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INCIDENCE ENERGY OF SUBGRAPH COMPLEMENTS OF GRAPHS

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Abstract

Incidence energy of the graph G, denoted by $IE(G)$, is defined as the sum of the singular values of its incidence matrix. The notion of incidence energy of subgraph complements of a graph has been proposed in this study. The expression for the sum of singular values and eigenvalues of $[I(G \oplus S)]$ and $[I(G \oplus S)][I(G \oplus S)]^T$, respectively has been obtained. Additionally, we have noted the changes in the trace of $[I(G \oplus S)][I(G \oplus S)]^T$ upon deleting an edge of $G \oplus S$. We have characterized incidence energy of subgraph complement of P_n and $K_{1,n-1}$ and also, obtained some bounds for incidence energy of subgraph complements of a graph. The incidence energy of subgraph complement of complete graph, complete bipartite graph and star has been computed.

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1. **Introduction**

Let $G=(V,E)$ be a finite, simple, undirected graph with the vertex and edge sets being $V=\{v_1,v_2,\dots,v_n\}$ and $E=\{e_1,e_2,\dots,e_m\}$, respectively. Suppose that the adjacency matrix of G is $A=[a_{ij}]$. Then, $|A-\mu I|=0$ is the characteristic equation of G. Let the eigenvalues of G, which are supposed to be in non-increasing order, be $\mu_1,\mu_2,\dots,\mu_n$ of A. According to [4], the energy of G is equal to the sum of its absolute eigenvalues, i.e., $E(G)=\sum\limits_{i=1}^{n}|\mu_i|$. M. Jooyandeh et al. [9] have expanded the notion of graph energy to arbitrary matrices.

The energy of B is the sum of its singular values. The absolute values of an eigenvalue correspond to the singular values of any symmetric matrix. Thus the energy of a graph G is the sum of the singular values of its adjacency matrix. Given a graph G and its singular values $\nu_1,\nu_2,\dots,\nu_n$, the incidence energy of G is given by $\sum\limits_{i=1}^{n}\nu_i$.

Readers are referred to [5, 7, 6, 14, 8, 15, 12, 11, 16] for more results on incidence energy.

The term subgraph complementation was first introduced by M. Kamiński et al. [10] in their study of clique-width of a graph. F. V. Fomin et al. [3] further studied subgraph complements of a graph. A graph that results from complementing every edge in one of the graph G's induced subgraphs is called a subgraph complement of G. Formally, let $G=(V,E)$ be a graph and $S\subseteq V$. $G\oplus S$ represents the subgraph complement of a graph G with regard to S. It is represented as a graph (V,E_S), where, for any two vertices u, $v\in V$, $uv\in E_S$ if and only if one of the following conditions matches [3]:

(1) $u\notin S$ or $v\notin S$ and $uv\in E$.
(2) $u,v\in S$ and $uv\notin E$.

Definition 1.1. Let $G\oplus S$ be subgraph complement of a graph G with respect to S. The subgraph complement adjacency matrix of $G\oplus S$ is an $n\times n$ matrix defined by $A_p(G\oplus S)=a_{ij}$ [18], where

$$a_{ij}=\begin{cases}1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } i\neq j,\\ 1, & \text{if } i=j \text{ and } v_i\in S,\\ 0, & \text{otherwise.}\end{cases}$$

For more information on subgraph complement of graphs, refer [18, 17].

Definition 1.2. Let $G\oplus S$ be subgraph complement of a graph G with respect to S. Then the incidence matrix of the graph $G\oplus S$ is $n\times m$ matrix defined by $I(G \oplus S) = d_{ij}$, where

$$d_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident with edge } e_j \text{ and } v_i \in S, \\ -1, & \text{if } v_i \text{ is not incident with edge } e_j \text{ and } v_i \in S, \\ 2, & \text{if } v_i \text{ is incident with edge } e_j \text{ and } v_i \notin S, \\ 0, & \text{if } v_i \text{ is not incident with edge } e_j \text{ and } v_i \notin S. \end{cases}$$

The incidence characteristic polynomial of subgraph complement of a graph G is defined by $\phi_p\{[I(G \oplus S)][I(G \oplus S)]^T\} = |\mu I - [I(G \oplus S)][I(G \oplus S)]^T|$ and incidence energy of $G \oplus S$ is denoted by $IE(G \oplus S)$, is defined as $\sum\limits_{i=1}^{n} |\nu_i|$, where $\nu_i's$ are singular values of incidence matrix of $G \oplus S$.

Throughout this paper, let I denote $I(G \oplus S)$ and I^T denote $[I(G \oplus S)]^T$.

Lemma 1.1. [2] *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers. If there exist real constants x, y, X and Y such that for each i, $i = 1, 2, \ldots, n$, $x \leq x_i \leq X$ and $y \leq y_i \leq Y$, then*

$$\left| n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i \right| \leq \alpha(n)(X - x)(Y - y),$$

where

$$\alpha(n) = n[\frac{n}{2}] \left(1 - \frac{1}{n}[\frac{n}{2}] \right).$$

The equality holds if and only if $a_i = a_j$ and $b_i = b_j$ for all $1 \leq i, j \leq n$.

Lemma 1.2. [2] *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers. If there exist real constants r and R such that for each i, $i = 1, 2, \ldots, n$, $r x_i \leq y_i \leq R x_i$, then*

$$\sum_{i=1}^{n} y_i^2 + rR \sum_{i=1}^{n} x_i^2 \leq (r + R) \sum_{i=1}^{n} x_i y_i.$$

The equality holds if $r x_i = y_i = R x_i$ for at least one $1 \leq i \leq n$.

Theorem 1.1. [2] *If $a_k > 0$ $(k = 1, 2, \ldots, n)$, then*

$$\left(\sum_{k=1}^{n} a_k \right) \left(\sum_{k=1}^{n} \frac{1}{a_k} \right) \geq n^2.$$

Equality holds if and only if all $a_k's$ are equal.

Theorem 1.2. [2] *If* $a_k > 0$ *for* $k = 1, 2, \ldots, n$ *and* $s = \sum\limits_{k=1}^{n} a_k$, *then*

$$\prod_{k=1}^{n}(1 + a_k) \leq \sum_{k=0}^{n} \frac{s^k}{k!}.$$

Theorem 1.3. [2] *Abel's Inequality: Let* a_i *and* $b_i \geq 0$, $i = 1$ *to* n *be two sequence of real numbers such that* $b_i's$ *are arranged in non-decreasing order and let* $s_k = \sum\limits_{i=1}^{k} a_i$, $k = 1$ *to* n.

If $m = \min\limits_{1 \leq k \leq n} s_k$ *and* $M = \max\limits_{1 \leq k \leq n} s_k$, *then*

$$mb_1 \leq a_1b_1 + a_2b_2 + \ldots + a_nb_n \leq Mb_1.$$

Theorem 1.4. [2] *If each of real numbers* x_i $(i = 1, 2, \ldots, n)$ *is greater than* -1, *and either all are positive or all are negative, then*

$$(1 + x_1)(1 + x_2) \ldots (1 + x_n) \geq 1 + x_1 + x_2 + \ldots x_n.$$

Theorem 1.5. [2] *If* $a = (a_1, a_2, \ldots, a_n)$ *and* $b = (b_1, b_2, \ldots, b_n)$ *are sequences of real numbers and* $0 \leq x \leq 1$, *then*

$$\left(\sum_{k=1}^{n} a_kb_k + x\sum_{i \neq j} a_ib_j\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2 + 2x\sum_{i<j} a_ia_j\right)\left(\sum_{k=1}^{n} b_k^2 + 2x\sum_{i<j} b_ib_j\right).$$

For $x = 0$, *inequality reduces to Cauchy's inequality.*

Theorem 1.6. [2] *Holder's Inequality: If* $a_i \geq 0$, $b_i \geq 0$ *for* $i = 1$ *to* n, *and* $\frac{1}{s} + \frac{1}{t} = 1$ *with* $s > 1$, *then*

$$\left(\sum_{i=1}^{n} a_i^s\right)^{\frac{1}{s}} \left(\sum_{i=1}^{n} b_i^t\right)^{\frac{1}{t}} \geq \sum_{i=1}^{n} a_ib_i,$$

with equality holding if and only if $\alpha a_i^s = \beta b_i^s$ *for* $i = 1$ *to* n, *where* α *and* β *are real non-negative constants with* $\alpha^2 + \beta^2 > 0$.

2. Properties of incidence energy of subgraph complement of graphs

Theorem 2.1. *Let* $G(n, m)$ *be a simple graph. If* $\nu_1, \nu_2, \ldots, \nu_n$ *represent incidence singular values of* I, *then*

(1) $\sum\limits_{i=1}^{n} \nu_i = IE(G \oplus S)$,

(2) $\sum\limits_{i=1}^{n} \nu_i^2 = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} d_{ij}^2$.

Proof. (1) Sum of singular values of I is equal to $IE(G \oplus S)$.

$\sum\limits_{i=1}^{n} \nu_i = \sum\limits_{i=1}^{n} \sqrt{\mu_i} = IE(G \oplus S)$.

(2) The sum of squares of singular values of $I = Trace\{II^T\}$.

$$\text{i.e.,} \quad \sum_{i=1}^{n} \nu_i^2 = Trace\{II^T\}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2.$$

□

Result: Let $G \oplus S$ be a graph on n vertices and m_s edges. Then $IE(G \oplus S) \geq 0$, equality holds if and only if $m_s = 0$.

THEOREM 2.2. *Let $G \oplus S$ be a $r-$regular graph on n vertices and m_s edges. Then $\sum\limits_{i=1}^{n} \nu_i^2 = m_s|S| + 4r(n - |S|)$.*

THEOREM 2.3. *Let $G \oplus S$ be its subgraph complement of graph G with respect to S. Then $\sum\limits_{i=1}^{n} \mu_i = a(|S|) + b(|S| + 4) + c(|S| + 8)$.*

Proof. Let $e = \{uv\}$ where $u, v \in V(G \oplus S)$. Then in any graph $G \oplus S$, we can calculate the trace of II^T as follows:

- Case 1. If an edge $e = G \oplus S$ is incident to both vertices which are in $\langle S \rangle$, i.e., u belong to S, and v belong to S, then we have
$$Trace[II^T] = |S|.$$
- Case 2. If an edge $e \in G \oplus S$ is incident to $u \in S$ and $v \in \langle V/S \rangle$, then we have
$$Trace[II^T] = |S| + 4.$$
- Case 3. If an edge $e \in G \oplus S$ is not incident to any vertex in $\langle S \rangle$, i.e., u and v belong to V/S, then we have
$$Trace[II^T] = |S| + 8.$$

Let a, b and c be the number of edges as in Case 1, Case 2 and Case 3, respectively. Then,

$$\sum_{i=1}^{n} \mu_i = a(|S|) + b(|S|+4) + c(|S|+8).$$

□

Corollary 2.1. *Let $G \oplus S$ be a subgraph complementation graph with respect to S and let e be an edge deleted from $G \oplus S$. Then,*

(1) *If $e \in G \oplus S$ is incident to both vertices which are in $\langle S \rangle$, then we have*

$$\sum_{i=1}^{n} \mu_i = (a-1)(|S|) + b(4+|S|) + c(8+|S|).$$

(2) *If $e \in G \oplus S$ is incident to one vertex in $\langle S \rangle$ and one vertex in $\langle V/S \rangle$, then we have*

$$\sum_{i=1}^{n} \mu_i = a(|S|) + (b-1)(4+|S|) + c(8+|S|).$$

(3) *If $e \in G \oplus S$ is not incident to any vertex in $\langle S \rangle$, then we have*

$$\sum_{i=1}^{n} \mu_i = a(|S|) + b(4+|S|) + (c-1)(8+|S|).$$

Theorem 2.4. *Let $P_n \oplus S_1$ and $P_n \oplus S_2$ be subgraph complementation of path P_n with respect to set S_1 and S_2. Let μ_i, σ_i, $i = 1$ to n be the eigenvalues of $[P_n \oplus S_1][P_n \oplus S_1]^T$ and $[P_n \oplus S_2][P_n \oplus S_2]^T$, respectively. If S_1 consists of pendant vertex and S_2 consists of middle vertex with $|S_1| = |S_2| = 1$, then*

$$\sum_{i=1}^{n} \mu_i \geq \sum_{i=1}^{n} \sigma_i.$$

Proof.

Case 1. Let $|S_1| = 1$, which is an end vertex. Then one column will be with one entry 1, one entry -1 and two entries 2. Remaining $n-2$ columns will be with one entry -1 and two entries 2.

Therefore,

$$\begin{aligned}\sum_{i=1}^{n} \mu_i &= 1(1)(1)^2 + 1(1)(2)^2 + (n-2)(1)(-1)^2 + (n-2)(2)(2)^2 \\ &= 5 + 9(n-2) \\ &= 9n - 13.\end{aligned}$$

Case 2. Let $|S_2| = 1$. Then two columns will be with one entry 1 and one entry 2. Remaining $n-3$ columns will be with one entry -1 and two entries 2.

Therefore,

$$\begin{aligned}\sum_{i=1}^{n} \sigma_i &= 2(1)(1)^2 + 2(1)(2)^2 + (n-3)(1)(-1)^2 + (n-3)(2)(2)^2 \\ &= 10 + 9(n-3) \\ &= 9n - 17.\end{aligned}$$

From Case 1 and Case 2, it is clear that $9n - 13 \geq 9n - 17, \forall n \geq 2$.

$\Longrightarrow \sum\limits_{i=1}^{n} \mu_i \geq \sum\limits_{i=1}^{n} \sigma_i$. □

3. Bounds for incidence energy of subgraph complement of graphs

Theorem 3.1. *Let $G \oplus S_1$ and $H \oplus S_2$ be two $r-$regular subgraph complements of graph G and H respectively on n vertices. Let m_{s_1} and m_{s_2} denote the number of edges of $G \oplus S_1$ and $H \oplus S_2$ respectively. If ν_i, $i = 1$ to n are singular values of $G \oplus S_1$ and ν_i', $i = 1$ to n are singular values of $H \oplus S_2$, then*

$$\sum_{i=1}^{n} \nu_i \nu_i' \leq \sqrt{(m_{s_1}|S_1| + 4r(n - |S_1|))(m_{s_2}|S_2| + 4r(n - |S_2|))}.$$

Proof. On substituting $s = t = 2$, $a_i = \nu_i$ and $b_i = \nu_i'$, where $i = 1$ to n in Theorem (1.6), we have

$$\sum_{i=1}^{n} \nu_i \nu_i' \leq \left(\sum_{i=1}^{n} \nu_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \nu_i'^2\right)^{\frac{1}{2}}.$$

We know that, for r-regular graph of order n $\sum\limits_{i=1}^{n} \nu_i^2 = (m_s|S| + 4r(n - |S|)$.

$$\Longrightarrow \sum_{i=1}^{n} \nu_i \nu_i' \leq \sqrt{(m_{s_1}|S_1| + 4r(n - |S_1|))(m_{s_2}|S_2| + 4r(n - |S_2|))}.$$

□

Theorem 3.2. *For singular values ν_i, $i = 1$ to n of $I(G \oplus S)$, we have*

$$IE(G \oplus S) \geq \frac{n^2}{\left(\sum\limits_{i=1}^{n} \dfrac{1}{\nu_i}\right)}.$$

Proof. On substituting $a_i = \nu_i$ in Theorem (1.1), we have

$$\left(\sum_{i=1}^{n} \nu_i\right)\left(\sum_{i=1}^{n} \frac{1}{\nu_i}\right) \geq n^2$$

$$\implies IE(G \oplus S) \geq \frac{n^2}{\left(\sum\limits_{i=1}^{n} \frac{1}{\nu_i}\right)}.$$

Equality holds if and only if all $\nu_i' s$ are equal. □

THEOREM 3.3. *If ν_i, $i = 1$ to n are the singular values of I, then*

$$\prod_{i=1}^{n}(1+\nu_i) \leq \sum_{i=0}^{n} \frac{(IE(G \oplus S))^i}{i!}.$$

Proof. On substituting $a_k = \nu_i$ in Theorem (1.2), we have

$$\prod_{i=1}^{n}(1+\nu_i) \leq \sum_{i=0}^{n} \frac{s^i}{i!}.$$

We know that $s = \sum\limits_{i=1}^{n} \nu_i = IE(G \oplus S)$.

Therefore,

$$\prod_{i=1}^{n}(1+\nu_i) \leq \sum_{i=0}^{n} \frac{(IE(G \oplus S))^i}{i!}.$$

□

THEOREM 3.4. *If μ_i and ν_i, $i = 1$ to n represent the eigenvalues and singular values of II^T and I respectively, then*

$$m\mu_1 \leq \nu_1\mu_1 + \nu_2\mu_2 + \ldots \nu_n\mu_n \leq M\mu_1.$$

Proof. On substituting $a_i = \nu_i$, $b_i = \mu_i$ in Theorem (1.3), where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$ and let

$$s_i = \nu_i + \nu_2 + \ldots + \nu_i \ (i = 1, 2, \ldots, n).$$

If $m = \min\limits_{1 \leq i \leq n} s_i$ and $M = \max\limits_{1 \leq i \leq n} s_i$, then

$$m\mu_1 \leq \nu_1\mu_1 + \nu_2\mu_2 + \ldots + \nu_n\mu_n \leq M\mu_1.$$

□

Theorem 3.5. *If ν_i, $i = 1$ to n are the singular values of I, then*

$$\prod_{i=1}^{n}(1+\nu_i) > 1 + \sum_{i=1}^{n}\nu_i.$$

Proof. Let $x_i = \nu_i$ in Theorem (1.4). Then, since all $\nu_i's$ are positive we have,

$$\prod_{i=1}^{n}(1+\nu_i) > 1 + \sum_{i=1}^{n}\nu_i.$$

□

Theorem 3.6. *If ν_i, $i = 1$ to n are the singular values of I, then*

$$IE(G \oplus S) \leq \frac{\sqrt{3n\left(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} d_{ij}^2 + 2\sqrt{(m_{s_1}|S_1| + 4r(n-|S_1|))(m_{s_2}|S_2| + 4r(n-|S_2|))}\right)}}{2}.$$

Proof. On substituting $a_i = 1$, $b_i = \nu_i$ and $x = 1$ in Theorem (1.5) , we have

$$\left(\sum_{i=1}^{n}\nu_i + \sum_{i=1}^{n}\nu_i\right)^2 \leq \left(\sum_{i=1}^{n}1 + 2\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\nu_i^2 + 2\sum_{i<j}\nu_i\nu_j\right),$$

$$\left(2\sum_{i=1}^{n}\nu_i\right)^2 \leq 3n\left(\sum_{i=1}^{n}\nu_i^2 + 2\sum_{i<j}\nu_i\nu_j\right).$$

From Theorem (3.1), we have

$\sum\limits_{i=1}^{n}\nu_i\nu_j \leq \sqrt{(m_{s_1}|S_1| + 4r(n-|S_1|))(m_{s_2}|S_2| + 4r(n-|S_2|))}.$

$(2IE(G \oplus S))^2 \leq 3n\left(\sum\limits_{i=1}^{n}\nu_i^2 + 2\sqrt{(m_{s_1}|S_1| + 4r(n-|S_1|))(m_{s_2}|S_2| + 4r(n-|S_2|))}\right)$

$(2IE(G \oplus S))^2 \leq 3n\left(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} d_{ij}^2 + 2\sqrt{(m_{s_1}|S_1| + 4r(n-|S_1|))(m_{s_2}|S_2| + 4r(n-|S_2|))}\right)$

$2IE(G \oplus S) \leq \sqrt{3n\left(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} d_{ij}^2 + 2\sqrt{(m_{s_1}|S_1| + 4r(n-|S_1|))(m_{s_2}|S_2| + 4r(n-|S_2|))}\right)}$

$IE(G \oplus S) \leq \dfrac{\sqrt{3n\left(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} d_{ij}^2 + 2\sqrt{(m_{s_1}|S_1| + 4r(n-|S_1|))(m_{s_2}|S_2| + 4r(n-|S_2|))}\right)}}{2}.$

□

Theorem 3.7. *For connected graph $G \oplus S$,*

$$\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2} \leq IE(G \oplus S) \leq \sqrt{n\sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2}.$$

Proof. On substituting $x = 0$ in Theorem (3.6), we get Cauchy-Schwarz inequality.

On replacing a_i and b_i by 1 and $|\nu_i|$, we get

$$\left(\sum_{i=1}^{n} |\nu_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\nu_i|^2\right)$$

$$\implies (IE(G \oplus S))^2 \leq n\sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2.$$

$$\implies IE(G \oplus S) \leq \sqrt{n\sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2}.$$

Also,

$$\left(\sum_{i=1}^{n} |\nu_i|\right)^2 \geq \sum_{i=1}^{n} |\nu_i|^2$$

$$\implies (IE(G \oplus S))^2 \geq \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2$$

$$\implies IE(G \oplus S) \geq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2}.$$

$IE(G \oplus S) = \sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} d_{ij}^2}$, whenever $G = K_n$ with $|S| = n$. $\square$

Theorem 3.8. *For connected graph $G \oplus S$,*

$$IE(G \oplus S) \geq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2 + n(n-1)P^{\frac{2}{n}}},$$

where $P = |II^T|$.

P r o o f. Use arithmetic-geometric mean inequality,

$$\sum_{i\neq j}|\nu_i||\nu_j| \geq n(n-1)P^{\frac{2}{n}}.$$

Now,

$$\implies (IE(G\oplus S))^2 = \left(\sum_{i=1}^{n}|\nu_i|\right)^2$$

$$\implies (IE(G\oplus S))^2 = \sum_{i=1}^{n}|\nu_i|^2 + \sum_{i\neq j}|\nu_i||\nu_j|$$

$$\implies IE(G\oplus S) \geq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}d_{ij}^2 + n(n-1)P^{\frac{2}{n}}}.$$

□

THEOREM 3.9. *Let ν_i, $i=1$ to n be the singular values of I. Then,*

$$IE(G\oplus S) \geq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}d_{ij}^2 - \frac{n^2}{4}\left(|\nu_1|-|\nu_n|\right)^2}.$$

P r o o f. Let $|\nu_i|$, $i=1$ to n be eigenvalues of I which are arranged in non-increasing order. On replacing $x_i = y_i = |\nu_i|$, $x = y = |\nu_n|$ and $X = Y = |\nu_1|$ in Lemma (1.1) and noting $\alpha(n) \leq \frac{n^2}{4}$,

$$\left| n\sum_{i=1}^{n}|\nu_i|^2 - \sum_{i=1}^{n}|\nu_i|^2 \right| \leq \frac{n^2}{4}\left(|\nu_1|-|\nu_n|\right)^2.$$

But,

$$\sum_{i=1}^{n}|\nu_i| = IE(G\oplus S) \quad \text{and} \quad \sum_{i=1}^{n}|\nu_i|^2 = \sum_{i=1}^{n}\sum_{j=1}^{m}d_{ij}^2.$$

Now,

$$n\left(\sum_{i=1}^{n}\sum_{j=1}^{m}d_{ij}^2\right) - (IE(G\oplus S))^2 \leq \frac{n^2}{4}\left(|\nu_1|-|\nu_n|\right)^2$$

$$\implies IE(G\oplus S) \geq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}d_{ij}^2 - \frac{n^2}{4}\left(|\nu_1|-|\nu_n|\right)^2}.$$

□

THEOREM 3.10. *Let ν_i, $i = 1$ to n be the singular values of I which are arranged in non-increasing order. Then*

$$IE(G \oplus S) \geq \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} d_{ij}^2 + n|\nu_n||\nu_1|}{(|\nu_n| + |\nu_1|)}.$$

Proof. Let $|\nu_1| \geq |\nu_2| \geq \ldots \geq |\nu_n| > 0$ be a non-increasing arrangement of singular values of I. By substituting $x_i = 1$, $y_i = |\nu_i|$, $r = |\nu_n|$, $R = |\nu_1|$ in Lemma (1.2), we procure

$$\sum_{i=1}^{n} |\nu_i|^2 + |\nu_n||\nu_1| \sum_{i=1}^{n} 1 \leq (|\nu_n| + |\nu_1|) \sum_{i=1}^{n} |\nu_i|.$$

But,

$$\sum_{i=1}^{n} |\nu_i|^2 = \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}^2 \quad \text{and} \quad \sum_{i=1}^{n} |\nu_i| = IE(G \oplus S).$$

Therefore,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2 + n|\nu_n||\nu_1| \leq (|\nu_n| + |\nu_1|) IE(G \oplus S).$$

This implies

$$IE(G \oplus S) \geq \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} d_{ij}^2 + n|\nu_n||\nu_1|}{(|\nu_n| + |\nu_1|)}.$$

□

4. Incidence energy of subgraph complement of some families of graphs

THEOREM 4.1. *For the complete graph K_n, with $|S| = k$,*

$$IE(K_n \oplus S) = (n-2) + \sqrt{(k-n)^2 - 4\{(k-n)(k^3-1) - 1\}}.$$

Proof. The incidence subgraph complement matrix of $[I_{K_n}][I_{K_n}]^T$ is given by

$$[I_{K_n}][I_{K_n}]^T = \begin{bmatrix} [m_s I + N(J-I)]_k & 2(3-n)J_{k \times n-k} \\ 2(3-n)J_{n-k \times k} & 4[(n-2)I + J]_{n-k} \end{bmatrix}_n,$$

where m_s= number of edges in $K_n \oplus S$= $\binom{n}{2} - \binom{k}{2}$ and $N = m_s - 4(n - |S|)$.

Consider

$$[[I_{K_n}][I_{K_n}]^T - \mu I] = \begin{bmatrix} (m_s - \mu)I + N(J - I) & 2(3 - n)J \\ 2(3 - n)J & [4(n - 2) - \mu]I + 4J \end{bmatrix}_n . \quad (1)$$

On changing R_i by $R_i - R_{i+1}$, $i = 1, 2, \ldots, k - 1, k + 1, \ldots, n - 1$, we have C_i by $C_i + C_{i-1} + \cdots + C_1$, $i = k, k-1, \ldots, 2, 1$ and C_j by $C_j + C_{j-1} + \cdots + C_1$, $j = n - k, n - k - 1, \ldots, k + 1$ in (1), and we obtain $det(D)$.

By expanding $det(D)$ along the rows from R_1 to R_{k-1} and R_{k+1} to R_{n-k-1}, we obtain

$$(m_s - \mu - N)^{k-1}(4(n - 2) - \mu)^{n-k-1} \begin{vmatrix} (k - 1)N + m_s - \mu & 2(3 - n)(n - k) \\ 2k(3 - n) & 4(n - 1) - \mu + 4(n - k - 1) \end{vmatrix} .$$

Therefore,

$\phi\{[I_{K_n}][I_{K_n}]^T\} = (m_s - \mu - N)^{k-1}(4(n - 2) - \mu)^{n-k-1}(\mu^2 + \mu(N(1 - k) - m_s + 4k + 8(1 - n)) + 8(knN - kN - nN + N + m_s n - m_s) - 4(k^2N + km_s + k(n - k)(3 - n)^2 - kN)$.

Let μ_1 and μ_2 be the roots of the equation $(\mu^2 + \mu(N(1 - k) - m_s + 4k + 8(1 - n)) + 8(knN - kN - nN + N + m_s n - m_s) - 4(k^2N + km_s + k(n - k)(3 - n)^2 - kN) = 0$.

Thus, the singular values of $[I_{K_n}]$ is given by, $\sqrt{m_s - N}$ with multiplicity of $(k - 1)$, $2\sqrt{n - 2}$ with multiplicity of $(n - k - 1)$, $\sqrt{\mu_1}$ and $\sqrt{\mu_2}$ with each multiplicity of 1.

Therefore,

$$\begin{aligned} IE(K_n \oplus S) &= \sqrt{m_s - N}(k - 1) + 2\sqrt{n - 2}(n - k - 1) + \sqrt{\mu_1}(1) + \sqrt{\mu_2}(1) \\ &= \sqrt{m_s - N}(k - 1) + 2\sqrt{n - 2}(n - k - 1) + \sqrt{\mu_1} + \sqrt{\mu_2}. \end{aligned}$$

□

Result: Let $G = K_n$.

- If $|S| = n - 1$, then $IE(G) + IE(\langle S \rangle) \geq IE(G \oplus S)$.
- If $|S| \leq n - 1$, then $IE(G) + IE(\langle S \rangle) \leq IE(G \oplus S)$.

THEOREM 4.2. *Let $K_{m,n}$ be complete bipartite graph with $m < n$ and $S = \{v_i / i = 1, \ldots, m\}$. Then*

$$IE(K_{m,n} \oplus S) = \sqrt{m_s - N}(m - 1) + 2\sqrt{n - 2}(n - 1) + \sqrt{\mu_1} + \sqrt{\mu_2}.$$

Proof. Let $K_{m,n}$ be complete bipartite graph of order $m+n$ with $m < n$ and $m \geq 3$, where $S = \{1, 2, \ldots, m\}$. Then,

$$[I_{K_{m,n}}][I_{K_{m,n}}]^T = \begin{bmatrix} [NJ + (m_s - N)I]_m & 2(2 - m)J_{m \times n} \\ 2(2 - m)J_{n \times m} & 4mI_n \end{bmatrix}_{m+n} ,$$

is the incidence matrix of $[K_{m,n} \oplus S][K_{m,n} \oplus S]^T$. Here, m_s is the number of edges in $K_{m,n} \oplus S$ and $N = 2(m+n-2) - m_s$.

Since $[I_{K_{m,n}}][I_{K_{m,n}}]^T$ is real and symmetric, geometric multiplicity is equal to algebraic multiplicity of each eigenvalue μ. So, result can be proved by showing $[I_{K_{m,n}}][I_{K_{m,n}}]^T Z = \mu Z$ for certain vector Z.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $m+n$ partitioned conformally with $[I_{K_{m,n}}][I_{K_{m,n}}]^T$.

Consider

$$[[I_{K_{m,n}}][I_{K_{m,n}}]^T - \mu I] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} [NJ + (m_s - N - \mu)I]X + [2(2-m)J]Y \\ [2(2-m)J]X + [4m - \mu]IY \end{bmatrix}_{m+n}. \tag{2}$$

Case 1. Let $X = E_j = e_1 - e_j$, $j = 2$ to m and $Y = 0_n$. Using equation (2), $[NJ + (m_s - N - \mu)IE_j + [2(2-m)J]0_n = (m_s - N - \mu)I]E_j$, then $\mu = m_s - N$ is the eigenvalue with multiplicity of at least $m-1$ as there are $m-1$ independent vectors of the form E_j.

Case 2. Let $X = 0_m$ and $Y = E_j$, $j = 2$ to n. Using equation (2), $[2(2-m)J]0 + [4m - \mu]IE_j$, then $\mu = 4m$ is the eigenvalue with multiplicity of at least $n-1$ as there are $n-1$ independent vectors of the form E_j.

Case 3. Let $X = 1_m$ and $Y = \frac{2m(m-2)}{4m-\mu} 1_n$, where μ is any root of the equation,

$$\mu^2 - \mu(mN - N + m_s + 4m) + 4m(mN - N + m_s - n(2-m)^2) = 0. \tag{3}$$

From equation (2),

$$[NJ + (m_s - N - \mu)I]1_m + [2(2-m)J]\frac{2m(m-2)}{4m-\mu}1_n = mN + m_s - N - \mu + 2n(2-m)\frac{2m(m-2)}{4m-\mu}1_m$$
$$= \frac{\mu^2 - \mu(mN - N + m_s + 4m) + 4m(mN - N + m_s - n(2-m)^2)}{4m - \mu} 1_m$$

So, μ_1, μ_2 are the roots of the equation (3) which are the eigenvalues with multiplicity of at least one.

Thus, the singular values of $[I_{K_{m,n}}]$ are given by
$\sqrt{m_s - N}$ with multiplicity of $m-1$, $2\sqrt{m}$ with multiplicity of $n-1$, $\sqrt{\mu_1}$ with multiplicity of 1 and $\sqrt{\mu_2}$ with multiplicity of 1.

Therefore,

$$\begin{aligned} IE(K_{m,n} \oplus S) &= \sqrt{m_s - N}(m-1) + 2\sqrt{m}(n-1) + \sqrt{\mu_1}(1) + \sqrt{\mu_2}(1) \\ &= \sqrt{m_s - N}(m-1) + 2\sqrt{n-2}(n-1) + \sqrt{\mu_1} + \sqrt{\mu_2}. \end{aligned}$$

$\square$

Corollary 4.1. *Let* $K_{1,n-1}$ *be star graph with* $S = \{v_1\}$, v_1 *be the vertex with degree* $n-1$. *Then*

$$IE(K_{1,n-1} \oplus S) = 4(n-2) + \sqrt{3+n}.$$

Proof. By using $m = 1$ and $n = n-1$ in Theorem (4.2), we obtain the singular values of $[I_{K_{1,n}}]$ are 4, $3+n$, 0 with multiplicities $n-2$, 1 and 1 respectively. Therefore,

$$\begin{aligned} IE(K_{1,n-1} \oplus S) &= 4(n-2) + 0(1) + \sqrt{3+n}(1) \\ &= 4(n-2) + \sqrt{3+n}. \end{aligned}$$

□

Theorem 4.3. *Let* $K_{1,n-1}$ *be star graph with* $S = \{v_2\}$, *where* v_2 *be pendant vertex.*

Then, $IE(K_{1,n-1} \oplus S) = 4(n-2) + \sqrt{3+n}$.

Proof. Let

$$\begin{bmatrix} 4(n-1)_1 & 2(3-n)_1 & 4J_{1\times n-2} \\ 2(3-n)_1 & (n-1)_1 & -2J_{1\times n-2} \\ 4J_{n-2\times 1} & -2J_{n-2\times 1} & 4I_{n-2} \end{bmatrix}_n$$

be the incidence subgraph complement matrix of $K_{1,n-1}$ with $\langle S\rangle$ as pendant vertex.

$$|[I_{K_{1,n-1}}][I_{K_{1,n-1}}]^T - \mu I| = \begin{bmatrix} 4(n-1) - \mu I & 2(3-n) & 4J \\ 2(3-n) & (n-1) - \mu I & -2J \\ 4J & -2J & 4I - \mu I \end{bmatrix}_n . \quad (4)$$

On using $R_i \longrightarrow R_i - R_{i-1}$, $i = 4, 5, \dots, n$, $C_i \longrightarrow C_i + C_{i+1} + C_{i+2} + \dots + C_{n-1} + C_n$, $i = 1, 2, \dots, n$ and then expanding along the rows from R_4 to R_n in equation (4), we have

$$\begin{aligned} |[I_{K_{1,n-1}}][I_{K_{1,n-1}}]^T - \mu I| &= (4-\mu)^{n-3} \begin{vmatrix} 6n-6-\mu & 2n-2 & 4n-8 \\ -3n+9-\mu & -n+3-\mu & -2n+4 \\ 6-\mu & 2-\mu & 4-\mu \end{vmatrix} \\ &= (4-\mu)^{n-3}(\mu^3 + \mu^2(1-5n) + 4\mu(4n-3)) \\ &= \mu(4-\mu)^{n-3}(\mu^2 + \mu(1-5n) + 4(4n-3)). \end{aligned}$$

Singular values are 2 with multiplicity of $n-3$, 0 with multiplicity of one,

$$\sqrt{\frac{5n-1+\sqrt{(n-1)(25n-49)}}{2}} \text{ and } \sqrt{\frac{5n-1-\sqrt{(n-1)(25n-49)}}{2}},$$

each with multiplicity one,

$$IE(K_{1,n-1} \oplus S) = 2(n-3)+$$
$$\sqrt{\frac{5n-1+\sqrt{(n-1)(25n-49)}}{2}} + \sqrt{\frac{5n-1-\sqrt{(n-1)(25n-49)}}{2}}.$$

□

Corollary 4.2. *Let* $S_1 = \{v_1\}$ *and* $S_2 = \{v_2\}$ *be two induced sets of* $K_{1,n-1}$, *where* v_1 *be the vertex with degree* $n-1$ *and* v_2 *be the vertex with degree* 1. *Then,* $IE(K_{1,n-1} \oplus S_1) \leq IE(K_{1,n-1} \oplus S_2)$.

Proof. Proof follows from Corollary 4.1 and Theorem 4.3. □

Conjecture: Let G be a tree of order n.

- Assuming $\langle S \rangle$ connected means $G \oplus S$ is disconnected, we have $IE(G) + IE(\langle S \rangle) \geq IE(G \oplus S)$.
- Assuming $\langle S \rangle$ disconnected means $G \oplus S$ is connected, we have $IE(G) + IE(\langle S \rangle) \leq IE(G \oplus S)$.

5. Conclusion

In this paper, we have introduced the concept of incidence energy of subgraph complements of graphs which depends on the subgraph complement set that we choose. We have proved some properties and obtained good bounds for incidence energy of subgraph complements of graphs. Also, computed incidence energy of complete graph, complete bipartite graph and star graph. In the last section, we have a conjecture relating incidence energy of graph, incidence energy of $\langle S \rangle$ that we choose and the incidence energy of subgraph complements of graphs.

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