International Journal of Applied Mathematics

Volume 38 No. 1 2025, 47–55
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v38i1.4

$\mathcal{PERM}$-GRAPH OF FINITE GROUPS

Bilal N. Al-Hasanat [1], Ibrahim A. Jawarneh [2]

Department of Mathematics

Al Hussein Bin Talal University Ma'an

JORDAN

[1] bilal_hasanat@yahoo.com

[2] ibrahim.a.jawarneh@ahu.edu.jo

Abstract

Let G be a finite group and K and H be two subgroups of G. Then K permutes with H if and only if $KH = HK$. The $\mathcal{P}erm$-graph of a finite group G is the graph Γ_G whose vertices set is the set of all subgroups of G, and two distinct vertices H_1 and H_2 are adjacent if and only if they are permute. In this article we will introduce the $\mathcal{P}erm$-graph of finite groups, which is a new graph representation of finite groups. Then we discuss some properties of such graph in order to show how it detect the properties of the group itself.

Math. Subject Classification: 20F99, 20E99, 20C99

Key Words and Phrases: finite group, permutable subgroup, simple graph, complete graph

1. Introduction

The first graph representation of finite groups was introduced by A. Cayley in 1878. In this representation, the vertices correspond to the elements of the group, and two vertices x and y are adjacent if there

Received: December 20, 2024

exists an element $z \in G$ such that $x = zy$. Another well-known graph associated with finite groups is the power graph. In the power graph of a group G, all its elements serve as vertices, and two distinct vertices are connected if one of them is a power of the other [1]. According to [2], some properties of the power graph have been investigated. Additionally, the commuting graph is another graph representation of a finite group G. In this graph, a non-empty subset H of G acts as the set of vertices, and two distinct vertices are adjacent if and only if they commute [3, 4]. The order graph is yet another representation of finite groups, which is constructed according to the order classes of the given group [5].

In this paper, we introduce a new graph representation of finite groups, referred to as the $\mathcal{P}erm$-graph. This graph is developed using a familiar concept in group theory, namely permutable subgroups [6]. First, we define the notion of a $\mathcal{P}erm$-graph, and then we explore some of its properties to demonstrate how this representation reveals characteristics of the group itself.

2. Preliminaries and notations

The considered groups in this research are finite groups. For a group G, the identity element is e and the order of G is $|G|$. Recall that a subgroup H of G permutes with a subgroup K of G if $HK = KH$, where $HK = \{xy \mid x \in H \ , \ y \in K\}$. In particular, a subgroup H of G is called permutable subgroup if $HK = KH$ for all $K \leq G$. A group G is called an Iwasawa group when every subgroup of G is permutable in G [7]. A subgroup H of G is normal subgroup if and only if $xH = Hx$ for all $x \in G$. A non-Abelian group G all subgroups of which are normal is called Hamiltonian group [8]. The normality of a subgroup H of a group G, supports the commutativity of H in G, that is every normal subgroup is permutable.

Recall that, a connected graph Γ is a graph for which there is at least one path between any two distinct vertices, and Γ is simple if it doesn't have any loops or parallel edges. All of the considered graphs in this research are connected simple graphs. The order of a graph Γ is the number of its vertices and denoted by $O(\Gamma)$, and the size of Γ is the number of its edges and denoted by $S(\Gamma)$. The trivial graph is the graph Γ of $O(\Gamma) = 1$ and $S(\Gamma) = 0$.

A graph representation of a finite group is the graph associated from the given group $(G, *)$, for which the set of vertices is chosen by a selective rule on G, as well as the adjacency on the indicated set of vertices is established from the group operation $*$.

DEFINITION 2.1. The $\mathcal{P}erm$-graph of a finite group G is the graph Γ_G whose vertices set is the set of all subgroups of G, and two distinct vertices H_1 and H_2 are adjacent if and only if they are permute.

EXAMPLE 2.1. Consider the dihedral group $G = D_8$, which has 10 subgroups. Moreover, 6 subgroups are normal subgroups:

$$H_1 = \{e\},\ H_2 = \{e, r^2\},\ H_3 = \{e, r, r^2, r^3\},$$

$$H_4 = \{e, r^2, s, r^2s\},\ H_5 = \{e, r^2, rs, r^3s\} \text{ and } H_6 = G.$$

Note that, every normal subgroup is permutable, which indicates that, these subgroups permute with all other subgroups. The other 4 subgroups are:

$$H_7 = \{e, s\}\ ,\ H_8 = \{e, rs\}\ ,\ H_9 = \{e, r^2s\}\ ,\ H_{10} = \{e, r^3s\}.$$

None of these subgroups is permute with each others. Now, set Γ_G be the graph of vertices set $V = \{H_i \mid i = 1, 2, \cdots, 10\}$ and H_m adjacent with H_n if and only if $H_mH_n = H_nH_m$. Therefore, the $\mathcal{P}erm$-graph of G can be presented in Figure 1.

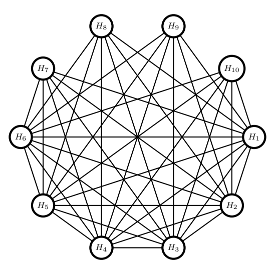

FIGURE 1. $\mathcal{P}erm$-graph of $G = D_8$

EXAMPLE 2.2. Let $G = D_{10} = D_{2(5)}$ be the dihedral group of 5-polygon. The group G is none-Abelian and it is easy to show that G has 8 subgroups, for which only 3 subgroups are normal; $H_1 = \{e\}$ $H_2 = \langle r\rangle \cong \mathbb{Z}_5$ and $H_3 = G$, indeed the normal subgroups are permutable. Moreover, $MN \neq NM$ for all other subgroups M and N of G. Therefore, the $\mathcal{P}erm$-graph of G can be shown in Figure 2.

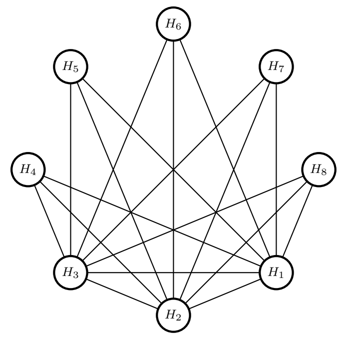

FIGURE 2. $\mathcal{P}erm$-graph of $G = D_{10}$

Note that, $H_4 = \{e, s\},\ H_5 = \{e, rs\},\ H_6 = \{e, r^2s\},\ H_7 = \{e, r^3s\}$ and $H_8 = \{e, r^4s\}$.

EXAMPLE 2.3. Consider the quaternion group

$$G = Q_8 = \left\langle a, b \mid a^4 = e,\ a^2 = b^2,\ aba = b \right\rangle.$$

The group G has 6 subgroups: $H_1 = \{e\}$, $H_2 = G$, $H_3 = \langle a^2 \rangle$, $H_4 = \langle a \rangle$, $H_5 = \langle b \rangle$ and $H_6 = \langle a, b \rangle$. Moreover, $H_iH_j = H_jH_i$ for all $i, j = 1, 2, \ldots, 6$. Therefore, the $\mathcal{P}erm$-graph of G is k_6 the complete graph of order 6.

3. $\mathcal{P}erm$-graph properties deduced from that of the group's

Clearly, for every Abelian group G, all subgroups are normal, so every subgroup is permutable, that is $HK = KH$ for every $K, H \leq G$. This can be formulated graphically on the $\mathcal{P}erm$-graph as every two distinct vertices are adjacent. This shows the next theorem.

THEOREM 3.1. *The $\mathcal{P}erm$-graph of an Abelian group of n subgroups is the complete graph K_n.*

P r o o f. Let G be a finite Abelian group of n subgroups. Then, every subgroup of G is normal, and hence is permutable. Thus, if Γ_G is the $\mathcal{P}erm$-graph of G, then the set of vertices is the subgroups of G, and any two distinct vertices are adjacent. That is, every vertex is of degree $n - 1$. Implies that Γ_G is the complete graph of n vertices. □

The converse of the previous theorem is not always true. See Example 2.3, which showed that the $\mathcal{P}erm$-graph of $G = Q_8$ is a complete graph, even though G is none-Abelian group.

PROPOSITION 3.1. *If G is a Hamiltonian group which has n subgroups, then the $\mathcal{P}erm$-graph of G is K_n.*

P r o o f. Since every subgroup of a Hamiltonian group is normal, then every subgroup of G permutes the others. Hence, the $\mathcal{P}erm$-graph of the Hamiltonian group is complete graph. □

EXAMPLE 3.1. The smallest Hamiltonian group is the quaternion group $G = Q_8$. The group G has 6 subgroups all are normal, then these subgroups are permutable. Thus, every two distinct vertices of the $\mathcal{P}erm$-graph Γ_G are adjacent. Implies that $\Gamma_G = K_6$.

On another point of view, any group G has at least two subgroups, the trivial subgroup $H_0 = \{e\}$ and the improper subgroup G and they are permutable. Therefore, the $\mathcal{P}erm$-graph of G has at least 2 vertices and at least one edge. This shows the next remark.

REMARK 3.1. The $\mathcal{P}erm$-graph Γ_G of a finite group G is of order $O(\Gamma_G) \geq 2$ and size $S(\Gamma_G) \geq 1$.

REMARK 3.2. The $\mathcal{P}erm$-graph of a finite group of k subgroups $k \geq 3$ is not acyclic connected graph, i.e is not a tree.

P r o o f. Let G be a finite group which has k subgroups. Let Γ_G be the $\mathcal{P}erm$-graph of G of vertices set V. Then, the order of Γ_G is k, and $a = \{e\}$, $b = G$ are two vertices in Γ_G which joint all other vertices. In particular, $\deg(a) = \deg(b) = k - 1$. Therefore, given any two vertices c and d in Γ_G, then c, a, d is a path in Γ_G that joint c and d. Thus, Γ_G is connected. Moreover, if $k \geq 3$, then $\{a, b, c\} \subseteq V$, and so Γ_G contains at least one cycle which is a, c, b, a, implies that Γ_G is acyclic graph. □

THEOREM 3.2. [9] *An edge e in a graph Γ is a bridge if and only if e is not in any cycle of Γ.*

Corollary 3.1. *The $\mathcal{P}erm$-graph of a finite group of k subgroups $k \geq 3$ has no bridges.*

P r o o f. Let Γ_G be the $\mathcal{P}erm$-graph of a group G which has $k \geq 3$ subgroups, then $|V| \geq 3$ where V is the set of vertices of Γ_G. Let $a = \{e\}$ and $b = G$ in V and let e be an edge in Γ_G. Then, we have the following cases:

Case 1: If $e = ab$, consider $c \in V$ for which ac and cb are two edges in Γ_G. Therefore, a, ac, c, cb, b, e, a is a cycle in Γ_G containing e.
Case 2: If $e = ac$ for some $c \in V$, then a, e, c, cb, b, ba, a is a cycle in Γ_G containing e.
Case 3: If $e = bc$ for some $c \in V$, it's the same as the previous case.
Case 4: If $e = cd$ for $c, d \in V$, then a, ac, c, e, d, da, a is a cycle in Γ_G containing e.

From the previous description, every edge in Γ_G is in at least one cycle of Γ_G. Using Theorem 3.2, implies that e is not a bridge. □

Remark 3.3. Let Γ_G be the $\mathcal{P}erm$-graph of a finite group G of k subgroups $k \geq 3$. Then $2 \leq \deg(v) \leq k-1$ for all $v \in \Gamma_G$.

P r o o f. The proof is easy, since the $\mathcal{P}erm$-graph Γ_G of a finite group G is simple, then $\deg(v) \leq k-1$ for all $v \in \Gamma_G$. Moreover, $a = \{e\}$ and $b = G$ are two permutable subgroups of any finite group G, and so any vertex v in Γ_G will be adjacent to a and b. Hence, $\deg(v) \geq 2$. □

4. The group properties deduced from that of its $\mathcal{P}erm$-graph

In this section, we show the group properties that can be deduced from its $\mathcal{P}erm$-graph.

Certainly, the trivial group $G = \{e\}$ has only one subgroup. Therefore, the $\mathcal{P}erm$-graph corresponding to the trivial group is a graph of order 1 and size 0, which is the trivial graph.

Lemma 4.1. *A trivial graph is the $\mathcal{P}erm$-graph of the trivial group.*

Combining both Theorem 3.1 and Proposition 3.1, one can show the following property.

THEOREM 4.1. *Let $\Gamma_G = K_n$ be the $\mathcal{P}erm$-graph of a finite group G. Then G is one of the following groups:*

(1) *G is Abelian group of n subgroups.*
(2) *G is Hamiltonian group of n subgroups.*
(3) *G is a none-Abelian Iwasawa group of n subgroups.*

P r o o f. Certainly, every vertex in $\Gamma_G = K_n$ is adjacent with all other vertices, and so all of the corresponding subgroups of the group G are permutable. Therefore, every subgroup of G is permutable. In particular, this can be satisfies only for all Abelian groups, Hamiltonian groups or finite none-Abelian Iwasawa groups. □

To keep this paper self-contained and for the next construction, we will list the following theorems.

THEOREM 4.2. [10] *Any none-Abelian group G of order $2p$ ($p > 3$ is prime) is isomorphic to D_{2p}.*

THEOREM 4.3. [10] *Consider the dihedral group $G = D_{2p}$, p is prime. Then:*

(1) *G has $p + 3$ subgroups.*
(2) *G has 3 normal subgroups.*
(3) *G has 3 permutable subgroups.*
(4) *G is an $\mathcal{A}$-group.*

Consider the $\mathcal{P}erm$-graph Γ_G in Example 1. For which the group $G = D_{10} = D_{2(5)}$ has been represented. One can see that: $O(\Gamma_G) = 8 = 5 + 3$ and there are five vertices in Γ of degree 3 and three vertices of degree 7. This will present the next result.

PROPOSITION 4.1. *A graph Γ of order $k = n+3$ for n is an odd prime and of degree sequence $\underbrace{3,\ 3,\ \ldots,\ 3}_{n}$, $n+2$, $n+2$, $n+2$ is the $\mathcal{P}erm$-graph of $G = D_{2n}$.*

P r o o f. Let n be an odd prime and Γ be a graph of order $k = n + 3$, with degree sequence $\underbrace{3,\ 3,\ \ldots,\ 3}_{n}$, $n + 2$, $n + 2$, $n + 2$. Then Γ can be considered as the $\mathcal{P}erm$-graph of a finite group G, for which G

has $O(\Gamma) = n + 3$ subgroups, three of these subgroups are permutable (represented in Γ as 3 vertices each of degree $n + 2$) and none of the other n subgroups are permuted with each other (represented in Γ as n vertices each of degree 3). The permutable subgroups of G are the trivial subgroup, the group itself and a subgroup H of index 2. Thus, $|H| = n$ (n is prime) and so H is cyclic, say $H = \langle x\rangle$. Therefore, G is a finite group of order $2n$, n is prime. Using Theorem 4.2 and Theorem 4.3, we conclude that $G \cong D_{2n}$. □

One of the special cases of graphs is the cycle graph for which every vertex is of degree 2. In particular, a cycle of 3 vertices and 3 edges which denoted by C_3. Corresponding to this case, the following theorem sets the structure of the group G for which the $\mathcal{P}erm$-graph of G is C_3.

THEOREM 4.4. *The $\mathcal{P}erm$-graph of a finite group G is C_3 if and only if $G \cong \mathbb{Z}_{p^2}$ for p is prime.*

P r o o f. Let $\Gamma_G = C_3$ be the $\mathcal{P}erm$-graph of a finite group G. Then, $O(\Gamma_G) = 3 = S(\Gamma_G)$. Thus the group G has only 3 subgroups, say $H_1 = \{e\}$, $H_2 = G$ and H_3. Indeed, the subgroups H_1 and H_2 are normal in G. The subgroup H_3 should not has any subgroup, and so it should be of prime order and prime index too. Set $|H_3| = p$ and $(G : H_3) = q$, then $p = q$, if else, then G has a subgroup of order q, but G has only 3 subgroups. Thus, G is of order $pp = p^2$ and every group of order p^2, p is prime is isomorphic to $\mathbb{Z}_{p^2}$.

Conversely, suppose that $G \cong \mathbb{Z}_{p^2}$ for p is prime. Then G has only 3 subgroups, which are $\{e\}$, G and $\mathbb{Z}_p$. Therefore, the $\mathcal{P}erm$-graph Γ_G is of order 3 and size 3 (because all the subgroups of G are permutable). Then, Γ_G is the cycle graph C_3. □

References

[1] A.V. Kelarev and S.J. Quinn, A combinatorial property and power graphs of groups, *Contributions to General Algebra*, **12**, No 58 (2000), 3–6.

[2] A.R. Moghaddamfar, S. Rahbariyan, and W.J. Shi, Certain properties of the power graph associated with a finite group, *Journal of Algebra and its Applications*, **13**, No 7 (2014), Art. 1450040.

[3] Y. Segev, On finite homomorphic images of the multiplicative group of a division algebra, *Ann. of Math.*, **149** (1999), 219–251.

[4] Y. Segev and G. Seitz, Anisotropic groups of type an and the commuting graph of finite simple groups, *Pacific J. Math.*, **202** (2002), 125–225.

[5] B.N. Al-Hasanat and A.S. Alhasanat, Order graph: A new representation of finite groups, *International Journal of Mathematics and Computer Science*, **14** (2019), 809-819.

[6] B.N. Al-Hasanat, A.F. Al-Dababseh, K.A. Al-Sharo and B.R. Al-Shuraifeen, On the direct product of nearly s-permutable subgroups, *International Journal of Applied Mathematics*, **35**, No 2 (2022), 307–315; http://dx.doi.org/10.12732/ijam.v35i2.9

[7] K. Iwasawa, Über die endlichen Gruppen und die Verbände ihrer Untergruppen, *J. Fac. Sci. Imp. Univ. Tokyo. Sect. I*, **4** (1941), 171–199.

[8] R. Dedekind, Uber gruppen, deren sammtliche teiler normalteiler sind, *Math. Ann.*, **48** (1897), 548–561.

[9] G. Chartrand, *Introductory Graph Theory*, Courier Corporation (1977).

[10] B.N. Al-Hasanat, A.H. Alkasasbeh, I.A. Jawarneh, K.A. Al-Sharo, and R.S. Al-Gounmeein, A-subgroups and A-groups, *Applied Mathematics & Information Sciences*, Accepted, 2024.