International Journal of Applied Mathematics

Volume 37 No. 6 2024, 591–596
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v37i6.2

A FORMULA FOR π WITH A NESTED RADICAL

Dubravko Sabolic

University of Zagreb, FER

Unska 3

10000 Zagreb, CROATIA

e-mail: dubravko.sabolic@gmail.com

Abstract

A formula for π is derived, using successive approximation of the area of a wedge of arbitrary central angle cut out from a unit-radius circle. The resulting formula takes the form of 2^n, multiplied by a nested radical of the order n. In general case, this radical splits into two separate nested radicals of the same shape and order, which are symmetric in a way that one of them stems from an arbitrary seed value, $s \in (0, 1)$, while the other is started from the complementary seed, $1 - s$. A reduced formula is also derived, with only one nested radical. The main computational characteristics of both types of the algorithm were briefly discussed, and compared to that of several other widely known π-formulas.

MSC 2020: 11B83, 33B10, 68Q25

Key Words and Phrases: π-formula, algorithm, nested radical

1. Introduction

There have been many formulas for calculation of π devised throughout the history [1]. In this article we derive another one, using a gradual approximation to the area of an arbitrary wedge cut out from a unit-radius circle. Naturally, the formula derived here, belonging to the antiquity era of π approximations, is by far not as computationally efficient as some other already existing ones,

Received: May 7, 2024

to mention Ramanujan-Sato algorithms [2] as typical examples. As today π is known to more than 202 trillion exactly computed and verified decimal places [2], this article merely presents another formula that is easily shown to converge to π. Relatively extensive lists of existing π formulas can be found in [1], [2], and [3].

Trigonometry rules enable calculation of π without calculating neither circumference of a polygon (Archimedes style), nor its area (similarly as in this article). For instance, the following equation is easily derived:

$$\sin\left(\frac{\pi}{2^n+1}\right) = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}, \tag{1}$$

where the square root operation is performed a total of n times. Since sine function of a very small argument is approximately equal to that argument, for large n the value of π can be estimated as:

$$\hat{\pi}(n) = 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}, \tag{2}$$

where the square root operation is performed, again, n times. Similarly, the validity of the following approximation of π is also easily shown:

$$\hat{\pi}(n) = \frac{3}{2}\times 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots\sqrt{3}}}}. \tag{3}$$

2. **The Algorithm**

Observe Figure 1. Define a unit circle with radius $r = 1$, and with the center in the node O. The central angle α defines a circular wedge whose area equals $r^2\pi(\alpha/(2\pi)) = \alpha/2$. That area can be successively approximated in the following way. Note the triangle OAB defined by α. It can be regarded as the roughest approximation to the actual wedge area. If we divide α by 2, the angle bisector crosses the circle in node C. By adding the area of the triangle ABC to that of the OAB, we obtain a better approximation of the wedge area. Next, if we divide the angle $\alpha/2$ by 2, the new bisector gives the new node D and creates a new triangle ACD. The successive approximation is obtained by adding two areas of ACD to the previously updated value. Each further division will generate twice as many new triangles that gradually fill up the area of the wedge, getting closer to the true value yet never overreaching it. Suppose, without a loss of generality, that $0 < \alpha < \pi/2$. Using elementary trigonometric operations with the above-mentioned triangles, it is easy to show that the sum of all of their areas after n divisions performed equals exactly:

$$\hat{P}(n) = 2^{n-1}\sin\left(\frac{\alpha}{2^n}\right). \tag{4}$$

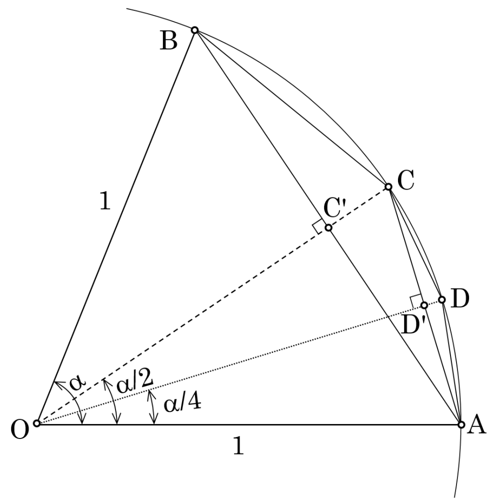

FIGURE 1. Definitions

Next, using the trigonometric formula for the sine of a half-angle and iterating it the appropriate number of times, it is easy to express the sine from (4) as a nested radical so that we can obtain:

$$\hat{P}(n,s) = 2^{n-1}\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{1}{2}\left(1-\sqrt{\cdots\left(1-\sqrt{s}\right)}\right)}\right)}\right)}, \tag{5}$$

or in an abbreviated form:

$$\hat{P}(n,s) = 2^{n-1}R(n,s). \tag{6}$$

Here, $R(n,s)$ stands for the n-th order nested radical generated from the quantity s that we shall call "seed." Next, $s = \cos^2(\alpha/2)$, and the total number of nested square-root operations to perform equals n (the square root in $\sqrt{s}$ is not included in that count).

Suppose that α is some fraction of π, so that $\alpha = \pi/t$, where t is an arbitrary real number larger than 2. Recall that the true area of the wedge equals $\alpha/2$. If we approximate that area with (6), we can try to approximate the π value with the following:

$$\hat{\pi}(n,\alpha) = 2t \times 2^{n-1}\sin\left(\frac{\alpha}{2^n}\right). \tag{7}$$

Since $s = \cos^2(\alpha/2) = \cos^2(\pi/2t)$, it applies: $2t = \pi/\arccos\sqrt{s}$:

$$\hat{\pi}(n,s) = \frac{\pi \times 2^{n-1}}{\arccos\sqrt{s}} R(n,s) = \frac{\pi}{\arccos\sqrt{s}} \hat{P}(n,s). \tag{8}$$

Note that in the right-hand sides of this equation, we operate with true π value. An n grows, we expect the approximations to improve monotonically. It is very easy to show that the first derivative of (4) with respect to n is negative for all n, while the second derivative is positive for all n, meaning that the marginal contributions to the total area are positive, yet diminishing. Therefore:

$$\lim_{n\to\infty} \frac{\hat{\pi}(n,s)}{\pi} = \frac{1}{\arccos\sqrt{s}} \times \lim_{n\to\infty} \hat{P}(n,s) = 1. \tag{9}$$

It follows: $\arccos\sqrt{s} = \lim_{n\to\infty} \hat{P}(n,s)$. However, from elementary trigonometry: $\arccos x = \arcsin\sqrt{1-x^2}$, where x is any real number. It follows: $\arcsin\sqrt{s} = \arccos\sqrt{1-s}$. The next trigonometric identity we will apply is: $\arcsin x + \arccos x = \pi/2$. We can deduce: $\arccos\sqrt{1-s} + \arccos\sqrt{s} = \pi/2$. Now we can utilize our observation about $\arccos\sqrt{s}$ derived from (9) to write down:

$$\pi = 2\lim_{n\to\infty} \Big(\hat{P}(n,s) + \hat{P}(n,1-s)\Big) = 2^n \lim_{n\to\infty} \Big(R(n,s) + R(n,1-s)\Big). \tag{10}$$

This is the expression for the exact value of π with infinitely nested radicals $R(\infty,s)$ and $R(\infty,1-s)$. Note the symmetry: s is a value between 0 and 1 because it is a squared cosine, and $1-s$ is its complement. If we have the nested radicals with $n < \infty$, we can finally derive formula for the n-th order approximation to π with the seed value s:

$$\hat{\pi}(n,s) = 2^n \Big(R(n,s) + R(n,1-s)\Big). \tag{11}$$

We tested actual algorithms for different values of s, and discovered no obvious advantages for using any particular value of s. Yet, it should not be equal to 1 because the algorithm collapses. If we pick $s = 1/2$, both radicals obviously become equal, so that (11) takes the form that is computationally the most efficient one because it needs to perform only a half of operations:

$$\hat{\pi}(n,s) = 2^{n+1} R(n,1/2). \tag{12}$$

3. **Computational characteristics**

The algorithm for calculation of π described above was tested using Matlab, in normal precision (16 digits). It turned out that this algorithm would require much better precision for calculation of bigger number of digits because the computation error accumulates too quickly. However, our goal was not to derive another slow algorithm for the calculation of additional digits of π. We just needed to verify numerically that the formula (11) exhibits a tendency to approach the true value of π as n grows larger. We established that the convergence rate equals that of the Archimedes algorithm, which is expected because both rely on the same principles. It is very modest compared to more modern methods, such as Ramanujan-Sato algorithms [2].

The main practical disadvantage of this algorithm is due to error accumulation—the increase in relative error caused by multiple subtractions of very similar numbers throughout the algorithm. Specifically, the $1-\frac{1}{2}\left(1-\sqrt{\cdots}\right)$ terms in (5), where $\frac{1}{2}\left(1-\sqrt{\cdots}\right)$ becomes increasingly similar to 1 as the algorithm progresses with n. Although our algorithm is theoretically as robust as Archimedes's, this issue makes it practically inferior. With standard 16-digit precision, it cannot produce more than 9 correct digits, despite the algorithm itself being theoretically sound and correctly performing.

A thorough comparative analysis of the computational characteristics of this algorithm can be found in [5] or obtained from the author upon request.

4. **Conclusions**

In this article, a formula for π involving nested radicals, derived from the successive approximation of the area of a wedge with an arbitrary central angle cut out from a unit circle, is presented. Its convergence properties, as well as the challenges posed by numerical error accumulation, were also analyzed and discussed.

References

[1] E.W. Weisstein, Pi Formulas, From: *MathWorld - A Wolfram Web Resource*, Retrieved 25 Jul. 2024; URL: `http://mathworld.wolfram.com/PiFormulas.html`.

[2] Approximations of π, From: *Wikipedia*, Retrieved 25 Jul. 2024; URL: `https://en.wikipedia.org/wiki/Approximations_of_pi`.

[3] D.H. Bailey, A collection of mathematical formulas involving π. Retrieved 25 Jul. 2024; URL: `https://www.davidhbailey.com/dhbpapers/pi-formulas.pdf`.

[4] K.A. Arras, An introduction to error propagation, *Swiss Federal Institute of Technology Lausanne*, Retrieved 25 Jul. 2024; URL: `https://infoscience.epfl.ch/record/97374/files/TR-98-01R3.pdf`.
[5] D. Sabolić and R. Malarić, Another formula for π with an infinitely nested radical (*In Croatian language*), *Poučak*, **80** (2019), 6-26. Retrieved 25 Jul. 2024; URL: `https://hrcak.srce.hr/file/347642`.