of FC" (in sense of Kiryakova [16]), appear to be also important in analysis, number theory, statistics, physics and various branches of applied mathematics. Such examples and other extensions of the Fox H-functions, Meijer G-functions, Wright-Fox ${}_p\Psi_q$- and ${}_pF_q$-functions, have been introduced in the end of 20th century, but somehow neglected since looking too much complicated or artificial. In our recent survey [19], we have attracted attention to the so-called *I-functions of Rathie* [30], and to their particular specifications *$\overline{H}$-functions of Inayat-Hussain* [7], see also Buschman and Srivastava [2], and to their interesting cases of the Le Roy type functions. All these are further extensions of the classes of the special functions beyond the cases of the Wright generalized hypergeometric functions ${}_p\Psi_q$ and Fox H-functions (1.1).

Definition 1.2. The *I-function* was defined by a kind of Mellin-Barnes type contour integral (Rathie [30]):

$$I^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_j, A_j, \alpha_j)_1^p \\ (b_k, B_k, \beta_k)_1^q \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{I}^{m,n}_{p,q}(s)\, z^s ds, \quad z \neq 0, \tag{1.4}$$

where $\mathcal{I}(s)$ stands for

$$\mathcal{I}^{m,n}_{p,q}(s) = \frac{\prod\limits_{k=1}^{m} \Gamma^{\beta_k}(b_k - B_k s) \prod\limits_{j=1}^{n} \Gamma^{\alpha_j}(1 - a_j + A_j s)}{\prod\limits_{k=m+1}^{q} \Gamma^{\beta_k}(1 - b_k + B_k s) \prod\limits_{j=n+1}^{p} \Gamma^{\alpha_j}(a_j - A_j s)} \tag{1.5}$$

with the power exponents for the Gamma-functions: $\alpha_j,\ j = 1, ..., p$ and $\beta_k, k = 1, ..., q$, that can have in general *arbitrary positive values*, not obligatory integer ones. The possible contours $\mathcal{L}$ are discussed below.

Clearly, for non-integer values of these "powers"-parameters, the I-function (1.4) is not expressible as a H-function, and as other "classical" special functions.

The I-function is, in general, a multi-valued function of z with a branch point at $z = 0$. Most of the denotations and some conditions on the orders and parameters in (1.4)-(1.5) are similar to these used for the H-function. Here, $a_j, j = 1, ..., p$ and $b_k, k = 1, ..., q$ can be also complex numbers such that no singularity of $\Gamma^{\beta_k}(b_k - B_k s), k = 1, ..., m$ coincides with any singularity of $\Gamma^{\alpha_j}(1 - a_j + A_j s), j = 1, ..., n$. Again (as in the above definition of the G-function), we prefer the definition with term z^s so to be in agreement with the denotations in the initial works on the topic of the I-function, and suppose $z^s = \exp[s\{\log|z| + i\arg z\}]$.

In general, for non-integer α_j and β_k, the mentioned singularities are *no more* poles but are converted to branching points of the multi-valued fractional powers of the Gamma-functions. The branch cuts should be chosen so that the path of integration can be distorted as one of the three type of contours $\mathcal{L}$, as follows:

(a) $\mathcal{L}$ goes from $c - i\infty$ to $c + i\infty$, with suitably chosen real abscissa c so that the singularities of $\Gamma^{\beta_k}(b_k - B_k s), k = 1, ..., m$, lie to the right of $\mathcal{L}$, and all singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s), j = 1, ..., n$, lie to the left of $\mathcal{L}$;

(b) $\mathcal{L}$ is a loop beginning and ending at $+\infty$ and encircling all the singularities of $\Gamma^{\beta_k}(b_k - B_k s), k = 1, ..., m$, once in the clock-wise direction, but none of the singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s), j = 1, ..., n$;

(c) $\mathcal{L}$ is a loop beginning and ending at $-\infty$ and encircling all the singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s), j = 1, ..., n$, once in the clock-wise direction, but none of the singularities of $\Gamma^{\beta_k}(b_k - B_k s), k = 1, ..., m$.

Rathie [30] noted that the contour **(a)** can be interpreted as a particular case of the contour **(c)**. Moreover, in case **(b)** the sign of the function is changed with respect to the case **(c)**. Hence, when more than one of the above definitions of $\mathcal{L}$ make sense, they lead to the *same* result.

Like for the properties of the Fox H-function, the following values related to the parameters are important *to characterize the behavior of the I-function* (as analogues of these in (1.2)):

$$\begin{gathered}
\Delta = \sum_{k=1}^{m} \beta_k B_k - \sum_{k=m+1}^{q} \beta_k B_k + \sum_{j=1}^{n} \alpha_j A_j - \sum_{j=n+1}^{p} \alpha_j A_j; \\
\mu = \sum_{k=1}^{q} \beta_k B_k - \sum_{j=1}^{p} \alpha_j A_j; \\
\nabla = \sum_{j=1}^{p} \alpha_j [\mathrm{Re}\,(a_j) - 1/2] - \sum_{k=1}^{q} \beta_k [\mathrm{Re}\,(b_k) - 1/2]; \\
R = \prod_{k=1}^{q} B_k^{\beta_k B_k} / \prod_{j=1}^{p} A_j^{\alpha_j A_j}.
\end{gathered} \tag{1.6}$$

In terms of (1.6), it is proved (Rathie [30]) that: the I-function (1.4) for contour $\mathcal{L}$ defined by **(a)**, converges when $|\arg z| < \Delta\pi/2$, if $\Delta > 0$; if $|\arg z| = \Delta\pi/2, \Delta \geq 0$, the integral (1.4) converges absolutely when: (i) $\mu = 0$ if $\nabla > 1$; (ii) $\mu \neq 0$, if with $s = \sigma + it$, σ and t real, σ chosen so that for $|t| \to \infty$, we have $\nabla + \sigma\mu > 1$.

(iii) Otherwise, the integral (1.4) *for contour $\mathcal{L}$ defined by* ***(b)*** converges for $q \geq 1$ and either $\mu > 0$, or $\mu = 0$ inside the circle $|z| < R$. Note that in this work we consider I-functions (the kernels of the introduced generalized fractional integrals) that fall especially in this case, with

$$q \geq 1 \text{ (in particular, } q = 1), \ \mu = 0, \ |z| < R = 1, \text{ contour } \mathcal{L} \text{ of type } \mathbf{(b)}. \quad (1.7)$$

Alternatively, the integral (1.4) for contour $\mathcal{L}$ defined by **(c)** converges when $p \geq 1$ and either $\mu < 0$, or $\mu = 0$ if $|z| > R$.

To determine single-valued branches of the multi-valued Γ-functions with fractional powers in (1.4), one needs to draw suitable branch cuts. Example for such a procedure has been proposed in a recent paper by Rogosin and Dubatovskaya [31, Sect.3], for a similar function denoted by HG (eq.(3.1) there), with $\mathcal{L}$ taken as a contour of the so-called Slater type ([33], see Marichev [22, Ch.4]): $\mathcal{L}_{+\infty}$ starting at $+\infty + i\varphi_2$ and finishing at $+\infty + i\varphi_1$, with suitable $\varphi_1 < \varphi_2$. Namely, the ideas of the Slater theorem (see same place) are expanded for cases of ratios of Γ-functions with arbitrary (fractional) powers.

2. "Classical" Erdélyi-Kober operators of FC

One of the basic definitions in Fractional Calculus (FC) is for the Riemann-Liouville (R-L) integral of fractional order $\alpha > 0$ ([32], etc.):

$$I^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-\tau)^{\alpha-1} f(\tau) d\tau = \frac{z^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(zt) dt. \quad (2.1)$$

For $\alpha = 0$ the R-L operator is assumed to be the identity.

Here, for brevity, we discuss only the so-called left-hand sided variants of the FC operators, for real values of all parameters, skip the definitions of the corresponding R-L and Caputo fractional derivatives, and the details on the conditions in various functional spaces. For these details one can see in [12], [14], as well as in [32], [36], etc.

Among the "classical" modifications of (2.1) with very wide range of applications, due to the bigger freedom of choice of the 2 additional parameters, are the *Erdélyi-Kober (E-K) fractional integrals* of order $\alpha > 0$, with "weight" $\nu \in \mathbb{R}$ and additional parameter $\beta^* > 0$ (*Attention*: in our original works we used denotation β, but here substitute it with β^*, just because in the generalization with I-function kernel, it is used β, that should be simpler than the fractions $1/\beta^*$ appearing in the GFC operators until now and below. Then one can take $\beta = 1/\beta^*$):

$$I_{\beta^*}^{\nu,\alpha} f(z) = z^{-\beta^*(\nu+\alpha)} \int_0^z \frac{(z^{\beta^*} - \tau^{\beta^*})^{\alpha-1}}{\Gamma(\alpha)} \tau^{\beta^* \nu} f(\tau)\, d(\tau^{\beta^*})$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\nu} f(zt^{1/\beta^*}) dt$$

$$= \frac{\beta^*}{\Gamma(\alpha)} \int_0^1 (1-\sigma^{\beta^*})^{\alpha-1} \sigma^{\beta^*\nu+\beta^*-1} f(z\sigma) d\sigma. \tag{2.2}$$

For $\alpha = 0$ (by default) (2.2) is the *identity*: $I_{\beta^*}^{\nu,0} f(z) = f(z)$. And *the semigroup property, as one of the main feature of FC operators*, is satisfied ([12, Ch.2], see also [17, eq. (17)])

$$I_{\beta^*}^{\nu,\alpha_1} I_{\beta^*}^{\nu+\alpha_1,\alpha_2} f(z) = I_{\beta^*}^{\nu+\alpha_1,\alpha_2} I_{\beta^*}^{\nu,\alpha_1} f(z) = I_{\beta^*}^{\nu,\alpha_1+\alpha_2} f(z), \ \alpha_1 > 0, \alpha_2 > 0. \tag{2.3}$$

Operators (2.2) appeared initially in works by Erdélyi [3], Kober [20] and Sneddon [34] with $\beta^* = 1$, $\beta^* = 2$, then have been widely used and studied with arbitrary $\beta^* > 0$, by Sneddon, later by Kiryakova [12] and next works, Luchko et al. as [36].

Note the basic fact that the E-K operator (2.2) preserves the power functions up to a multiplier:

$$I_{\beta^*}^{\nu,\alpha}\{z^p\} = \vartheta_p \, z^p, \text{ with } \vartheta_p = \frac{\Gamma(p/\beta^* + \nu + 1)}{\Gamma(p/\beta^* + \nu + \alpha + 1)}, \ p > -\beta^*(\nu+1), \tag{2.4}$$

resp. $\alpha \geq 0$, $\nu > -\frac{p}{\beta^*} - 1$ are the conditions for the operator's parameters.

Often, the E-K operator is considered in a form more typical for the operators of integration, with the feature to increase the powers, and as used in operational / convolutional calculus, as follows:

$$\mathbb{I} f(z) := z^{\alpha_0} I_{\beta^*}^{\nu,\alpha} f(z), \quad \alpha_0 \geq 0. \tag{2.5}$$

Evidently, for the particular choice $\nu = 0, \beta^* = 1$, $\alpha_0 = \alpha$, one has the R-L integral

$$I^{\alpha} f(z) = z^{\alpha} I_1^{0,\alpha} f(z).$$

The theory of operators (2.2), (2.5) and the corresponding Erdélyi-Kober fractional derivatives $D_{\beta^*}^{\nu,\alpha}$ lies in the base of the Generalized Fractional Calculus (GFC), Kiryakova [12] (see also survey [14]). These operators are used essentially and studied also by Luchko et al., see e.g. [36]. In the works as Kiryakova [12], [15, Lemmas 1-2, Th.2], there are provided results for the E-K and multiple E-K images of the so-called Special Functions of Fractional Calculus (SF of FC) in the general cases, namely of the Fox H-functions, Meijer G-functions, Wright generalized hypergeometric functions ${}_p\Psi_q$ and of all their particular cases mentioned in [16], for E-K operators of the Le Roy type functions, see Paneva-Konovska [26], and Kiryakova, Paneva-Konovska et al. [18]. This allowed further to propose an unified approach (see [15]) to foresee and give explicitly the images of very wide classes of SF under the GFC operators in the sense of [12], i.e. under multiple compositions of E-K operators. These include, as very particular cases, the results for many particular variants of the E-K operators, popular as the Saigo operators, Marichev-Saigo-Maeda, etc.

We skip the details on the various functional spaces (of weighted continuous functions, weighted Lebesgue integrable functions or weighted analytical functions) where the theory of E-K operators is developed, [12, Ch.1, Ch.2, Ch.5]. The corresponding E-K fractional derivatives $D_{\beta^*}^{\nu,\alpha}$ such that $D_{\beta^*}^{\nu,\alpha} I_{\beta^*}^{\nu,\alpha} f(z) = f(z)$ in these spaces, are introduced and studied by Kiryakova [12, Ch.2] by means of explicit integro-differential expression (for R-L type):

$$D_{\beta^*}^{\nu,\alpha} f(z) = [\prod_{j=1}^{n} (\frac{1}{\beta^*} z \frac{d}{dz} + \nu + j)]\, I_{\beta^*}^{\nu+\alpha, n-\alpha} f(z),\ n-1 < \alpha \leq n, \tag{2.6}$$

or by differential-integral (for Caputo type) representation, one can see e.g. [14], [17].

Let us note that *the kernel of the E-K integral* (2.2) is an important *particular case of the Meijer G-function and of Fox H-function*, namely:

$$\beta^* \frac{(1-\sigma^{\beta^*})^{\alpha-1}}{\Gamma(\alpha)} \sigma^{\beta^*\nu+\beta^*-1} = \beta^* \sigma^{\beta^*-1} G_{1,1}^{1,0} \left[\sigma^{\beta^*} \middle| \begin{array}{c} \nu+\alpha \\ \nu \end{array} \right]$$

$$= H_{1,1}^{1,0} \left[\sigma \middle| \begin{array}{c} \nu+1-\frac{1}{\beta^*}+\alpha, \frac{1}{\beta^*} \\ \nu+1-\frac{1}{\beta^*}, \frac{1}{\beta^*} \end{array} \right] = H_{1,1}^{1,0} \left[\sigma \middle| \begin{array}{c} \nu+1-\beta+\alpha, \beta \\ \nu+1-\beta, \beta \end{array} \right], \tag{2.7}$$

when take $\beta := 1/\beta^* > 0$ like in the operator considered next in this paper.

Then, the E-K fractional integral $I_{\beta^*}^{\nu,\alpha}$ appears as the simplest case with $m = 1$ of the generalized fractional calculus operators from Kiryakova [12] with kernels $G_{m,m}^{m,0}$, resp. $H_{m,m}^{m,0}$.

Definition 2.1. Let $m \geq 1$ be an integer; $\alpha_i \geq 0$, $\nu_i \in \mathbb{R}$, $\beta_i > 0$, $i = 1, ..., m$. In analogy with the definition of (2.2), consider the sets of parameters: $\boldsymbol{\alpha} = (\alpha_1 \geq 0, ..., \alpha_m \geq 0)$ as a multi-order of fractional integration, $\boldsymbol{\nu} = (\nu_1, ..., \nu_m)$ as a multi-weight, and additional multi-parameter $\boldsymbol{\beta^*} = (\beta_1^* := 1/\beta_1, ..., \beta_m^* := 1/\beta_m)$. The integral operator defined as follows:

$$I_{(\beta_k^*),m}^{(\nu_k),(\alpha_k)} f(z) = \int_0^1 H_{m,m}^{m,0} \left[t \middle| \begin{array}{c} (\nu_i+\alpha_i+1-\frac{1}{\beta_i^*}, \frac{1}{\beta_i^*})_1^m \\ (\nu_i+1-\frac{1}{\beta_i^*}, \frac{1}{\beta_i^*})_1^m \end{array} \right] f(zt) dt \tag{2.8}$$

$$= \int_0^1 H^{m,0}_{m,m}\left[t \,\middle|\, \begin{matrix} (\nu_i+\alpha_i+1-\beta_i,\beta_i)_1^m \\ (\nu_i+1-\beta_i,\beta_i)_1^m \end{matrix}\right] f(zt)dt, \tag{2.9}$$

for $\sum\limits_{i=1}^{m}\alpha_i > 0$, and $I^{(\nu_k),(\alpha_k)}_{(\beta_k^*),m} f(z) = f(z)$ for $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, is called a *multiple (m-tuple) Erdélyi-Kober fractional integration operator*. More generally, the operators of the form

$$If(z) = z^{\alpha_0}\, I^{(\nu_k),(\alpha_k)}_{(\beta_k^*),m} f(z) \quad \text{with} \quad \alpha_0 \geq 0, \tag{2.10}$$

are called *generalized* (m*-tuple*) *fractional integrals of multi-order* ($\alpha_1 \geq 0, \ldots,$ $\alpha_m \geq 0$).

Note that for $\forall \beta_i^* := 1/\beta > 0, i = 1, ..., m$, we have the little simpler generalized fractional integrals with the kernel as Meijer $G^{m,0}_{m,m}$-function (1.3) ([12, Ch.1]),

$$I^{(\nu_k),(\alpha_k)}_{1/\beta,m} f(z) = \int_0^1 G^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{matrix} (\nu_k+\alpha_k)_1^m \\ (\nu_k)_1^m \end{matrix}\right] f(z\sigma^\beta)d\sigma. \tag{2.11}$$

Our operators as (2.9) and (2.11) from [12] and next works on GFC, are an illustration of the idea of Kalla [8] to introduce *generalized operators of fractional integration* of the general form

$$Rf(z) = z^{-\eta-1}\int_0^z \Phi(\xi/z)\,\xi^\eta\, f(\xi)d\xi = \int_0^1 \Phi(\sigma)\,\sigma^\eta\, f(z\sigma)d\sigma, \tag{2.12}$$

where the kernel Φ can be some (special) function for which the integral makes sense in a wide class of functions f. There have been many works treating such operators when Φ is taken as an arbitrary H- or G-function, or Bessel, Gauss etc. functions. But for such general kernels no full theory of corresponding GFC can be developed.

However, our suitable choice of kernels as Fox $H^{m,0}_{m,m}$-functions and Meijer $G^{m,0}_{m,m}$-functions, allowed the main feature that was of crucial importance for having a GFC with a theory: the property that the single-integral operators (2.8) can be equivalently represented by means of *commutative compositions of classical E-K integrals* ($m = 1$):

$$I^{(\nu_k),(\alpha_k)}_{(\beta_k^*),m} f(z) = \left[\prod_{i=1}^{m} I^{\nu_i,\alpha_i}_{\beta_i^*}\right] f(z)$$

$$= \int_0^1 \dots \int_0^1 \left[\prod_{i=1}^{m} \frac{(1-t_i)^{\alpha_i - 1} t_i^{\nu_i}}{\Gamma(\alpha_i)} \right] f\left(z\, t_1^{\frac{1}{\beta_1^*}} \dots \sigma_m^{\frac{1}{\beta_m^*}} \right) dt_1 \dots dt_m, \tag{2.13}$$

that is, by iterated integrals without special functions involved in the kernel. The frequent appearance of compositions like (2.13) in problems and models related to applications, combined with the simple but effective tools of the theory of the kernel special functions (H- and G-functions) in definition (2.8), explains the wide and efficient use of the operators of the GFC from [12].

The generalized fractional derivatives $D_{(\beta_k^*),m}^{(\nu_k),(\alpha_k)}$ of multi-order $(\alpha_1, ..., \alpha_m)$, corresponding to (2.8), (2.10) have been also introduced explicitly, in a way similar to that for D^α and $D_{\beta^*}^{\nu,\alpha}$ but with more complicated differ-integral representations. See [12], also [14] and [17].

Following an analogy with (2.7), we now introduce generalizations of the E-K fractional integrals (2.2), taking as a kernel an $I_{1,1}^{1,0}$-function of Rathie, instead of the $G_{1,1}^{1,0}$- and $H_{1,1}^{1,0}$-functions. To go further, in a next work, we will develop a generalized fractional calculus with Φ-kernel in (2.12) being $I_{m,m}^{m,0}$-function, thus introducing new fractional integrals of multi-order $(\alpha_1, ..., \alpha_m)$, similar to these in (2.8) with $H_{m,m}^{m,0}$.

3. Generalized Erdelyi-Kober fractional integrals with I-function

Definition 3.1. Let us take real parameters $\alpha \geq 0$, $\beta > 0$, $\gamma > 0$, $\nu \in \mathbb{R}$. By means of these 4 parameters and an $I_{1,1}^{1,0}$-function as a kernel, we define the following *generalized Erdélyi-Kober (g. E-K) integral operator*

$$\mathbb{I}_{\beta,\gamma}^{\nu,\alpha} f(z) = \int_0^1 I_{1,1}^{1,0} \left[\sigma \,\middle|\, \begin{matrix} \nu - \beta + \alpha, \beta, \gamma \\ \nu - \beta, \beta, \gamma \end{matrix} \right] f(z\sigma) d\sigma \tag{3.1}$$

of (fractional) order $\alpha > 0$; and for $\alpha = 0$: $\mathbb{I}_{\beta,\gamma}^{\nu,0} f(z) := f(z)$.

Note that for $\underline{\gamma := 1}$, we have a 3-parameters' $(\alpha \geq 0, \beta > 0$, real $\nu)$ operator of the form

$$\mathbb{I}_{\beta,1}^{\nu,\alpha} f(z) = \int_0^1 I_{1,1}^{1,0} \left[\sigma \,\middle|\, \begin{matrix} \nu - \beta + \alpha, \beta, 1 \\ \nu - \beta, \beta, 1 \end{matrix} \right] f(z\sigma) d\sigma$$

$$= \int_0^1 H_{1,1}^{1,0} \left[\sigma \,\middle|\, \begin{matrix} \nu - \beta + \alpha, \beta \\ \nu - \beta, \beta \end{matrix} \right] f(z\sigma) d\sigma$$

$$= \frac{1}{\beta\,\Gamma(\alpha)} \int_0^1 (1-\sigma^{1/\beta})^{\alpha-1}\,\sigma^{\nu/\beta-1}\,f(z\sigma)d\sigma = \frac{1}{\beta}\,I_{1/\beta}^{\nu-1,\alpha}\,f(z), \tag{3.2}$$

that is, a "*classical*" *E-K fractional integral*, according to definition (2.2) and relation (2.7).

First, let us analyse the above definition with respect to its kernel's behaviour and also, for the case $\alpha = 0$.

For $I_{1,1}^{1,0}\left[\sigma \middle| \begin{array}{c} \nu-\beta+\alpha,\beta,\gamma \\ \nu-\beta,\beta,\gamma \end{array}\right]$, as mentioned in (1.7), the values of parameters in (1.6) are as follows:

$$\Delta = \gamma\beta - \gamma\beta = \mathbf{0};\ \boldsymbol{\mu} = \gamma\beta - \gamma\beta = \mathbf{0}; \nabla = \gamma(\alpha+1); \mathbf{R} = \beta^{\gamma\beta}/\beta^{\gamma\beta} = \mathbf{1}.$$

Therefore, case (iii) (see Sect.1) is applicable for the definition of this $I_{1,1}^{1,0}$ with contour of the form **(b)**, and it is convergent in $|z|<1$, vanishes outside the unit disc, and for $|\arg z| = 0$ has a well described behaviour at the singular points $z=0$ and $z=1$ that ensures the convergence of integral (3.1) for functions of the form

$$f(z)\!=\!z^{\kappa}\widetilde{f}(z),\ \widetilde{f} \text{ continuous (integrable, analytic) on } (0,\infty),\ \kappa\!>\!-\nu/\beta.$$

Namely, according to (69) in Rathie [30],

$$I(\sigma) = I_{1,1}^{1,0}\left[\sigma \middle| \begin{array}{c} \nu-\beta+\alpha,\beta,\gamma \\ \nu-\beta,\beta,\gamma \end{array}\right] \sim \sigma^c,\ \ c=\nu/\beta-1, \ \ \text{and so,}$$

$$I(\sigma)\,\sigma^{\kappa}\widetilde{f}(z\sigma) \sim \sigma^d,\ d=\nu/\beta-1+\kappa > \nu/\beta-1-\nu/\beta=-1, \ \ \text{for } \sigma\to 0,$$

since $\widetilde{f}$ is bounded. Near the other singularity $\sigma = 1$, the behaviour is more delicate, but as mentioned in our previous work [19, (65)], and compare with representation (2.7),

$$I(\sigma) \sim (1-\sigma^{1/\beta})^{\eta-1} \ \text{ with } \ \eta-1=\alpha\gamma-1>-1, \ \text{ as } \ \sigma\to 1, \sigma<1.$$

This generalized E-K integral *preserves the power functions up to a scalar multiplier*, similarly like in (2.4), namely:

$$\mathbb{I}_{\beta,\gamma}^{\nu,\alpha}\,\{z^p\} = \Theta_p\,z^p,\ \text{ with }\ \Theta_p = \frac{\Gamma^{\gamma}(\nu+p\beta)}{\Gamma^{\gamma}(\nu+p\beta+\alpha)},\ \ p>-\nu/\beta, \tag{3.3}$$

resp. $\alpha \geq 0$, $\nu > -p\beta$ as conditions for the operator's parameters.

To derive (3.3), we have used the auxiliary result (63) from our recent survey [19], being an analogue of the corresponding integral of an $H^{m,0}_{m,m}$-function from [12, App.E, (E.21)]:

$$\int_0^1 I^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (a_i, A_i, \alpha_i)_1^m \\ (b_i, B_i, \beta_i)_1^m \end{matrix} \right.\right] d\sigma = \int_0^\infty \text{... same I-f. ...}\, d\sigma$$

$$= \prod_{i=1}^m \frac{\Gamma^{\beta_i}(b_i + B_i)}{\Gamma^{\alpha_i}(a_i + A_i)},\ \ \forall a_i > b_i > 0,\ \ \text{now for } m = 1, \text{ namely :} \tag{3.4}$$

$$\int_0^1 I^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} \nu - \beta + \alpha + p\beta, \beta, \gamma \\ \nu - \beta + p\beta, \beta, \gamma \end{matrix} \right.\right] d\sigma = \frac{\Gamma^\gamma(\nu + p\beta)}{\Gamma^\gamma(\nu + p\beta + \alpha)}. \tag{3.5}$$

It is easy to present the *Mellin transform of the operator* (3.1). We have

$$\mathfrak{M}\{\mathbb{I}^{\nu,\alpha}_{\beta,\gamma} f(z); s\} = \int_0^\infty z^{s-1} ds \left\{ \int_0^1 I^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} \nu - \beta + \alpha, \beta, \gamma \\ \nu - \beta, \beta, \gamma \end{matrix} \right.\right] f(z\sigma) d\sigma \right\}$$

$$= \int_0^1 I^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} \nu - \beta + \alpha, \beta, \gamma \\ \nu - \beta, \beta, \gamma \end{matrix} \right.\right] d\sigma \left\{ \int_0^\infty z^{s-1} f(z\sigma) ds \right\}$$

$$= \int_0^1 I^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} \nu - \beta + \alpha, \beta, \gamma \\ \nu - \beta, \beta, \gamma \end{matrix} \right.\right] \mathfrak{M}\{f(\sigma \cdot z; s\} d\sigma$$

$$= \int_0^1 I^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} \nu - \beta + \alpha, \beta, \gamma \\ \nu - \beta, \beta, \gamma \end{matrix} \right.\right] \sigma^{-s} \mathfrak{M}\{f(z; s\} d\sigma$$

$$= \mathfrak{M}\{f(z; s\} \cdot \int_0^1 I^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} \nu - \beta + \alpha - s\beta, \beta, \gamma \\ \nu - \beta - s\beta, \beta, \gamma \end{matrix} \right.\right] \sigma^{-s} d\sigma,$$

if we apply a well-known property of the Mellin transform as $\mathfrak{M}\{f(\sigma \cdot z); s\} = \sigma^{s-1} \cdot \mathfrak{M}\{f(z); s\}$, also (7.3) from [30] and (3.5). Therefore,

$$\mathfrak{M}\{\mathbb{I}^{\nu,\alpha}_{\beta,\gamma} f(z); s\} = \mathfrak{M}\{f(z; s\} \cdot \frac{\Gamma^\gamma(\nu - s\beta)}{\Gamma^\gamma(\nu + \alpha - s\beta)}. \tag{3.6}$$

Then, as a reasoning of the default assumption that $\mathbb{I}^{\nu,0}_{\beta,\gamma}$ is set to be the identity operator for $\alpha = 0$, one can use either the representation (3.3) that $\mathbb{I}^{\nu,\mathbf{0}}_{\beta,\gamma}\{z^p\} = z^p$, or $\mathfrak{M}\{\mathbb{I}^{\nu,\mathbf{0}}_{\beta,\gamma} f(z); s\} = \mathfrak{M}\{f(z); s\}$.

Let us discuss the properties of the introduced generalized E-K integral (3.1) *as an operator of fractional calculus*, i.e. as an integral of fractional order α.

It is easy to check the bilinearity, and also the commutativity of two different generalized E-K operators, using the properties of the I-function, as the symmetry in the sets of triplets, see (i) in [30, Sect.7].

LEMMA 3.1. *Semigroup property (Law of indices) for the generalized E-K integral operator,* $\alpha_1 \geq 0, \alpha_2 \geq 0$*:*

$$\mathbb{I}_{\beta,\gamma}^{\nu+\alpha_1,\alpha_2}\,\mathbb{I}_{\beta,\gamma}^{\nu,\alpha_1}\,f(z)\left(=\mathbb{I}_{\beta,\gamma}^{\nu,\alpha_1}\,\mathbb{I}_{\beta,\gamma}^{\nu+\alpha_1,\alpha_2}\,f(z)\right)=\mathbb{I}_{\beta,\gamma}^{\nu,\alpha_1+\alpha_2}\,f(z). \tag{3.7}$$

P r o o f. For shortness, we introduce the notations

$${}^2\mathbb{I}\,F(z)=\mathbb{I}_{\beta,\gamma}^{\nu+\alpha_1,\alpha_2}\,F(z)=\int_0^1 I_{1,1}^{1,0}\left[\tau\,\middle|\,\begin{matrix}\nu+\alpha_1-\beta+\alpha_2,\beta,\gamma\\ \nu+\alpha_1-\beta,\beta,\gamma\end{matrix}\right]F(z\tau)d\tau$$

and

$$F(z\tau)={}^1\mathbb{I}\,f(z\tau)=\int_0^1 I_{1,1}^{1,0}\left[\sigma\,\middle|\,\begin{matrix}\nu-\beta+\alpha_1,\beta,\gamma\\ \nu-\beta,\beta,\gamma\end{matrix}\right]f(z\tau\sigma)d\sigma.$$

Then,

$$\begin{aligned}{}^2\mathbb{I}\,{}^1\mathbb{I}\,f(z)=\mathbb{I}_{\beta,\gamma}^{\nu+\alpha_1,\alpha_2}\,\mathbb{I}_{\beta,\gamma}^{\nu,\alpha_1}\,f(z)=&\int_0^1 I_{1,1}^{1,0}\left[\tau\,\middle|\,\begin{matrix}\nu+\alpha_1-\beta+\alpha_2,\beta,\gamma\\ \nu+\alpha_1-\beta,\beta,\gamma\end{matrix}\right]d\tau\\ &\times\left\{\int_0^1 I_{1,1}^{1,0}\left[\sigma\,\middle|\,\begin{matrix}\nu-\beta+\alpha_1,\beta,\gamma\\ \nu-\beta,\beta,\gamma\end{matrix}\right]f(z\tau\sigma)d\sigma\right\}\\ =&\int_0^1 I_{1,1}^{1,0}\left[\sigma\,\middle|\,\begin{matrix}\nu-\beta+\alpha_1,\beta,\gamma\\ \nu-\beta,\beta,\gamma\end{matrix}\right]d\sigma\\ &\times\left\{\int_0^1 I_{1,1}^{1,0}\left[\tau\,\middle|\,\begin{matrix}\nu+\alpha_1-\beta+\alpha_2,\beta,\gamma\\ \nu+\alpha_1-\beta,\beta,\gamma\end{matrix}\right]f(z\tau\sigma)d\tau\right\},\end{aligned}$$

interchanging the order of integrations. Let us note the limits in which the variables change: $\tau|_0^1$ and $\sigma|_0^1$, and substitute $y := \tau\sigma|_0^1$ in the inner integral, with $\tau=\sigma^{-1}y, d\tau=\sigma^{-1}dy$. Then,

$${}^2\mathbb{I}\,{}^1\mathbb{I}\,f(z)=\int_0^1 \sigma^{-1}\,I_{1,1}^{1,0}\left[\sigma\,\middle|\,\begin{matrix}\nu-\beta+\alpha_1,\beta,\gamma\\ \nu-\beta,\beta,\gamma\end{matrix}\right]d\sigma$$

$$\times\ \left\{ \int_0^1 I_{1,1}^{1,0}\left[\sigma^{-1} y \,\middle|\, \begin{array}{c} \nu+\alpha_1-\beta+\alpha_2, \beta, \gamma \\ \nu+\alpha_1-\beta, \beta, \gamma \end{array} \right] f(zy)dy \right\}$$

$$= \int_0^1 I_{1,1}^{1,0}\left[\sigma \,\middle|\, \begin{array}{c} \nu-\beta+\alpha_1-\beta, \beta, \gamma \\ \nu-\beta-\beta, \beta, \gamma \end{array} \right] d\sigma$$

$$\times\ \left\{ \int_0^1 I_{1,1}^{0,1}\left[y^{-1}\sigma \,\middle|\, \begin{array}{c} 1-\nu-\alpha_1+\beta, \beta, \gamma \\ 1-\nu-\alpha_1+\beta-\alpha_2, \beta, \gamma \end{array} \right] f(zy)dy \right\}$$

$$= \int_0^1 f(zy)dy \left\{ \widetilde{\mathbb{I}} \right\},$$

where we have used 2 properties of the I-functions, that are very similar to these for the H-functions, namely (7.3) and (7.5) from Rathie [30] (to enter the multiplier σ^{-1} inside the $I_{1,1}^{1,0}$-function in the first integral, and to replace $I_{1,1}^{1,0}$ by $I_{1,1}^{0,1}$ of reciprocal argument in the second one). It remains to evaluate the integral denoted above by $\widetilde{\mathbb{I}}$ as integral of product of two different I-functions. In general, such a result comes from the evaluation of the Mellin transform of such a product, as from Vellaisamy-Kataria [35, Prop.3.1, p.288]. But in our work [19], we have rewritten this in a simpler form as Lemma 1, eq.(67), for $\lambda \neq 0$, $\nu \neq 0$, $s > 0$:

$$\int_0^\infty \sigma^{s-1} \cdot I_{p,q}^{m,n}\left[\lambda\sigma \,\middle|\, \begin{array}{c} (a_i, A_i, \alpha_i)_1^p \\ (b_j, B_j, \beta_j)_1^q \end{array} \right] \cdot I_{u,v}^{k,l}\left[\nu\sigma \,\middle|\, \begin{array}{c} (c_i, C_i, \varphi_i)_1^u \\ (d_j, D_j, \psi_j)_1^v \end{array} \right] d\sigma$$

$$= \frac{1}{s} \cdot I_{q+u,p+v}^{n+k,m+l}\left[\frac{\nu}{\lambda} \,\middle|\, \begin{array}{c} (c_i, C_i, \varphi_i)_1^l, (1-b_j-B_j, B_j, \beta_j)_1^q, (c_i, C_i, \varphi_i)_{l+1}^u \\ (d_j, D_i, \psi_j)_1^k, (1-a_i-A_i, A_i, \alpha_i)_1^p, (d_j, D_j, \psi_j)_{k+1}^v \end{array} \right]. \quad (3.8)$$

We take in mind that the first involved I-function vanishes identically for arguments of $\sigma > 1$, then $\int_0^\infty$ reduces to $\int_0^1$. Namely, we have:

$$\widetilde{\mathbb{I}} = \int_0^\infty \ldots d\sigma = \int_0^1 I_{1,1}^{1,0}\left[\sigma \,\middle|\, \begin{array}{c} \nu-\beta+\alpha_1-\beta, \beta, \gamma \\ \nu-\beta-\beta, \beta, \gamma \end{array} \right]$$

$$\times\ I_{1,1}^{0,1}\left[y^{-1}\sigma \,\middle|\, \begin{array}{c} 1-\nu+\beta-\alpha_1, \beta, \gamma \\ 1-\nu+\beta-\alpha_1-\alpha_2, \beta, \gamma \end{array} \right] d\sigma$$

$$= I_{2,2}^{0,2}\left[y^{-1} \left| \begin{array}{c} \underline{(1-\nu+\beta-\alpha_1,\beta,\gamma)}, (1-\nu+\beta+\underline{\beta}-\underline{\beta},\beta,\gamma) \\ (1-\nu+\beta-\alpha_1-\alpha_2,\beta,\gamma), \underline{(1-\nu+\beta+\underline{\beta}-\alpha_1-\underline{\beta},\beta,\gamma)} \end{array} \right. \right]$$

$$= I_{1,1}^{0,1}\left[y^{-1} \left| \begin{array}{c} 1-\nu+\beta,\beta,\gamma \\ 1-\nu+\beta-\alpha_1-\alpha_2,\beta,\gamma \end{array} \right. \right] = I_{1,1}^{1,0}\left[y \left| \begin{array}{c} \nu-\beta+\alpha_1+\alpha_2,\beta,\gamma \\ \nu-\beta,\beta,\gamma \end{array} \right. \right],$$

the "striken" (underlined) triplets annulate each other and so, reduce the orders of the I-function (another property (7.1), [30], analogous to that for H-function). The result, after using again the mentioned property (7.5), is exactly the kernel-function of the operator in r.h.s. of (3.7),

$$\mathbb{I}_{\beta,\gamma}^{\nu,\alpha_1+\alpha_2} f(z) = \int_0^1 I_{1,1}^{1,0}\left[y \left| \begin{array}{c} \nu-\beta+\alpha_1+\alpha_2,\beta,\gamma \\ \nu-\beta,\beta,\gamma \end{array} \right. \right] f(zy)dy.$$

□

Having in mind the established properties of the introduced generalized E-K integrals, *the axioms an operator to be operator of fractional calculus are well satisfied*. Therefore, we can consider (3.1) as *operator of integration of fractional order* $\alpha \geq 0$.

4. **Particular cases**

As already noted, when the 4th parameter $\gamma > 0$ is taken as $\underline{\gamma = 1}$, the kernel $I_{1,1}^{1,0}$-function of the generalized Erdélyi-Kober operator (3.1) reduces to a $H_{1,1}^{1,0}$-function, also to a $G_{1,1}^{1,0}$-function, and especially to the elementary function $(1-\sigma^{1/\beta})^{\alpha-1}\sigma^{\nu/\beta-1}$, and we have the *"classical" Erdélyi-Kober integral of fractional order* α, see (3.2):

$$\mathbb{I}_{\beta,1}^{\nu,\alpha} f(z) = \frac{1}{\beta} I_{1/\beta}^{\nu-1,\alpha} f(z).$$

Additionally, if $\underline{\beta = 1}$,

$$\mathbb{I}_{1,1}^{\nu,\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{\nu-1} f(z\sigma)d\sigma,$$

it is a *"weighted" variant of the Riemann-Liouville fractional integral as introduced by Kober* [20] ; while for $\underline{\beta = 1/2}$ $(\beta^* = 2)$, this is

$$\mathbb{I}_{2,1}^{\nu,\alpha} f(z) = \frac{2}{\Gamma(\alpha)} \int_0^1 (1-\sigma^2)^{\alpha-1} \sigma^{2\nu-1} f(z\sigma)d\sigma$$

$$= 2z^{2-2\alpha-2\nu} \frac{1}{\Gamma(\alpha)} \int_0^z (z^2-\tau^2)^{\alpha-1} \tau^{2\nu-1} f(\tau) d\tau,$$

up to a multiplier, the *Erdélyi-Kober integral as initially introduced by Sneddon*, see e.g. [34].

In Kiryakova's works as [12, 1.1.iii], and on, we provided a long list of examples of particular cases of the Erdélyi-Kober operators that were studied by different authors (operators of Hardy-Littlewood, Uspensky, Sonine, etc.) and employed in various areas, including to provide a Gelfond-Leontiev type operator, for which the Mittag-Leffler function appears as an eigenfunction.

Indeed, in [12, Ch.2], see details also in [13], we have considered the so-called *Gelfond-Leontiev (G-L) generalized integration and differentiation operators* generated by an entire function $\varphi(z)$, [6]. The core of these notions is that to an analytic function $f(z)$ defined by means of a power series, we put in correspondence its images constructed as Hadamard products by quotients of"shifted" coefficients of $\varphi(\lambda)$. In this case we take as entire function φ the *Mittag-Leffler function* (which is an example of a Fox H-function)

$$\varphi(z) := E_{\alpha,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\nu)}, \quad \alpha>0, \nu \in \mathbb{R},$$

we have proved that the power series for the corresponding G-L integration w.r.t. to this φ can be presented also by means of an integral operator. This is nothing else but an Erdélyi-Kober integral:

$$l_{\alpha,\nu} f(z) = \frac{z}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{\nu-1} f(z\sigma^\alpha) d\sigma = z\, I_{1/\alpha}^{\nu-1,\alpha} f(z). \tag{4.1}$$

And the following relations hold:

$$l_{\alpha,\nu} E_{\alpha,\nu}(\lambda z) = \frac{1}{\lambda} E_{\alpha,\nu}(\lambda z) - \frac{1}{\lambda \Gamma(\nu)}, \tag{4.2}$$

$$d_{\alpha,\nu} E_{\alpha,\nu}(\lambda z) = \lambda\, E_{\alpha,\nu}(\lambda z), \quad \lambda \neq 0, \tag{4.3}$$

where $d_{\alpha,\nu} f(z) = D_{1/\alpha}^{\nu-1,\alpha} z^{-1} f(z)$ is the Erdélyi-Kober fractional derivative, left inverse to (4.1). So, the M-L function $E_{\alpha,\nu}(z)$ *is eigenfunction for the Erdélyi-Kober operators* (4.2)-(4.3).

Similar situation arises with the generalized Erdélyi-Kober integral (3.1). Recently, an interest has been risen to the not so-popular *Le Roy function* $F^{(\gamma)}(z)$ that has been introduced in [21] almost same time as the Mittag-Leffler function ([24]) and for rather similar purposes not related then to fractional calculus. Several authors have considered a Mittag-Leffler type analogue $F^{(\gamma)}_{\alpha,\nu}$ of the Le Roy function, called *M-L type Le Roy (MLR) function*:

$$F^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[(k!)]^{\gamma}}, \quad F^{(\gamma)}_{\alpha,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k+\nu)]^{\gamma}}, \quad \forall \alpha, \beta, \gamma > 0. \tag{4.4}$$

This MLR function is an entire function of order $1/\alpha\gamma$, and can be represented also in terms of the Rathie I-function (see [19]), namely:

$$F^{(\gamma)}_{\alpha,\nu}(z) = I^{1,1}_{1,2}\left[-z \,\middle|\, \begin{array}{c} (0,1,1) \\ (0,1,1),(1-\nu,\alpha,\gamma) \end{array}\right].$$

In Kiryakova and Paneva-Konovska [18, Sect.3] we presented (classical) Erdélyi-Kober integral's images of the M-L type Le Roy function and of its multi-index extensions.

Then, in our work [19], we considered the problem to find suitable integral and differential operators for which the Le Roy type functions are eigenfunctions. As a consequence, we can expect in what kind of integral / differential equations they would appear as solutions.

Again we started with the apparatus of the Gelfond-Leontiev operators [6], this time generated by these functions. For the multi-index extensions ($m = 1, 2, 3, ...$) of (4.4) (which is the case for $m = 1$), the G-L operators are presented by (57)-(58) there by means of power series, and Th. 5 states that these are their eigen operators. In the case $m = 1$ related to $F^{(\gamma)}_{\alpha,\nu}(z)$, the G-L integration L_{MLR} is presented also by the integral operator

$$Lf(z) := L_{MLR}f(z) = \int_0^1 I^{1,0}_{1,1}\left[\sigma \,\middle|\, \begin{array}{c} \nu,\alpha,\gamma \\ \nu-\alpha,\alpha,\gamma \end{array}\right] f(z\sigma)d\sigma. \tag{4.5}$$

The corresponding "eigen"-relations are as follows:

$$L\, F^{(\gamma)}_{\alpha,\nu}(\lambda z) = \frac{1}{\lambda} F^{(\gamma)}_{\alpha,\nu}(\lambda z) - \frac{1}{\lambda \Gamma^{\gamma}(\nu)}, \quad D\, F^{(\gamma)}_{\alpha,\nu}(\lambda z) = \lambda F^{(\gamma)}_{\alpha,\nu}(\lambda z), \;\; \lambda \neq 0,$$

where $D := D_{MLR}$ is the corresponding G-L differentiation operator. It is now observed that the *"eigen" integral operator* for $F^{(\gamma)}_{\alpha,\nu}$, (4.5), *is a case of the introduced generalized Erdélyi-Kober fractional integral* (3.1):

$$L_{MLR}\, f(z) = \mathbb{I}^{\nu+\beta-\alpha,\alpha}_{\alpha,\gamma}\, f(z), \quad \alpha > 0, \gamma > 0, \nu \in \mathbb{R}.$$

Unfortunately, the problem to give an explicit differential-integral representation of a generalized Erdélyi-Kober fractional derivative, like the definition (2.6) of the Erdélyi-Kober fractional derivative $D_{\beta}^{\nu,\alpha}$, is yet resolved. This will need available a complicated differential property for the Rathie I-function, so to define properly the *formal inversion*, following from the semigroup property (3.7) for $\alpha_1 = \alpha, \alpha_2 = -\alpha$: $\mathbb{D}_{\beta,\gamma}^{\nu,\alpha} := \left\{\mathbb{I}_{\beta,\gamma}^{\nu,\alpha}\right\}^{-1} = \mathbb{I}_{\beta,\gamma}^{\nu+\alpha,-\alpha}$.

To go further, in a next work, we will develop a generalized fractional calculus with Φ-kernel in (2.12) being an $I_{m,m}^{m,0}$-function, thus introducing new fractional integrals of multi-order $(\alpha_1, ..., \alpha_m)$, similar to these in (2.8) with $H_{m,m}^{m,0}$-function.

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