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ON A SYMBOLIC METHOD FOR SECOND-ORDER BOUNDARY VALUE PROBLEMS OVER ALGEBRAS

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Abstract

This study presents a symbolic approach for solving second order boundary value problems with Stieltjes boundary conditions (integral, differential, and generic boundary conditions). The proposed symbolic method computes the Green's operator and the Green's function of the provided boundary value problem on the level of operators by applying the algebra of integro-differential operators. The suggested algorithm will aid in implementing manual calculations in mathematical software programs like Mathematica, Matlab, Singular, Scilab, Maple and others.

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1. Introduction

Boundary value problems (BVPs) are fundamental in scientific computing and have numerous applications across various fields. However, current computer algebra systems lack systematic support for solving them symbolically. Over the past five decades, researchers and engineers have focused on developing applications for general BVPs of higher-order ordinary differential equations with general boundary conditions, such as point evaluations. The symbolic analysis of BVPs, along with the formulation of Green's operator and Green's function for semi non-homogeneous BVPs (non-homogeneous differential equations with homogeneous boundary conditions), was pioneered by Markus Rosenkranz et al. in 2004 [1], also see [2, 26, 8, 9, 10, 11]. In literature, there are several symbolic algorithms available for solving differential equations, integro-differential equations, differential-algebraic equations [20, 21, 22, 23, 24, 18] and references therein.

In this paper, we introduce a symbolic algorithm for fully non-homogeneous second-order BVPs, which involves non-homogeneous differential equations with non-homogeneous boundary conditions. Our method is similar to the symbolic approach used for semi non-homogeneous BVPs in integro-differential algebras. We present the Green's operator and Green's function for fully non-homogeneous second-order BVPs with general, differential, and integral boundary conditions separately. Additionally, we extend our results to include the Green's operator and Green's function for BVPs with Stieltjes boundary conditions, which combine general, differential, and integral boundary conditions.

The structure of this paper is as follows: Section 1.1 reviews the essential concepts related to the algebra of integro-differential operators. In Section 2, the suggested algorithm for BVP is presented with four different boundary conditions: in Section 2.1, BVP with general boundary conditions; in Section 2.2, BVP has differential boundary conditions; in Section 2.3, BVP has integral boundary conditions; and in Section 2.4, BVP with Stieltjes boundary conditions. Section 3 demonstrates the algorithm through real-world applications, and Section 4 concludes with final remarks.

1.1. Algebra of Integro-differential Operators.

The BVP, along with Green's operator and Green's function in operator-based notations, is formulated by revisiting the foundational concepts of integro-differential algebras and the algebra of integro-differential operators. For more details, refer to [12] or [14, 19, 15, 17, 13]. Throughout this section, we assume that $\mathbb{K}$ is a field of characteristic zero, and $\mathcal{F} = C^{\infty}[a, b]$ is considered for simplicity.

Definition 1.1. [12, 17] An algebraic structure $(\mathcal{F}, \partial, \int)$ is termed an *integro-differential algebra* over $\mathbb{K}$ if $\mathcal{F}$ is a commutative $\mathbb{K}$-algebra equipped with $\mathbb{K}$-linear operators ∂ and $\int$, satisfying the following conditions:

- $\partial(\int f) = f$,
- $\partial(fg) = (\partial f)g + f(\partial g)$,
- $(\int\partial f)(\int\partial g) + \int\partial(fg) = (\int\partial f)g + f(\int\partial g)$.

Here $\partial : \mathcal{F} \to \mathcal{F}$ and $\int : \mathcal{F} \to \mathcal{F}$ are two maps defined by $\partial = \frac{d}{dx}$, a derivation, and $\int = \int_a^x dx$, a $\mathbb{K}$-linear right inverse of ∂, i.e. $\partial \circ \int = 1$ (the identity map). The map $\int$ is called an *integral* for ∂ and $\int \circ \partial = 1 - \Sigma$, where Σ is called the *evaluation* operator of $\mathcal{F}$ defined as $\Sigma : f \mapsto f(a)$, evaluates at initial point a. An integro-differential algebra over $\mathbb{K}$ is called *ordinary* if $\mathrm{Ker}(\partial) = \mathbb{K}$.

For a standard integro-differential algebra, evaluation can be considered as a multiplicative linear functional $\Sigma : \mathcal{F} \to \mathbb{K}$, meaning that $\Sigma(fg) = (\Sigma f)(\Sigma g)$ for all $f, g \in \mathcal{F}$. Let $\Phi \subseteq \mathcal{F}^*$ represent the set of all multiplicative linear functionals, including Σ. To define a BVP, we also need to introduce *point evaluations* as additional generators. For instance, the boundary conditions $u(0) = 5$, $u'(2) = -1$, $\int_0^1 u\,dx = 0$ and $u'(1) + \int_0^2 u\,dx = -2$ on a function $u \in \mathcal{F} = C^\infty[a, b]$ correspond to the functionals $\Sigma_0 u = 5$, $\Sigma_2\partial u = -1$, $\Sigma_1\int u = 0$ and $\Sigma_1\partial u + \Sigma_2\int u = -2$ in $\mathcal{F}^*$.

Definition 1.2. [12, 17] Let $(\mathcal{F}, \partial, \int)$ be an ordinary integro-differential algebra over $\mathbb{K}$ and $\Phi \subseteq \mathcal{F}^*$. The *integro-differential operators* $\mathcal{F}[\partial, \int]$ are defined as the $\mathbb{K}$-algebra generated by the symbols ∂ and $\int$, the functions $f \in \mathcal{F}$ and the characters (functionals) $\Sigma_c, \phi, \chi \in \Phi$, modulo the Noetherian and confluent rewrite system given in Table 1.

Table 1. Rewrite rules for integro-differential operators

$fg \to f \cdot g$	$\partial f \to f\partial + f'$	$\int f \int \to (\int f)\int - \int(\int f)$
$\chi\phi \to \phi$	$\partial\phi \to 0$	$\int f\partial \to f - \int f' - (\Sigma f)\Sigma$
$\phi f \to (\phi f)\phi$	$\partial\int \to 1$	$\int f\phi \to (\int f)\phi$

2. Proposed Symbolic Algorithm

For an integro-differential algebra $\mathcal{F}$, a fully non homogeneous second-order BVP is given by a differential operator $L = \partial^2 + a_1\partial + a_0$ and the boundary conditions $b_1, b_2 \in \mathcal{F}[\partial, \int]$ with boundary data $c_1, c_2 \in \mathbb{R}$. Given a forcing function $f \in \mathcal{F}$ and a set of boundary data $c_1, c_2 \in \mathbb{R}$, we want to find $u \in \mathcal{F}$ such that

$$Lu = f \text{ and } b_1 u = c_1,\ b_2 u = c_2. \tag{1}$$

The quantities $\{f, c_1, c_2\}$ are known as the *data* for the BVP. As mentioned in [8], the data can be decomposed as

$$\{f, c_1, c_2\} = \{f, 0, 0\} + \{0, c_1, c_2\}. \tag{2}$$

The data $\{f, 0, 0\}$ indicates the BVP with semi non-homogeneous equations and the data $\{0, c_1, c_2\}$ indicates the BVP with semi homogeneous equations. Symbolically, the solution of (1) can be written as

$$u = G(f, c_1, c_2),$$

where G is a linear operator (known as Green's operator) that transforms the data into the solution.

In [12], Rosenkranz et al. presented a symbolic solution of the form $G_{nh}(f, 0, \dots, 0)$ for a BVP with data $\{f; 0, \dots, 0\}$ (semi non-homogeneous BVP). In this paper we find the solution of fully non-homogeneous BVP of order two with data $\{f; c_1, c_2\}$. To motivate the solution, we briefly recall the symbolic solution $G_{nh}(f, 0, 0)$.

Consider a semi non-homogeneous BVP

$$Lu = f \text{ and } b_1 u = 0, b_2 u = 0, \tag{3}$$

where $L = \partial^2 + a_1\partial + a_0$ is a surjective linear map and $B = \{b_1, b_2\} \subseteq \mathcal{F}^*$ is a closed subspace of the dual space. We call $G_{nh}(f, 0, 0) \in \mathcal{F}$ a solution of (3) for a given forcing function $f \in \mathcal{F}$, if $LG_{nh}(f, 0, 0) = f$ and $G_{nh}(f, 0, 0) \in B^{\perp}$. In operator notations $LG_{nh} = 1$ and $BG_{nh} = 0$, and the operator G_{nh} is called *Green's operator* for the semi non-homogeneous BVP. As mentioned in [12, 17], the Green's operator and Green's function can be computed as $G_{nh} = (1 - P)L_F^{-1}$ and $u = G_{nh}(f)$ respectively, where $P \in \mathcal{F}[\partial, \mathcal{C}]$ is the projector operator onto Ker(L) along $B^{\perp}$, and L_F^{-1} is the fundamental right inverse of L that can be computed using the classical method namely various of parameters.

Now, consider a semi homogeneous BVP

$$Lu = 0 \text{ and } b_1 u = c_1, b_2 u = c_2, \tag{4}$$

where $\{c_1, c_2\} \subset \mathbb{R}$ is set of boundary data. Let $\mathcal{H}$ be any function (not necessarily satisfying the differential operator L) such that $b_1\mathcal{H} = c_1$ and $b_2\mathcal{H} = c_2$. Set $u = \mathcal{H} + v$, then one can observe that v satisfies the semi non-homogeneous BVP

$$Lv = -L\mathcal{H} \text{ and } b_1 v = 0, b_2 v = 0,$$

then, the solution $u = \mathcal{H} + v$ of (4) is computed similar to the Green function of semi non-homogeneous BVP (3), and it is denoted by $G_h(0, c_1, c_2) = \mathcal{H} +$

$G_{nh}(-L\mathcal{H})$. In details,

$$\begin{aligned} G_h(0, c_1, c_2) &= \mathcal{H} + G_{nh}(-L\mathcal{H}) = \mathcal{H} + (1-P)L_F^{-1}(-L\mathcal{H}) \\ &= \mathcal{H} - L_F^{-1}L\mathcal{H} + PL_F^{-1}L\mathcal{H} = P\mathcal{H}, \ (\because \ L_F^{-1}L = 1). \end{aligned}$$

The function $\mathcal{H} \in \mathcal{F}$ is such that $b_1\mathcal{H} = c_1$ and $b_2\mathcal{H} = c_2$. We call $\mathcal{H}$ as a *right inverse* of B such that $b_1\mathcal{H} = c_1$ and $b_2\mathcal{H} = c_2$. Since $\mathcal{H}$ is depending only on the boundary data, this amounts to an *interpolation problem* [3, 4, 7] with the given boundary conditions. The Green's operator maps each f to its unique solution $G_{nh}(f, 0, 0)$. The BVP (3) is called *regular* if and only if $B^\perp$ is complement of $\mathrm{Ker}(L)$ so that $\mathcal{F} = \mathrm{Ker}(L) \oplus B^\perp$ as a direct sum. The regularity of a BVP can be tested algorithmically [12, pg. 31] as follows: If $\{u_1, u_2\}$ is a basis for $\mathrm{Ker}(L)$ and $\{b_1, b_2\}$ is a basis for B, then the BVP is regular if and only if the *evaluation matrix*

$$b(u) = \begin{pmatrix} b_1(u_1) & b_1(u_2) \\ b_2(u_1) & b_2(u_2) \end{pmatrix} \tag{5}$$

is regular (non-singular).

Finally, the solution of the given BVP (1) is

$$G(f, c_1, c_2) = G_{nh}(f, 0, 0) + G_h(0, c_1, c_2) = (1-P)L_F^{-1}f + P\mathcal{H}, \tag{6}$$

where P is the projector operator, L_F^{-1} is the fundamental right inverse and $\mathcal{H}$ is the function satisfying the boundary data.

2.1. **BVPs with General Boundary Conditions.** In this section, we consider a BVP of the following type:

$$Lu = f \text{ and } \Sigma_a y = c_1, \ \Sigma_b y = c_2, \tag{7}$$

where $L = \partial^2 + a_1\partial + a_0$ is differential operator, $f \in \mathcal{F}$ is a given forcing function, $c_1, c_2 \in \mathbb{R}$ are boundary data, and Σ_a, Σ_b are evaluation operators at a and b respectively. We want to find $u \in \mathcal{F}$ such that the given BVP (7) is satisfied.

Let $\{y_1, y_2\}$ be fundamental system for L, i.e., $y_1, y_2 \in \mathrm{Ker}(L)$, then the given BVP is regular if and only if the evaluation matrix $\begin{pmatrix} y_1(a) & y_1(b) \\ y_2(a) & y_2(b) \end{pmatrix}$ is regular. As mentioned in [1, 8], the solution u is computed as $u = (1-P)L_F^{-1}(f) + P\mathcal{H}$, where

$$\begin{aligned} P &= \left(\frac{x-b}{a-b}\right)\Sigma_a + \left(\frac{a-x}{a-b}\right)\Sigma_b, \\ L_F^{-1} &= y_2 \varrho \frac{y_1}{y_1y_2' - y_2y_1'} - y_1 \varrho \frac{y_2}{y_1y_2' - y_2y_1'}, \\ \mathcal{H} &= \left(\frac{c_1-c_2}{a-b}\right)x + \frac{ac_2 - bc_1}{a-b}. \end{aligned}$$

Now, the solution is

$$u = \left(1 - \left(\frac{x-b}{a-b}\right)\Sigma_a - \left(\frac{a-x}{a-b}\right)\Sigma_b\right)\left(y_2\varrho\frac{y_1 f}{y_1y_2' - y_2y_1'} - y_1\varrho\frac{y_2 f}{y_1y_2' - y_2y_1'}\right)$$
$$+\left(\left(\frac{x-b}{a-b}\right)\Sigma_a + \left(\frac{a-x}{a-b}\right)\Sigma_b\right)\left(\left(\frac{c_1-c_2}{a-b}\right)x + \frac{ac_2-bc_1}{a-b}\right).$$

After simplification, we have

$$\begin{aligned} u(x) = {} & y_2(x)\int_a^x \frac{y_1(x)f(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)}\,dx \\ & - y_1(x)\int_a^x \frac{y_2(x)f(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)}\,dx \\ & - \left(\frac{a-x}{a-b}\right)y_2(b)\int_a^b \frac{y_1(x)f(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)}\,dx \\ & + \left(\frac{a-x}{a-b}\right)y_1(b)\int_a^b \frac{y_2(x)f(x)}{y_1(x)y_2'(x) - y_2(x)y_1'(x)}\,dx \\ & + \left(\frac{c_1-c_2}{a-b}\right)x + \frac{ac_2-bc_1}{a-b}. \end{aligned} \tag{8}$$

The following example illustrate the solution of a given BVP as in the equation (8).

Example 2.1. Consider the differential equation $Lu = f$ with boundary conditions $\Sigma_a u = c_1, \Sigma_b u = c_2$, where $L = \partial^2$ is a differential operator, f is the forcing function. For simplicity, we take $f(x) = e^x$, evaluation points $a = 0, b = 1$, and the boundary data $c_1 = 1, c_2 = 2$. Following the proposed methods, we have

Fundamental system $\{y_1, y_2\} = \{1, x\}$,
Fundamental right inverse operator $L_F^{-1} = x\varrho - \varrho x$,
Projector operator $P = (1-x)\Sigma_0 + x\Sigma_1$,
Boundary data interpolating function $\mathcal{H} = x + 1$.

Now the solution of the given BVP, computed as in equation (8), is

$$u(x) = 2x + e^x - xe.$$

It is verified that $Lu(x) = \partial^2 u(x) = e^x$ and $\Sigma_0 u(x) = 1, \Sigma_1 y(x) = 2$.

The following section presents the solution of the given BVP with differential boundary conditions.

2.2. **BVPs with Differential Boundary Conditions.** In this section, we consider a BVP of the following type:

$$Lu = f \text{ with } \Sigma_a\partial u = c_1,\ \Sigma_b u = c_2, \text{ or } \Sigma_a u = c_1,\ \Sigma_b\partial u = c_2, \tag{9}$$

where $L = \partial^2 + a_1\partial + a_0$ is differential operator, $f \in \mathcal{F}$ is a given forcing function, $c_1, c_2 \in \mathbb{R}$ are boundary data, and $\Sigma_a\partial, \Sigma_b\partial$ are derivative evaluation operators at a and b respectively. We want to find $u \in \mathcal{F}$ such that the given BVP (9) is satisfied. Let $\{y_1, y_2\}$ be fundamental system for L, i.e., $y_1, y_2 \in \mathrm{Ker}(L)$. For sake of simplicity, we consider the first set of boundary conditions i.e., $\Sigma_a\partial u = c_1, \Sigma_b u = c_2$ for the given differential operator L. As mentioned in [1, 8] and similar to the process in Section 2.1, the solution u is computed $u = (1-P)L_F^{-1}(f) + P\mathcal{H}$, where

$$\begin{aligned}
&P = (x-b)\Sigma_a\partial + \Sigma_b,\\
&L_F^{-1} = y_2\varrho\frac{y_1}{y_1y_2' - y_2y_1'} - y_1\varrho\frac{y_2}{y_1y_2' - y_2y_1'},\\
&\mathcal{H} = c_1x + (c_2 - c_1b).
\end{aligned}$$

Now, the solution is

$$\begin{aligned}
u =& (1-(x-b)\Sigma_a\partial - \Sigma_b)\left(y_2\varrho\frac{y_1f}{y_1y_2' - y_2y_1'} - y_1\varrho\frac{y_2f}{y_1y_2' - y_2y_1'}\right)\\
&+ ((x-b)\Sigma_a\partial + \Sigma_b)(c_1x + (c_2 - c_1b)). \qquad (10)
\end{aligned}$$

If we consider the other set of boundary conditions i.e., $\Sigma_a u = c_1, \Sigma_b\partial u = c_2$ for the given differential operator L, then the solution is

$$\begin{aligned}
u =& (1-\Sigma_a - (x-a)\Sigma_b\partial)\left(y_2\varrho\frac{y_1f}{y_1y_2' - y_2y_1'} - y_1\varrho\frac{y_2f}{y_1y_2' - y_2y_1'}\right)\\
&+(\Sigma_a + (x-a)\Sigma_b\partial)(c_2x + (c_1 - c_2a)). \qquad (11)
\end{aligned}$$

The following example illustrate the solution of a given BVP as in the equation (10).

Example 2.2. Consider a boundary value problem $Lu = f$ with boundary conditions $\Sigma_a\partial u = c_1, \Sigma_b u = c_2$, where $L = \partial^2$ is a differential operator, f is the forcing function. As in Example 2.1, we take $f(x) = e^x$, evaluation points $a = 0, b = 1$, and the boundary data $c_1 = 1, c_2 = 2$. Following the proposed methods, we have

Fundamental system $\{y_1, y_2\} = \{1, x\}$,
Fundamental right inverse operator $L_F^{-1} = x\varrho - \varrho x$,
Projector operator $P = (x-1)\Sigma_0\partial + \Sigma_1$,
Boundary data interpolating function $\mathcal{H} = x + 1$.

Now the solution of the given BVP, computed as in equation (10), is

$$u(x) = 2 + e^x - e.$$

It is verified that $Lu(x) = \partial^2 u(x) = e^x$ and $\Sigma_0\partial u(x) = 1, \Sigma_1 u(x) = 2$.

The following section presents the solution of the given BVP with integral boundary conditions.

2.3. **BVPs with Integral Boundary Conditions.** In this section, we consider a BVP of the following type.

$$Lu = f \text{ with } \Sigma_a u = c_1,\ \Sigma_b \varrho u = c_2, \tag{12}$$

where $L = \partial^2 + a_1\partial + a_0$ is differential operator, $f \in \mathcal{F}$ is a given forcing function, $c_1, c_2 \in \mathbb{R}$ are boundary data, and $\Sigma_b\varrho$ is integrate evaluation operators at b. We want to find $u \in \mathcal{F}$ such that the given BVP (12) is satisfied. Let $\{y_1, y_2\}$ be fundamental system for L, i.e., $y_1, y_2 \in \mathrm{Ker}(L)$. As mentioned in [1, 8] and similar to the process in Section 2.1 and Section 2.2, the solution u is computed as $u = (1-P)L_F^{-1}(f) + P\mathcal{H}$, where

$$\begin{aligned} P &= \left(\frac{2x}{a-b} - \frac{a^2-b^2}{(a-b)^2}\right)\Sigma_a + \left(\frac{2(x-a)}{(a-b)^2}\right)\Sigma_b\varrho, \\ L_F^{-1} &= y_2\varrho\frac{y_1}{y_1y_2' - y_2y_1'} - y_1\varrho\frac{y_2}{y_1y_2' - y_2y_1'}, \\ \mathcal{H} &= \frac{2(a-b)c_1 + c_2}{(a-b)^2}x - \frac{(a^2-b^2)c_1 + 2ac_2}{(a-b)^2}. \end{aligned}$$

Now, the solution is

$$\begin{aligned} u = &\left(1 - \left(\frac{2x}{a-b} - \frac{a^2-b^2}{(a-b)^2}\right)\Sigma_a + - \left(\frac{2(x-a)}{(a-b)^2}\right)\Sigma_b\varrho\right) \\ &\times \left(y_2\varrho\frac{y_1 f}{y_1y_2' - y_2y_1'} - y_1\varrho\frac{y_2 f}{y_1y_2' - y_2y_1'}\right) \\ &+ \left(\left(\frac{2x}{a-b} - \frac{a^2-b^2}{(a-b)^2}\right)\Sigma_a + \left(\frac{2(x-a)}{(a-b)^2}\right)\Sigma_b\varrho\right) \\ &\times \left(\frac{2(a-b)c_1 + c_2}{(a-b)^2}x - \frac{(a^2-b^2)c_1 + 2ac_2}{(a-b)^2}\right). \end{aligned} \tag{13}$$

The following example illustrate the solution of a given BVP as in the equation (13).

Example 2.3. Consider a boundary value problem $Lu = f$ with boundary conditions $\Sigma_a u = c_1, \Sigma_b\varrho u = c_2$, where $L = \partial^2$ is a differential operator, $f(x) = e^x$ is the forcing function, evaluation points $a = 0, b = 1$, and the boundary data $c_1 = 1, c_2 = 2$. Following the proposed methods, we have

Fundamental system $\{y_1, y_2\} = \{1, x\}$,
Fundamental right inverse operator $L_F^{-1} = x\varrho - \varrho x$,
Projector operator $P = (1-2x)\Sigma_0 + 2x\Sigma_1\varrho$,
Boundary data interpolating function $\mathcal{H} = 2x + 1$.

Now the solution of the given BVP, computed as in equation (13), is

$$u(x) = (6 - 2e)x + e^x.$$

It is verified that $Lu(x) = \partial^2 u(x) = e^x$ and $\Sigma_0 u(x) = 1, \Sigma_1 \varrho u(x) = 2$.

The following section presents the solution of the given BVP with Stieltjes boundary conditions (combination of general, differential and integral conditions).

2.4. **BVPs with Stieltjes Boundary Conditions.** In this section, we consider a BVP of the following type.

$$Lu = f \text{ with } b_1 u = c_1,\ b_2 u = c_2, \tag{14}$$

where $L = \partial^2 + a_1\partial + a_0$ is differential operator, $f \in \mathcal{F}$ is a given forcing function, $c_1, c_2 \in \mathbb{R}$ are boundary data, and $b_1, b_2 \in \mathcal{F}[\partial, \varrho]$ are stieltjes boundary operators at a and b respectively. We want to find $u \in \mathcal{F}$ such that the given BVP (14) is satisfied. Let $\{y_1, y_2\}$ be fundamental system for L, i.e., $y_1, y_2 \in \operatorname{Ker}(L)$. As mentioned in [1, 8] and similar to the process in Section 2.1, Section 2.2 and Section 2.3, the solution u is computed as $u = (1 - P)L_F^{-1}(f) + P\mathcal{H}$. The following example illustrates the solution of a given BVP as in the equation (13).

Example 2.4. Consider a BVP of type $Lu = f$ with boundary conditions $\Sigma_0 u + \Sigma_0 \partial u = 2, \Sigma_1 u + \Sigma_1 \varrho u = 1$, where the differential operator $L = \partial^2$, the forcing function $f(x) = e^x$. Now the solution of the given BVP is computed similar to previous examples as

$$u(x) = e^x + 4(e - 1)(x - 1).$$

It is verified that $Lu(x) = \partial^2 u(x) = e^x$ and $\Sigma_0 u + \Sigma_0 \partial u = 2, \Sigma_1 u + \Sigma_1 \varrho u = 1$.

3. **Examples**

Two-point BVPs with Stieltjes boundary conditions appear in various real-world applications, particularly in fields such as physics, engineering, and finance. These problems involve finding a function that satisfies a differential equation and certain boundary conditions. One of the real-world applications (Quantum Particle in a Potential Well) is presented in Example 3.2. Several numerical examples are presented in this section to illustrate the proposed method.

Example 3.1. [8] Consider the one-dimensional problem of a thin rod occupying the interval $(0, a)$ on the x-axis. This is one of the classical examples of the ordinary linear BVPs [25]. We solve

$$\frac{d^2u}{dx^2} = f,\ 0 < x < 1;\ u(0) = \alpha,\ u(1) = \beta, \tag{15}$$

for the temperature $u \in C^\infty[0,1]$, where $f \in C^\infty[0,1]$ is the prescribed source density (per unit length of the rod) of heat and α, β are the prescribed end temperatures.

The operator representation of the given BVP (15) is

$$\begin{aligned} &Lu = f \\ &\Sigma_0 u = \alpha,\ \Sigma_1 u = \beta, \end{aligned}$$

where the differential operator $L = \partial^2$ with $\mathrm{Ker}(L) = \{1, x\}$, and the set of boundary operators is $B = \{\Sigma_0, \Sigma_1\}$ with boundary data $\{\alpha, \beta\}$. The null space projector P is computed as

$$P = (1-x)\Sigma_0 + x\Sigma_1.$$

The fundamental right inverse of L_F^{-1}, computed is given by

$$L_F^{-1} = x\varrho - \varrho x.$$

The right inverse $\mathcal{H}$ of B is computed as follows: For a given fundamental system $\{1, x\}$, boundary operators $\{\Sigma_0, \Sigma_1\}$ with boundary data $\{\alpha, \beta\}$, the operator H calculated as

$$\mathcal{H} = (1, x)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha(1-x) + \beta x.$$

Now, the solution of the given BVP (15) is

$$u = (1-x)\int_0^x \xi f(\xi)\ d\xi + x\int_x^1 (1-\xi)f(\xi)\ d\xi + \alpha(1-x) + \beta x.$$

In particular, $f(x) = e^x$, then the solution is

$$u(x) = e^x + (\beta - \alpha - e + 1)x + \alpha - 1.$$

One can verify that $u''(x) = e^x$ and $u(0) = \alpha, u(1) = \beta$.

Example 3.2. (Quantum Particle in a Potential Well) In quantum mechanics, the Schrödinger equation is fundamental in describing the quantum state of a system. Stieltjes boundary conditions can be particularly useful in modeling quantum systems where the boundary behavior is not just pointwise but involves integral constraints, reflecting more complex interactions or distributions of physical quantities.

Consider a particle in a one-dimensional potential well described by the time independent Schrödinger equation [5]

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=\mathbb{E}\psi(x)$$

with Stieltjes boundary conditions

$$\alpha_1\psi(a)+\beta_1\int_a^b g_1(x)\psi(x)=\gamma_1,$$
$$\alpha_2\psi(b)+\beta_2\int_a^b g_2(x)\psi(x)=\gamma_2,$$

where $\psi(x)$ is the wave function of the particle, $V(x)$ is the potential, $\mathbb{E}$ is the energy eigenvalue, $\hbar$ is the reduced Planck constant, m is the mass of the particle, $g_1(x)$ and $g_2(x)$ are given functions that might represent distributions of some physical quantity, and α_i, β_i and γ_i are constants. These integral boundary conditions can represent physical constraints such as the normalization of the wave function or the presence of distributed sources or sinks at the boundaries. In a potential well with distributed sources, $g_1(x)$ and $g_2(x)$ could represent the strength of these sources over the interval $[a,b]$.

Consider a particle in a potential well with $V(x)=0$ for simplicity, in the interval $[0,1]$, and boundary conditions with $\alpha_1=\alpha_2=\beta_1=\beta_2=1$, $g_1(x)=x$, $g_2(x)=1$ and $\gamma_1=0$, $\gamma_2=1$. Indeed, for $V(x)=0$, the time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=\mathbb{E}\psi(x),\ \text{or} \tag{16}$$

$$\frac{d^2\psi(x)}{dx^2}=-k^2\psi(x), \tag{17}$$

where $k^2=\frac{2m\mathbb{E}}{\hbar^2}$, and the boundary conditions are

$$\begin{aligned}\psi(0)+\int_0^1 x\psi(x)&=0,\\ \psi(1)+\int_0^1 \psi(x)&=1.\end{aligned} \tag{18}$$

Now, the symbolic representation of the BVP (17)–(18) is

$$T\psi=f,\ \text{with}\ b_1\psi=c_1,\ b_2\psi=c_2, \tag{19}$$

where $T=\partial^2+k^2$, $b_1=\Sigma_0+\Sigma_1\varrho x$, $b_2=\Sigma_1+\Sigma_1\varrho$, $c_1=0$, $c_2=1$, and $f=0$. If we take $k=1$, then we have $T=\partial^2+1$. Following the proposed method, we have

$$\psi(x) = \left(\frac{\cos(1) - \sin(1)}{\cos(1) + \sin(1)}\right)\cos(x) + \sin(x).$$

It is also verified that $T\psi = 0$ and $b_1\psi = 0, b_2\psi = 1$.

EXAMPLE 3.3. A BVP for a damped oscillator typically involves a second-order linear differential equation with given boundary conditions [16]. The general form of the equation for a damped oscillator is:

$$m\ddot{x} + c\dot{x} + kx = 0, \tag{20}$$

where m is the mass of the oscillator, c is the damping coefficient, k is the spring constant, $x(t)$ is the displacement as a function of time, $\dot{x}(t)$ is the velocity, $\ddot{x}(t)$ is the acceleration. For a BVP, we need to specify boundary conditions at two different points in time, t_1 and t_2. For example, $x(t_1) = x_1$, $x(t_2) = x_2$.

Consider a damped oscillator with $m = 1$, $c = 3$, and $k = 2$. Suppose the boundary conditions are $x(0) = 2$ and $x(1) = 0$. Then the equation (20) becomes $\ddot{x} + 3\dot{x} + 2x = 0$ with boundary conditions are $x(0) = 2$ and $x(1) = 0$.

The symbolic representation of the BVP for damped oscillator is

$$Tx = f, \text{ with } b_1x = c_1, \; b_2x = c_2, \tag{21}$$

where $T = \partial^2 + 3\partial + 2$, $\partial = \frac{dx}{dt}$, $b_1 = \Sigma_0$, $b_2 = \Sigma_1$, $c_1 = 2$, $c_2 = 0$, and $f = 0$. Following the proposed method, we have

$$x(t) = \frac{2e^{\frac{1}{2}}\left(e^{\left(-1-\frac{t}{2}\right)} - e^{\left(-t-\frac{1}{2}\right)}\right)}{e^{-\frac{1}{2}} - 1}.$$

This is the solution to the boundary value problem for the given damped oscillator. It is also verified that $Tx = f$ and $b_1x = 2, b_2x = 0$.

4. **Conclusion**

This study introduces a symbolic method that effectively addresses Stieltjes boundary conditions for second-order boundary value problems. By leveraging the algebra of integro-differential operators, this approach computes the Green's operator and Green's function at the operator level. The developed algorithm enhances the implementation of these calculations in mathematical software, making it a valuable tool for software like Mathematica, Matlab, Singular, Scilab, Maple, and others. This contribution facilitates more efficient and accurate problem-solving in mathematical and engineering contexts.

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