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COMPUTATIONAL APPROACHES FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATIONS

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Abstract

In this research paper, we propose two accurate and efficient techniques for solving nonlinear Volterra integral equations. One proposed techniques based on the expansion of unknown function into a series of the bases functions of Chebyshev wavelets of the second kind, whereas second technique is based on the expansion of the bases function of Haar wavelets. To illustrate the simplicity, accuracy and efficacy of both the techniques, some numerical examples have been performed.

MSC 2020: 65N99

Key Words and Phrases: Chebyshev wavelets of the second kind; Haar wavelets; integral equations; test examples

1. Introduction

Integral equations are fundamental tools for mathematical modeling in both pure and applied numerical analysis, with broad applications across various fields. They play a key role in modeling physical, biological, and chemical phenomena, as well as in scientific and engineering disciplines like astrophysics.

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In physics, they model electromagnetic wave propagation and quantum scattering. In engineering, they are used for structural analysis, signal processing, and control systems design. In biology, integral equations describe population dynamics and disease spread. They are also applied in finance for option pricing and risk assessment, in acoustics for sound propagation and noise control, in environmental science for pollution dispersion modeling, and in medical imaging for tomography and optical imaging. These equations are essential for solving problems where the current state depends on past influences or spans a range of values.

In this paper, the authors present a direct method for solving integral equations, applicable to both continuous and discontinuous solutions, by utilizing the Chebyshev wavelet basis in Galerkin equations. Furthermore, they introduce an approach for solving Volterra-type integral equations that leverages the operational matrix of integration (OMI) for Chebyshev wavelets, as discussed in [1]. A numerical algorithm employing both Chebyshev wavelet and Haar wavelet has been developed to solve stochastic Ito Volterra integral equations in [2, 3]. A second Chebyshev wavelet method has been discussed for solving Fredholm and Volterra integral equations in [4]. Additionally, the product operational matrix and operational matrix of integration have been derived. The Haar wavelet introduced in [5] are quite frequently used for solving differential and integral equation. In this paper, the authors have provided an overview of the literature on Haar wavelets and the solutions of integral as well as differential equations using these wavelets in [6]. The author in [9] discussed the application of integral equations with the use of wavelet methods. An efficient Hermite wavelet method for the numerical solution of nonlinear Fredholm integral equations has been proposed in[8]. In [7] , a simple and efficient Haar wavelet method for the numerical solution of nonlinear first kind Volterra integral equations has been presented. Numerical solution of second kind Fredholm, Volterra and mixed Volterra integral equations with the help of legendre multi wavelets and Haar wavelets has been developed in [10, 12]. In [11], the authors investigated the use of Bernstein polynomials for numerically solving nonlinear Volterra, Fredholm, and Hammerstein integral equations. A spectral collocation method based on Jaccobi wavelets has been implemented for the numerical solution of Volterra integral equations of third kind in [13]. The authors in [14] formulated a new computational technique for solving stochastic Ito Volterra integral equations by using Chebyshev wavelets. In this research paper, the authors introduced a efficient technique for solving second kind Fredholm integral equations by using Fibonacci wavelets in [15]. For the investigation of numerical integration, an effective technique has been presented utilizing the Chebyshev wavelets in[16].

2. Chebyshev wavelets of the second kind

In the last few decades, wavelets based numerical techniques have been used extensively for the solution of various problems of science, engineering and technology. Wavelets are a group of functions generated by dilating and translating a fundamental function known as the mother wavelet. When the dilation parameter a and the translation parameter b changes continuously, they form the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{1/2}\psi\Big(\frac{t-b}{a}\Big), \ a, b \in R, \ a \neq 0. \tag{1}$$

The second kind wavelets $\psi_(n, m) = \psi(k, n, m, t)$ have four arguments; k denotes the positive integer, $n = 1, 2, 3, 4, ..., 2^{(k-1)}$; the degree of second kind Chebyshev polynomials is denoted by m and t represents the normalized time. It is defined on the interval $[0, 1)$ as follows:

$$\psi_{n,m}(t) = \begin{cases} 2^{k/2}\tilde{U_m}(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where

$$\tilde{U_m}(t) = \sqrt{\frac{2}{\pi}}U_m(t), \tag{3}$$

$m = 0, 1, 2, 3, ..., M-1$ and M is a fixed integer. In relation given by (2) is for orthonormality. Here $U_m(t)$ represents the second kind Chebyshev polynomials of degree m and are orthogonal with respect to the weight function $\omega(t) = \sqrt{(1-t^2)}$ on the interval $[-1, 1]$, and the following recursive formula has been satisfied:

$$U_0(t) = 1, U_1(t) = 2t, U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \ m = 1, 2, 3, 4, ...$$

It is noted that in case of Chebyshev wavelets of second kind, the weight function has to be translated and dilated as $\omega_n(t) = \omega(2^k t - 2n + 1)$. Here we find out the integral of second kind Chebyshev wavelets functions with $k = 1$ and $M = 6$. For this, the six basis functions in $[0, 1]$ are as follows:

$$\begin{cases} \psi_{1,0}(x) = \frac{2}{\sqrt{\pi}}, \\ \psi_{1,1}(x) = \frac{2}{\sqrt{\pi}}(4x-2), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(16x^2-16x+3), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(64x^3-96x^2+40x-4), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(256x^4-512x^3+336x^2-80x+5), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(1024x^5-2560x^4+2304x^3-896x^2+140x-6). \end{cases} \tag{4}$$

3. Haar wavelets and its properties

Haar functions consists of an orthogonal set of switched rectangular waveforms, each with function potentially having different amplitudes. The Haar wavelet is a sequence of rescaled square- shaped functions that collectively forms a wavelet family or basis. The Haar wavelet function $h_i(x)$ is defined as follows:

$$[\ h_i(x) = \begin{cases} 1, & \alpha \le x < \beta \\ -1, & \beta \le x < \gamma\ ] \\ 0, & \text{elsewhere} \end{cases}, \tag{5}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, $m = 2^j$, and $j = 0, 1, 2, 3, \ldots, J$. The level of resolution is denoted by J. The translation parameter is represented by the integer $k = 0, 1, 2, \ldots, m-1$. The index i is calculated as $i = m + k + 1$. The minimum value of i is 2, and the maximum value of i is 2^{j+1}.

The collocation points are determined as

$$x_l = \frac{(l-0.5)}{2M}, \quad l = 1, 2, 3, \ldots, 2M. \tag{4}$$

The operational matrix P, which is $2M \times 2M$, is calculated as below:

$$P_{1,i}(x) = \int_0^{x_l} h_i(x)\,dx \tag{6}$$

$$P_{n+1,i}(x) = \int_0^{x} P_{n,i}(x)\,dx, \quad n = 1, 2, 3, \ldots \tag{7}$$

From (5), we obtain:

$$P_{i,1}(x) = \begin{cases} x-\alpha, & \alpha \le x < \beta \\ \gamma - x, & \beta \le x < \gamma \\ 0, & \text{elsewhere.} \end{cases}$$

4. **Computational analysis**

In this section, we will discuss about the computational and error analysis of the proposed scheme and perform some numerical experiments.

Example 1: Consider the nonlinear integral equation

$$\frac{e^{2t}-1}{2} = \int_0^t y^2(s)\, ds,$$

with the initial condition $y(0) = 1$. The exact solution to the problem is:

$$y(t) = e^t.$$

First, substitute $y^2(s) = w(s)$, then the integral equation becomes

$$\frac{e^{2t}-1}{2} = \int_0^t w(s)\, ds. \tag{7}$$

Assume that

$$w(s) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\, \phi_{n,m}(s). \tag{8}$$

From (7), we obtain

$$\frac{e^{2t}-1}{2} = \int_0^t \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\, \phi_{n,m}(s) \right) ds. \tag{9}$$

This implies

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_0^t \phi_{n,m}(s)\, ds = \frac{e^{2t}-1}{2}. \tag{10}$$

After discretizing (10), we obtain a system of linear equations that can be solved using any classical method. Solving this system of equations provides the wavelet coefficients. For $k = 1$ and $M = 8$, the wavelet coefficients are:

$$\begin{cases} c_{1,0} = 2.7151619253, \\ c_{1,1} = 1.3080719230, \\ c_{1,2} = 0.3204236219, \\ c_{1,3} = 0.0527492111, \\ c_{1,4} = 0.0065386655, \\ c_{1,5} = 0.0006492345, \\ c_{1,6} = 0.0000532401, \\ c_{1,7} = 0.0000034075. \end{cases}$$

Substituting the values of wavelet coefficients into (8), we obtain the solution $w(s)$. The required solution of the given problem is obtained from the relation $y^2(s) = w(s)$.

s	Exact Solutions	Chebyshev wavelet solutions	Haar wavelet solutions
0.0625	1.0644944589	1.0644945731	1.0320792724
0.1875	1.2062302494	1.2062302051	1.2286399260
0.3125	1.3668379411	1.3668379633	1.3403284204
0.4375	1.5488302986	1.5488302817	1.5647857417
0.5625	1.7550546569	1.7550546766	1.7326794342
0.6875	1.9887374695	1.9887374333	1.9991750723
0.8125	2.2535347872	2.2535349036	2.2338391938
0.9375	2.5535894580	2.5535886057	2.5591332753

TABLE 1. Comparison of numerical solutions of Example 1.

s	Absolute Errors Chebyshev	Absolute Errors Haar
0.0625	1.1426×10^{-7}	3.2415×10^{-2}
0.1875	4.4250×10^{-8}	2.2410×10^{-2}
0.3125	2.2225×10^{-8}	2.6510×10^{-2}
0.4375	1.6887×10^{-8}	1.5955×10^{-2}
0.5625	1.9649×10^{-8}	2.2375×10^{-2}
0.6875	3.6241×10^{-8}	1.0438×10^{-2}
0.8125	1.1641×10^{-7}	1.9696×10^{-2}
0.9375	8.5226×10^{-7}	5.5438×10^{-3}

TABLE 2. Comparison of absolute errors of Example 1.

Table 1 shows the comparison of exact solutions and solutions obtained with the help of Chebyshev wavelets and Haar wavelets. Table 2 shows

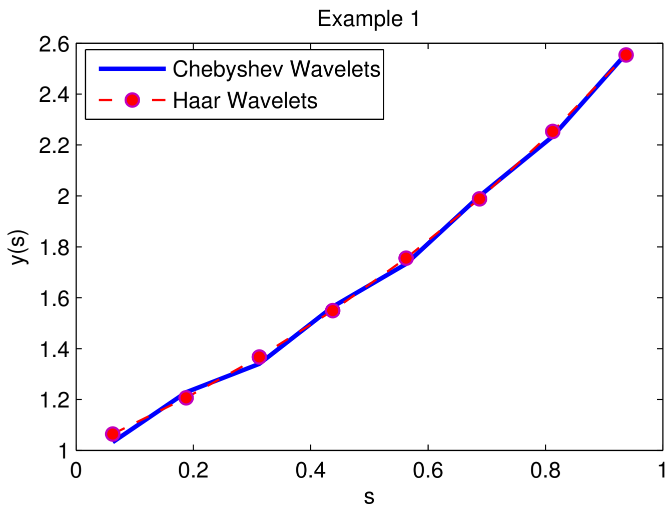

FIGURE 1. Comparison of Chebyshev Wavelets and Haar Wavelets

the comparison of absolute errors obtained by Chebyshev wavelets and Haar wavelets. Figure 1 shows the comparison of numerical results obtained by Chebyshev wavelets and Haar wavelets of Example 1.

Example 2: Consider the nonlinear integral equation

$$e^t - 1 = \int_0^t \sqrt{y(s)}\, ds,$$

with the initial condition $y(0) = 1$. The exact solution to the problem is:

$$y(s) = e^{2s}.$$

Table 3 shows the comparison of exact solutions and solutions obtained with the help of Chebyshev wavelets and Haar wavelets. Table 4 shows the comparison of absolute errors obtained by Chebyshev wavelets and Haar wavelets. Figure 2 shows the comparison of numerical results obtained by Chebyshev wavelets and Haar wavelets of Example 2.

Example 3: Consider the nonlinear integral equation

$$\frac{t^2}{2} = \int_0^t \sqrt{\cos^{-1} y(s)}\, ds,$$

s	Exact Solutions	Chebyshev wavelet solutions	Haar wavelet solutions
0.0625	1.1331484530	1.1331484543	1.0648410191
0.1875	1.4549914146	1.4549914139	1.5273531660
0.3125	1.8682459574	1.8682459578	1.7791872019
0.4375	2.3988752939	2.3988752935	2.4901343595
0.5625	3.0802168489	3.0802168495	2.9638162680
0.6875	3.9550767229	3.9550767215	4.0697059666
0.8125	5.0784190371	5.0784190428	4.9257623566
0.9375	6.5208191203	6.5208190681	6.6639815624

Table 3. Comparison of numerical solutions of Example 2.

s	Absolute Errors Chebyshev	Absolute Errors Haar
0.0625	1.3290×10^{-9}	6.8307×10^{-2}
0.1875	6.5331×10^{-10}	7.2362×10^{-2}
0.3125	4.1634×10^{-10}	8.9059×10^{-2}
0.4375	4.0128×10^{-10}	9.1259×10^{-2}
0.5625	5.9198×10^{-10}	1.1640×10^{-1}
0.6875	1.3840×10^{-9}	1.1463×10^{-1}
0.8125	5.6315×10^{-9}	1.5266×10^{-1}
0.9375	5.2214×10^{-8}	1.4316×10^{-1}

Table 4. Comparison of absolute errors of Example 2.

with the initial condition $y(0) = 1$. The exact solution to the problem is:

$$y(s) = \cos(s^2).$$

s	Exact Solutions	Chebyshev wavelet solutions	Haar wavelet solutions
0.0625	0.9999923706	0.9999923706	0.9999995231
0.1875	0.9993820826	0.9993820826	0.9988553324
0.3125	0.9952354167	0.9952354167	0.9968731023
0.4375	0.9817376814	0.9817376814	0.9759570850
0.5625	0.9503597595	0.9503597595	0.9604377334
0.6875	0.8903621651	0.8903621651	0.8695027811
0.8125	0.7898964223	0.7898964223	0.8194465684
0.9375	0.6379937585	0.6379937585	0.5910176044

Table 5. Comparison of numerical solutions of Example 3.

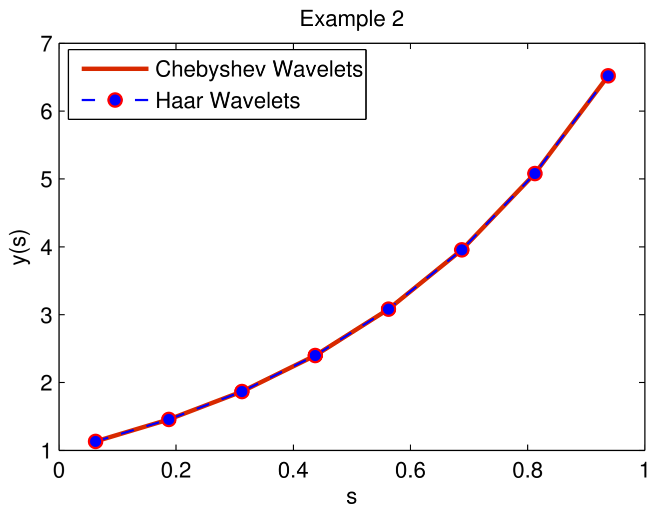

FIGURE 2. Comparison of solutions obtained by two Wavelets

s	Absolute Errors (Chebyshev)	Absolute Errors (Haar)
0.0625	0	7.1525×10^{-6}
0.1875	1.1102×10^{-16}	5.2675×10^{-4}
0.3125	4.4409×10^{-16}	1.6377×10^{-3}
0.4375	8.8818×10^{-16}	5.7806×10^{-3}
0.5625	2.4425×10^{-15}	1.0078×10^{-2}
0.6875	3.9968×10^{-15}	2.0859×10^{-2}
0.8125	9.1038×10^{-15}	2.9550×10^{-2}
0.9375	1.6875×10^{-14}	4.6976×10^{-2}

TABLE 6. Comparison of absolute errors of Example 3.

Example 4: Consider the nonlinear integral equation

$$\frac{t^2}{2} = \int_0^t \sqrt{\sqrt{y(s)}}\, ds,$$

with the initial condition $y(0) = 0$. The exact solution of the problem is:

$$y(s) = s^4.$$

Substitute $\sqrt{\sqrt{y(s)}} = w(s)$ in the given integral equation. Table 7 shows the comparison of exact solutions and solutions obtained with the help of Chebyshev wavelets and Haar wavelets of Example 4. Table 8 shows the comparison of absolute errors obtained by Chebyshev wavelets and Haar wavelets

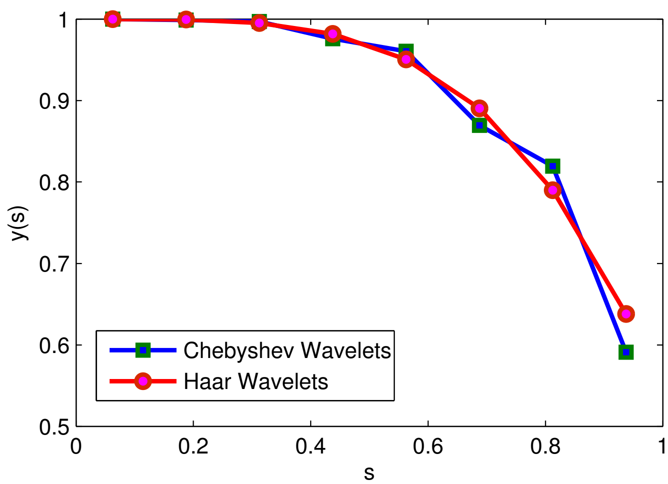

FIGURE 3. Comparison of solutions found by two Wavelets

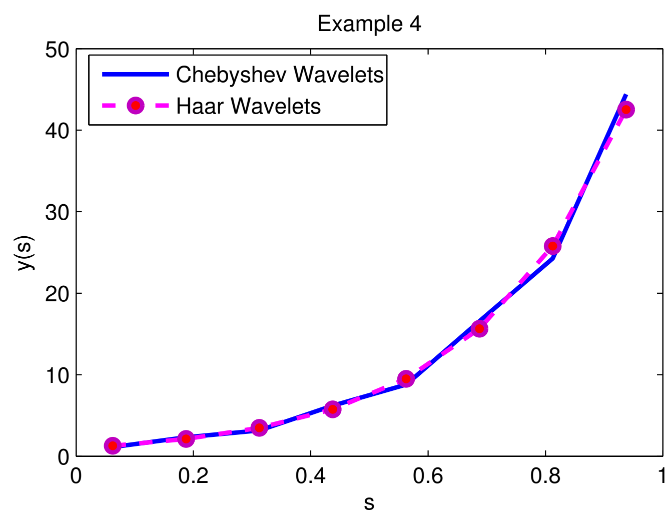

FIGURE 4. Comparison of numerical results obtained by Wavelets

of Example 4. Figure 4 shows the comparison of numerical results obtained by Chebyshev wavelets and Haar wavelets.

5. Conclusion

In this study, we have presented two numerical techniques based on the basis functions of Chebyshev wavelets of the second kind and the Haar wavelets for solving some special type Volterra integral equations arising in various applications of sciences and engineering. From the above numerical study, it is concluded that Chebyshev wavelets of the second kind gives better results in comparison to Haar wavelets. The numerical results are much closer to the exact solutions, when number of collocation points may increase.

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