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A NOTE ON THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME FOR THE ELLIPTIC-TELEGRAPH IDENTIFICATION PROBLEM WITH DIRICHLET BOUNDARY CONDITION

Ahmad Al-Hammouri [1] **, Allaberen Ashyralyev** [2,3,4]

[1] Department of Mathematics, College of Science and Technology
Irbid National University
Irbid, 21110, JORDAN
e-mail: alhammouri.math@gmail.com

[2] Department of Mathematics, Bahcesehir University
34353, Istanbul, TURKIYE

[3] Peoples' Friendship University of Russia (RUDN University)
Ul Mikluko Maklaya 6, Moscow 117198, RUSSIAN FEDERATION

[4] Institute of Mathematics and Mathematical Modeling
Almaty, 050010, KAZAKHSTAN
e-mail: allaberen.ashyralyev@bau.edu.tr

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Abstract

In the present paper, the second-order of accuracy difference scheme (DS) for the approximate solution of a source identification problem (SIP) for the multidimensional elliptic-telegraph equations is constructed. Theorem on stability estimates for the solution of this DS and second-order difference derivatives is presented. Numerical results are given for the solutions of the one-dimensional SIP for the elliptic-telegraph equation.

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1. **Introduction**

The SIP for partial differential equations is essential in modeling a wide range of biological, physical, engineering, and sociological processes (see [1]-[3]). The telegraph equation, in particular, is significant for addressing key issues like signal analysis and wave propagation. For example, the study in [3] focuses on developing, analyzing, and implementing stable numerical methods for solving second-order hyperbolic equations. Extensive research has examined both local and nonlocal challenges associated with hyperbolic-elliptic differential and difference equations (for a comprehensive overview, see references [4]-[21]).

Studies [4] and [5] thoroughly investigated the stability of local and nonlocal problems for hyperbolic-elliptic differential equations, introducing first- and second-order of accurate DSs. These studies also provided stability estimates for the solution of DSs and both first- and second-order of difference derivatives.

Additionally, paper [6] examined the mixed elliptic-hyperbolic equation within a rectangular domain, exploring periodic conditions and a nonlocal problem proposed by A. A. Desin. This work proved theorems on the convergence of constructed series within the class of regular solutions and established the stability of these solutions. Similarly, in [7], the existence of traveling wave solutions for a hyperbolic-elliptic system of partial differential equations was demonstrated using the geometric theory of singular perturbations. Reference [19] addressed a linear hyperbolic equation with nonlocal integral boundary conditions, establishing stability conditions under a specific matrix norm.

The study of space dependent SIPs for elliptic-telegraph differential equations has drawn significant attention from researchers (see [22]-[25] and related references).

The main goal of this study is to construct and investigate the second-order of accuracy stable DS for the approximate solution of the SIP

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-\sum\limits_{r=1}^{n}\left(a_r(x)u_x(t,x)\right)_{x_r}+\delta u(t,x) \\ =p(x)+f(t,x), x=(x_1,...,x_n)\in\Omega,\ 0<t<1, \\ -u_{tt}(t,x)-\sum\limits_{r=1}^{n}\left(a_r(x)u_x(t,x)\right)_{x_r}+\delta u(t,x) \\ =p(x)+g(t,x),\ x=(x_1,...,x_n)\in\Omega,\ -1<t<0, \\ u\left(t,x\right)=0, x\in S, -1\leq t\leq 1, \\ u(0,x)=\varphi(x),\ u_t(0^+,x)=u_t(0^-,x), \\ u(-1,x)=\psi(x), u(1,x)=\xi(x),\ x\in\overline{\Omega} \end{cases} \tag{1}$$

for the multidimensional elliptic-telegraph equations. Here, Ω is the unit open cube in $n-$ dimensional Euclidean space $\Re^n$ with boundaries defined by $0<y_k<1$, for $1\leq k\leq n$, and S is the boundary of $\overline{\Omega}=\Omega\cup S$. Assume that $\delta>0$ is a suitably large constant, $\alpha_r(x)\geq a_0>0$, and $f(t,x)(x\in\Omega,\ 0<t<1)$, $g(t,x)(x\in\Omega,\ -1<t<0), \varphi(x), \psi(x), \xi(x)(\ x\in\overline{\Omega})$, and $\alpha_r(x)(1\leq r\leq n, x\in\Omega)$ are sufficiently smooth functions that meet all the necessary compatibility conditions to guarantee that the SIP (1) has a smooth solution $u(t,x)$ and $p(x)$.

The second-order of accuracy DS for the approximate solution of the SIP (1) for the multidimensional elliptic-telegraph equations is constructed. Theorem on stability estimates for the solution of this DS and second-order difference derivatives is established. Numerical results are presented for the solutions of the one-dimensional SIP for the elliptic-telegraph equation.

2. **Stability of DS**

In this section, we study the second-order of accuracy DS in t for the approximate solution of SIP (1). The discretization of SIP (1) is carried out in two stages. In the first stage, we introduce the grid spaces

$$\overline{\Omega_h}=\{x=x_r=h_1j_1,...,h_nj_n, j=(j_1,...,j_n)\, 0\leq j_r\leq N_r,$$

$$N_rh_r=1, r=1,...,n\}\,, \Omega_h=\overline{\Omega_h}\cap\Omega, S_h=\overline{\Omega_h}\cap S$$

and introduce the Hilbert space $L_{2h} = L_2\left(\overline{\Omega_h}\right)$ of the grid functions $\phi^h(x) = \{\phi(h_1 j_1, ..., h_n j_n)\}$ defined on $\overline{\Omega_h}$ equipped with the norm

$$\left\|\phi^h\right\|_{L_{2h}} = \left(\sum_{x\in\Omega_h} \left|\phi^h(x)\right|^2 h_1...h_n\right)^{1/2}.$$

Moreover, we introduce the difference operator A_h^x given by the formula

$$A_h^x u^h(x) = -\sum_{r=1}^{n}(\alpha_r(x)u^h_{x_r})_{x_r,j_r} + \delta u^h(x), \tag{2}$$

where A_h^x is known as self-adjoint and positive-definite operator in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of the difference operator A_h^x, we arrive at the following SIP

$$\begin{cases} u^h_{tt}(t,x) + \alpha u^h_t(t,x) + A_h^x u^h(t,x) \\ = p^h(x) + f^h(t,x),\ x\in\Omega_h,\ 0<t<1, \\ -u^h_{tt}(t,x) + A_h^x u^h(t,x) \\ = p^h(x) + g^h(t,x),\ x\in\Omega_h,\ -1<t<0, \\ u^h(0,x) = \varphi^h(x), u^h_t(0^+,x) = u^h_t(0^-,x), \\ u^h(-1,x) = \psi^h(x), u^h(1,x) = \xi^h(x), x\in\overline{\Omega}_h. \end{cases} \tag{3}$$

In the second stage, we replace SIP (3) with a second-order of accuracy DS

$$\begin{cases} \frac{u^h_{k+1}(x)-2u^h_k(x)+u^h_{k-1}(x)}{\tau^2} + \alpha\frac{u^h_{k+1}(x)-u^h_{k-1}(x)}{2\tau} \\ +\frac{1}{2}A_h^x\left(u^h_{k+1}(x)+u^h_{k-1}(x)\right) = p^h(x) + f^h_k(x), f^h_k(x) \\ = f^h(t_k,x), 1\le k\le N-1, x\in\Omega_h, \\ \frac{u^h_{k+1}(x)-2u^h_k(x)+u^h_{k-1}(x)}{\tau^2} + A_h^x u^h_k(x) = p^h(x) + g^h_k, \\ g^h_k(x) = g(t_k,x), -N+1\le k\le -1, x\in\Omega_h, \\ u^h_0(x) = \varphi^h(x), -3u^h_0(x) + 4u^h_1(x) - u^h_2(x) \\ = 3u^h_0(x) - 4u^h_{-1}(x) + u^h_{-2}(x), u^h_{-N}(x) = \psi^h(x), \\ u^h_N(x) = \xi^h(x), x\in\overline{\Omega}_h. \end{cases} \tag{4}$$

THEOREM 2.1. *Suppose that* $\alpha \ge 4, \left(\frac{\alpha}{2}+1\right)^2 \ge \delta \ge \left(\frac{\alpha}{2}\right)^2 + 1$. *Then, for the solution* $\left\{\left\{u^h_k(x)\right\}_{-N}^N, p^h(x)\right\}$ *of problem (4) the following stability estimates hold:*

$$\max_{-N\le k\le N}\left\|u_k\right\|_{L_{2h}} + \left\|\left(A_h^x\right)^{-1}p^h\right\|_{L_{2h}} \tag{5}$$

$$\leq M_1(\alpha,\delta)\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\xi^h\right\|_{L_{2h}}\right.$$

$$\left.+\max_{-N+1\leq k\leq -1}\left\|g_k^h\right\|_{L_{2h}}+\max_{1\leq k\leq N-1}\left\|f_k^h\right\|_{L_{2h}}\right],$$

$$\max_{-N+1\leq k\leq N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}}+\max_{-N+1\leq k\leq N}\left\|u_k^h\right\|_{W_{2h}^2} \tag{6}$$

$$\leq M_2(\alpha,\delta)\left[\left\|\varphi^h\right\|_{W_{2h}^2}+\left\|\psi^h\right\|_{W_{2h}^2}+\left\|\xi^h\right\|_{W_{2h}^2}+\left\|g_{-1}^h\right\|_{L_{2h}}+\left\|f_1^h\right\|_{L_{2h}}\right.$$

$$\left.+\max_{-N+1\leq k\leq -2}\left\|\frac{1}{\tau}\left(g_k^h-g_{k-1}^h\right)\right\|_{L_{2h}}+\max_{2\leq k\leq N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right]$$

hold, where $M_1(\alpha,\delta), M_2(\alpha,\delta)$ *do not depend on* f_k^h, $1\leq k\leq N-1$, g_k^h, $-N+1\leq k\leq -1$, $\varphi^h(x)$, $\psi^h(x)$ *and* $\xi^h(x)$.

P r o o f. DS (4) can be written in the abstract form

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_{k-1}}{2\tau}+\frac{1}{2}Au_{k+1}+\frac{1}{2}Au_{k-1}=p+f_k, \\ f_k=f(t_k), 1\leq k\leq N-1, \\ \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=p+g_k, \\ g_k=g(t_k), -N+1\leq k\leq -1, \\ u_0=\varphi, -3u_0+4u_1-u_2=3u_0-4u_{-1}+u_{-2}, \\ u_{-N}=\psi, u_N=\xi \end{cases} \tag{7}$$

for the approximate solution of the space dependent SIP (3) in a Hilbert space $H=L_{2h}$ with self-adjoint and positive-definite operator $A=A_h$ defined by formula (2). Here, $f_k=f_k^h(x), g_k=g_k^h(x)$ are given abstract mesh functions and $u_k=u_k^h(x)$ is unknown abstract mesh function defined on $\overline{\Omega_h}$ and $p=p^h(x)$ is the element of L_{2h}. Therefore, estimates (5) and (6) follow from the following Theorem 2 on the stability inequalities for the solution of DS (7), and the Theorem 3 on the coercivity stability estimate for the solution of the elliptic difference problem generated by (2) in L_{2h}. □

THEOREM 2.2. *Suppose that* $\varphi,\psi,\xi\in D(A)$ *and* $\alpha\geq 4, \left(\frac{\alpha}{2}+1\right)^2\geq\delta\geq\left(\frac{\alpha}{2}\right)^2+1$. *Then, for the solution of DS (7), the stability inequalities*

$$\max_{-N\leq k\leq N}\|u_k\|_H+\left\|A^{-1}p\right\|_H \tag{8}$$

$$\leq M_3(\alpha,\delta)\left[\|\varphi\|_H+\|\psi\|_H+\|\xi\|_H\right.$$

$$+\max_{-N+1\le k\le -1}\left\|A^{-1/2}g_k\right\|_H+\max_{1\le k\le N-1}\left\|A^{-1/2}f_k\right\|\Big]_H,$$

$$\max_{-N+1\le k\le N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H+\max_{-N\le k\le N}\|Au_k\|_H+\|p\|_H \tag{9}$$
$$\le M_4(\alpha,\delta)\left[\|A\varphi\|_H+\|A\psi\|_H+\|A\xi\|_H+\|g_{-1}\|_H+\|f_1\|_H\right.$$
$$\left.+\sum_{k=-N+1}^{-2}\|g_k-g_{k-1}\|_H+\sum_{k=2}^{N-1}\|f_k-f_{k-1}\|_H\right]$$

hold, where $M_3(\alpha,\delta), M_4(\alpha,\delta)$ *do not depend on* $f_k, 1\le k\le N-1, g_k, -N+1\le k\le -1$, φ,ψ *and* ξ.

THEOREM 2.3. [26] *For the solution of the elliptic differential problem*

$$\begin{cases} A_h^x u^h(x)=\mu^h(x), x\in\Omega_h, \\ u^h(x)=0, x\in S_h \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n}\left\|u^h_{x_r x_r}\right\|_{L_2(\mathbf{\Omega})}\le M||\mu^h||_{L_2(\mathbf{\Omega})}.$$

Here M *does not depend on* h *and* μ^h.

3. Numerical results

The numerical methods for obtaining the approximate solutions of partial differential equations play an important role in applied mathematics. In this section, we will use the second-order of accuracy DS to approximate the solution of a simple test problem. We will apply a procedure of modified Gauss elimination method to solve the problem. Finally, the error analysis of second-order of accuracy DS will be given.

The SIP

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2}+2\frac{\partial u(t,x)}{\partial t}-\frac{\partial^2 u(t,x)}{\partial x^2}=p(x)-\sin x, \\ x\in(0,\pi), t\in(0,1), \\ -\frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial^2 u(t,x)}{\partial x^2}+u(t,x)=p(x)-\sin x, \\ x\in(0,\pi), t\in(-1,0), \\ u(0,x)=\sin x, u(-1,x)=e\sin x,\ u(1,x)=e^{-1}\sin x, \\ x\in[0,\pi], u(t,0)=0, u(t,\pi)=0, t\in[-1,1] \end{cases} \tag{10}$$

for the elliptic-telegraph equation with the Dirichlet condition is considered, where

$$f(t,x) = -\sin x, g(t,x) = -\sin x.$$

The exact solution pair of this problem is

$$(u(t,x), p(x)) = \left(e^{-t}\sin x, \sin x\right), \ 0 \le x \le \pi, -1 \le t \le 1.$$

Here, we denote the set $[-1,1]_\tau \times [0,\pi]_h$ of all grid points

$$\begin{aligned}[-1,1]_\tau \times [0,\pi]_h &= \{(t_k, x_n) : t_k = k\tau, -N \le k \le N, \\ &\quad N\tau = 1, x_n = nh, 0 \le n \le M, Mh = \pi\}.\end{aligned}$$

The solution of SIP (10) can be written as

$$u(t,x) = \omega(t,x) + q(x), \tag{11}$$

where $q(x)$ is the solution of the problem

$$-q''(x) = p(x), \ 0 < x < \pi, q(0) = q(\pi) = 0 \tag{12}$$

for the numerical solution of SIP (10), we construct the second-order of accuracy DS in t

$$\begin{cases} \dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + 2\dfrac{u_n^{k+1} - u_n^{k-1}}{2\tau} \\ -\frac{1}{2}\left(\dfrac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} + \dfrac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2}\right) \\ = p_n - \sin x_n, 1 \le k \le N-1, 1 \le n \le M-1, \\ -\dfrac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} - \dfrac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \\ = p_n - \sin x_n, -N+1 \le k \le -1, 1 \le n \le M-1, \\ -3u_n^0 + 4u_n^1 - u_n^2 = 3u_n^0 - 4u_n^{-1} + u_n^{-2}, 0 \le n \le M, \\ u_n^0 = 0, u_n^{-N} = (e-1)\sin x_n, u_n^N = (e^{-1}-1)\sin x_n, \\ 0 \le n \le M, u_0^k = u_M^k = 0, -N \le k \le N, \end{cases} \tag{13}$$

where u_n^k and p_n denote the numerical approximations of $u(t,x)$ at $(t,x) = (t_k, x_n)$ and $p(x)$ at $x = x_n$, respectively.

The solution of DS (13) can be found in the form

$$u_n^k = \omega_n^k + q_n, n = 0, 1, ..., M, k = -N, ..., N.$$

Using (12), we get

$$\begin{aligned} p_n &= \frac{\omega_{n+1}^{2N+1} - 2\omega_n^{2N+1} + \omega_{n-1}^{2N+1}}{h^2} \\ &-e^{-1}\frac{\sin x_{n+1} - 2\sin x_n + \sin x_{n-1}}{h^2}, 1 \le n \le M-1. \end{aligned}$$

In the third step, using (11), we get

$$-\frac{q_{n+1}-2q_n+q_{n-1}}{h^2}=p_n, 1\leq n\leq M-1, q_0=q_M=0.$$

Now, we will obtain $\left\{\left\{\omega_n^k\right\}_{k=-N}^{N}\right\}_{n=0}^{M}$ as solution of nonlocal boundary value problem

$$\begin{cases} \frac{\omega_n^{k+1}-2\omega_n^k+\omega_n^{k-1}}{\tau^2}+2\frac{\omega_n^{k+1}-\omega_n^{k-1}}{2\tau} \\ -\frac{1}{2}\left(\frac{\omega_{n+1}^{k+1}-2\omega_n^{k+1}+\omega_{n-1}^{k+1}}{h^2}+\frac{\omega_{n+1}^{k-1}-2\omega_n^{k-1}+\omega_{n-1}^{k-1}}{h^2}\right) \\ =p_n-\sin x_n, 1\leq k\leq N-1, 1\leq n\leq M-1, \\ -\frac{\omega_n^{k+1}-2\omega_n^k+\omega_n^{k-1}}{\tau^2}-\frac{\omega_{n+1}^k-2\omega_n^k+\omega_{n-1}^k}{h^2} \\ =p_n-\sin x_n, -N+1\leq k\leq -1, 1\leq n\leq M-1, \\ -3\omega_n^0+4\omega_n^1-\omega_n^2=3\omega_n^0-4\omega_n^{-1}+\omega_n^{-2}, 0\leq n\leq M, \\ \omega_n^0=0, \omega_n^{-N}=(e-1)\sin x_n, \omega_n^N=(e^{-1}-1)\sin x_n, \\ 0\leq n\leq M, \omega_0^k=\omega_M^k=0, -N\leq k\leq N. \end{cases} \tag{14}$$

Here, ω_k^n denotes the numerical approximation of $\omega(t,x)$ at (t_k,x_n). For obtaining the solution of DS (14), we can write it in the matrix form as

$$\begin{cases} A\omega_{n+1}+B\omega_n+C\omega_{n-1}=F_n, 1\leq n\leq M-1, \\ \omega_0=\omega_M=0, \end{cases} \tag{15}$$

where A,B,C are $(2N+1)\times(2N+1)$ square, and $F_n, \omega_s, s=n,n\pm 1$ are $(2N+1)\times 1$ column matrices

$$A=C=\begin{bmatrix} 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & t & 0 & t & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 & t & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & t & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & t & 0 & t \\ 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & b & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & b & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & b & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)}$$

with $A_{N+i,i} = b$ and $A_{i,N+i-1} = A_{i,N+i+1} = t$ for $i = 2, ..., N$.

$$B = \begin{bmatrix} -1 & 0 & \cdot & 0 & 0 & 1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & d & c & a & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 & d & c & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & c & a & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & d & c & a \\ 0 & 0 & \cdot & -1 & 4 & -6 & 4 & -1 & \cdot & 0 & 0 & 0 \\ g & j & \cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & j & g & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & g & j & g & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & 1 & 0 & 0 & \cdot & 0 & 0 & -1 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

with $B_{N+i,i} = j, B_{N+i,i-1} = B_{N+i,i+1} = g$, and $B_{i,N+i-1} = d, B_{i,N+i} = c, B_{i,N+i+1} = a$ for $i = 2, ..., N$.

$$F_n = \begin{bmatrix} (1-e^1)\sin x_n \\ -\sin x_n \\ \cdot \\ -\sin x_n \\ 0 \\ -\sin x_n \\ \cdot \\ -\sin x_n \\ (1-e^{-1})\sin x_n \end{bmatrix}_{(2N+1)\times 1}, \omega_s = \begin{bmatrix} \omega_s^1 \\ \omega_s^2 \\ \cdot \\ \omega_s^{N-1} \\ \omega_s^N \\ \omega_s^{N+1} \\ \cdot \\ \omega_s^{2N} \\ \omega_s^{2N+1} \end{bmatrix}_{(2N+1)\times 1}$$

Here,

$$a = \tfrac{1}{\tau^2} + \tfrac{1}{\tau} + \tfrac{1}{h^2}, b = -\tfrac{1}{h^2}, c = -\tfrac{2}{\tau^2}, t = -\tfrac{1}{2h^2},$$
$$d = \tfrac{1}{\tau^2} - \tfrac{1}{\tau} + \tfrac{1}{h^2}, g = -\tfrac{1}{\tau^2}, z = \tfrac{2}{\tau^2} + \tfrac{2}{h^2}.$$

For the solution of the matrix equation (15), we use the modified Gauss elimination method. We seek a solution of the matrix equation (15) by the following form

$$\omega_n = \alpha_{n+1}\omega_{n+1} + \beta_{n+1},\ n = M-1, ..., 1, 0, \tag{16}$$

where α_n $(1 \le n \le M-1)$ are $(2N+1)\times(2N+1)$ square matrices and β_n $(1 \le n \le M-1)$ are $(2N+1)\times 1$ column vectors, calculated as

$$\begin{cases} \alpha_{n+1} = -Q_n A,\ \beta_{n+1} = Q_n(DF_n - C\beta_n), \\ Q_n = (B + C\alpha_n)^{-1},\ n = 1, 2, ..., M-1. \end{cases}$$

Here, D and α_1 are identity $(2N+1)\times(2N+1)$ square matrix, and β_1 is $(2N+1)\times 1$ column vector with zero elements. Finally, we compute the error between the exact solution and numerical solution by

$$\begin{cases} \|E_\omega\|_\infty = \max\limits_{-N\le k\le N, 0\le n\le M} \left|\omega\left(t_k, x_n\right) - \omega_n^k\right|, \\ \|E_u\|_\infty = \max\limits_{-N\le k\le N, 0\le n\le M} \left|u\left(t_k, x_n\right) - u_n^k\right|, \\ \|E_p\|_\infty = \max\limits_{-N\le k\le N, 0\le n\le M} \left|p\left(x_n\right) - p_n\right|, \end{cases}$$

where $\omega\left(t,x\right), u\left(t,x\right), p(x)$ represent the exact solutions, ω_n^k and u_n^k represent the numerical solutions at (t_k, x_n), and p_n represent the numerical solutions at x_n. The numerical results are given in the Table 1.

Table 1. Errors

Errors	$\|E_\omega\|_\infty$	$\|E_p\|_\infty$	$\|E_u\|_\infty$
$N = M = 10$	0.0211	0.0109	0.0029
$N = M = 20$	0.0054	0.0028	$7.5025e-04$
$N = M = 40$	0.0014	$7.2547e-04$	$1.9041e-04$
$N = M = 80$	$3.4447e-04$	$1.8454e-04$	$4.7966e-05$

As it is seen in Table 1, if N and M are doubled, the values of errors decrease by a factor of approximately 1/4.

4. Conclusion

In the present paper, the absolute stable DS of the second-order of accuracy DS for the approximate solution of the SIP for the multidimensional elliptic-telegraph differential equation with Dirichlet condition is constructed. Theorem on stability of this DS is established. Numerical results are presented for the solutions of the one-dimensional SIP for the elliptic-telegraph equation.

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