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FRACTIONAL MULTISTEP DIFFERENTIAL TRANFORMATION METHOD USED TO ANALYZE A MODIFIED FORM OF FRACTIONAL ORDER LORENZ SYSTEM

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Abstract

Dynamics of nonlinear fractional-order Lorenz system is investigated by employing Fractional Multistep Differential Transformation Method (FMDTM). In order to illustrate the new technique, the numerical algorithm is applied in the 3D solution of modified Lorenz system by adding the forth varied parameter $a = 40, =3, =10$, considered as a highly simplified model for the weather. Parameter fixed dynamical analysis method and chaos diagram are used. Results show that the fractional order Lorenz system has rich dynamical behaviour and it is a potential model for application. Investigation of dynamics is realized by fixing the parameters (system has chaotic behaviour) and by changing the added parameter $d \in [5, 38]$, implemented with the aid of Mathematica symbolic package. For $d = 25$, the minimal fractional order, for which the system shows chaotic behaviour is $v = 0.8726$, for $v = 0.998$, the minimal value of d, for which system shows chaotic behaviour is $d > 12.05219$. The fractional derivatives are described in the Caputo sense. Based on FMDTM, is shown that the system has rich dynamical characteristics, it changes from a non-chaotic system to a chaotic one, using fractional order $v \in (0, 1]$. The method deals with the approximated solutions to integer-order differential equations and is based on polynomial approximations, with good results (based on numerical experiments) for fractional order closed to 1.

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Key Words and Phrases: Caputo fractional derivative, modified Lorenz system, fractional multistep differential transformation method, nonlinear system, dynamical behaviour

1. Introduction

Fractional calculus is not a new topic, it has almost the same history as that of classical calculus. The theory of derivatives of fractional-order (or non-integer order) goes back to Leibniz's note in his list to L'Hospital, dated 30 September 1695, in which the meaning of derivative of order one-half was discussed [1, 2]. Although fractional calculus has a 300-year-old history, its applications to physics and engineering are just recent focus of interest. Different from the typical derivative, there are more than six kinds of definitions of fractional derivatives, not mutually equivalent. The Caputo derivative is defined on the basis of fractional integral and used in this paper. Is analyzed numerically the behaviour of a dynamical system, which exhibits high sensitivity to initial conditions, known as chaos [2, 6]. This behaviour was first observed by Edward Lorenz while solving a system of three differential equations governing weather prediction on a computer.

DEFINITION 1.1. [3] The Caputo fractional derivative of a function $f(t)$, of order $v \in (0, 1]$ is defined by:

$$\overline{D_t}^v f(t) = \frac{1}{\Gamma(1-v)} \int_0^t (t-\tau)^{v-1} f'(\tau) d\tau. \tag{1}$$

Usually, it is preferred to choose the Caputo fractional derivative, because it allows conventional initial conditions to be included in the formulation of the problem, [2], [3].

DEFINITION 1.2. [3] An n-dimensional fractional-order system with Caputo derivative is described by:

$$\overline{D_t}^v \mathbf{y} = \mathbf{f}(\mathbf{y}) \tag{2}$$

with $v \in [v_1, v_2, ..., v_n]^T$, $v_i \in (0, 1]$ $(i = 1, 2, ..., n)$ and $\mathbf{y} \in \mathbf{R}$. The equilibrium points $E^* = (y_1^*, y_2^*, ..., y_n^*)$, of (2) are the solutions of the equation $\mathbf{f}(\mathbf{y}) = \mathbf{0}$.

DEFINITION 1.3. [5] The trajectory of (2) is said to be stable if for any initial conditions $y_i(t_0) = c_i$ $(i = 1, 2, ..., n)$, exist $\epsilon > 0$, that for any solution $y(t)$ of (2) to satisfy the condition $\| y(t) \| < \epsilon$.

$y(t) = 0$ is asymptotically stable if it is stable and satisfy the condition $\lim_{t \to \infty} y \| y(t) \| = 0$.

THEOREM 1.1. *The equilibrium point* $E^* = (y_1^*, y_2^*, ..., y_n^*)$ *of the system (2) is locally asymptotically stable if all the eigenvalues* λ_i $(i = 1, 2, ..., n)$ *of the Jacobian matrix* $J = \frac{\partial f}{\partial y}$, $f = [f_1, f_2, ..., f_n]^T$ *satisfy the condition:*

$$|\arg(eig(J))| = |\arg(\lambda_i)| > v\frac{\pi}{2}, \quad i = 1, 2, ..., n. \tag{3}$$

The equilibrium point is called as a non-hyperbolic if

$$|\arg(eig(J))| = |\arg(\lambda_i)| \neq v\pi.$$

DEFINITION 1.4. [5, 7] For three dimensional system, the equilibrium point $E^* = (y_1^*, y_2^*, y_3^*)$ is:

a. Node: when all the values λ_i $(i = 1, 2, ..., n)$ of $J = \frac{\partial f}{\partial y}$ are real with the same sign;

b. Saddle: when all eigenvalues are real and at least one of them is positive and at least one is negative; Saddles are always unstable;

c. Focus-Node: when it has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign; The equilibrium is stable (unstable) when the sign is negative (positive);

d. Saddle-Focus: when it has one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues; This type of equilibrium is always unstable.

DEFINITION 1.5. [5, 6, 11] Lorenz system is described as:

$$\begin{aligned} \overline{D}_t^v x(t) &= a(y(t) - x(t)), \\ \overline{D}_t^v y(t) &= x(t)(c - z(t)) + dy(t), \\ \overline{D}_t^v z(t) &= x(t)y(t) - bz(t), \end{aligned} \tag{4}$$

$x_0 = 1, y_0 = 1, z_0 = 1,\ a, b, c > 0,\ a = 40, b = 3,\ c = 10$ and varied $d \in [5, 38]$, with equilibrium points of (4) the trivial one $E_0 = (0, 0, 0)$ and $E_{1/2} = (\pm\sqrt{b(c+d)}, \pm\sqrt{b(c+d)}, c+d)$ and Jacobian matrix [3]:

$$J = \begin{bmatrix} -a & a & 0 \\ c - z^* & d & -x^* \\ y^* & x^* & -b \end{bmatrix}.$$

2. Numerical method

In recent years, the numerical approximation for the solutions of fractional order dynamical systems has attracted increasing attention in many fields of applied sciences and engineering. Many researchers considered the trapezoidal method, predictor-corrector method, extrapolation method, and spectral method [1]. These methods are appropriate options if the resulting system of equations is linear but they present a high computational cost when the problem we are solving is badly conditioned or nonlinear.

In this paper the dynamics of a generalized form of fractional-order Lorenz system is investigated by employing Fractional Multistep Differential Transformation Method (FMDTM) [8, 9].

DEFINITION 2.1. [9, 10] The problem

$$\overline{D}_t^v y(t) = f(t, y(t)), \quad y(0) = y_0,\ v \in (0, 1] \tag{5}$$

is called initial value problem (IVP).

We will construct the methods, assuming that a solution of J is sought on some time interval $[0, T]$ arbitrary $v \in (0, 1]$ and $f : [0, T] \times D \to R$, $D \subseteq R$.

The interval $[0, T]$ is divided into l subintervals. Consider an equi-spaced grid with step length h, $t_j = jh$, $j = 1, 2, ...$ Let y_j denote the approximated solution at t_j and $y(t_j)$ denote the exact solution of the initial value problem (5), [1, 2, 8, 11].

2.1. **Fractional Differential Transformation Method.** The differential transform method is derived from the Taylor series expansion. The DTM does not require a symbolic evaluation of derivatives. Instead, relative derivatives are computed iteratively [1, 11].

The k-th differential transform of a function $y(t)$ is defined as follows:

$$F(k) = \frac{1}{k!}[\frac{d^k f(t)}{dt^k}]_{t=t_0} \tag{6}$$

for $k = 1, 2, 3, ...$ The differential inverse transform is defined as [7, 10]:

$$y(t) = \sum_{n=0}^{k} F(k)(t-t_0)^k. \tag{7}$$

The following are the basic properties of the Caputo time-fractional derivative and the differential transformation, [9]:

1. For $f(t) = g(t) \pm h(t)$, then $F(k) = G(k) \pm H(k)$.
2. For $f(t) = g(t)h(t)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$.
3. For $f(t) = g_1(t)g_2(t)...g_n(t)$, then

$$F(k) = \sum_{k_{n-1}=0}^{k} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)g_2(k_2-k_1)...g_n(k_{n-1}-k_n).$$

4. For $f(t) = (t-t_0)^p$, then $F(k) = \sigma(k - \alpha p)$, where

$$\sigma(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} .$$

5. For $f(t) = \overline{D}_{t_0}^{v}[g(t)]$, then

$$F(k) = \frac{\Gamma(v + 1 + \frac{k}{\alpha})}{\Gamma(1 + \frac{k}{\alpha})} G(k + \alpha v).$$

The multi-step approach introduces a new idea for constructing the approximate solution. Assume that the interval $[t_0, T]$ is divided into M subintervals $[t_0, t_1], [t_1, t_2], \ , ..., [t_{M-1}, t_M]$, where $t_0 < t_1 < t_2 < ... < t_{M-1} < t_M, \ t_0 = t_0, t_M = T$ and $t_i - t_{i-1} = h > 0, \ h = \frac{T}{M}, \ i = 0, 1, 2, ...M$, [1, 6, 3]. The main idea of the FMDTM is in the following. First, we apply the DTM to (5) over the interval $[t_0, t_1]$, we will obtain the first approximate solution. The same process is repeated in the following intervals $[t_{i-1}, t_i], \ i = 1, 2, .., M$ to obtain the solutions $f_i(t)$ so: $y_i(t) = \sum_{n=0}^{k} a_{ni}(t - t_{i-1})^n, \ t \in [t_i, t_{i+1}]$, where

$N = KM$. In fact, the FMDTM assumes the following solution:

$$y(t) = \begin{cases} y_1(t), & t \in [t_0, t_1] \\ y_2(t), & t \in [t_1, t_2] \\ \vdots \\ y_M(t), & t \in [t_{M-1}, t_M] \end{cases} . \tag{8}$$

The new algorithm, FMDTM, is simple for computational performance for all values of h. It is easily observed that if the step size $h = T$, then the FMDTM reduces to the classical DTM.

3. **Numerical Simulations**

In this section we present simulation results of the modified Lorenz's system (1.3) on three dimensional plot, taking standard parameters $a = 40, b = 8, c = 10$ [9], varied $d \in [5, 38]$ with fractional order $v \in (0, 1]$ and initial conditions $x(o) = 1, y(0) = 1, z(0) = 1$ to analyze its chaotic behavior using FMDTM.

For fixed value of $d = 25$, the eigenvalues of Jacobian matrix for the equilibrium point $E_0 = (0, 0, 0)$ are $(-45.6608, 30.6608, -3.0000)$ and for two others $E_{1/2} = (\pm\sqrt{105}, \pm\sqrt{105}, 35)$ are

$$(-25.2415, 3.6207 + 178795i, 3.6207 - 178795i).$$

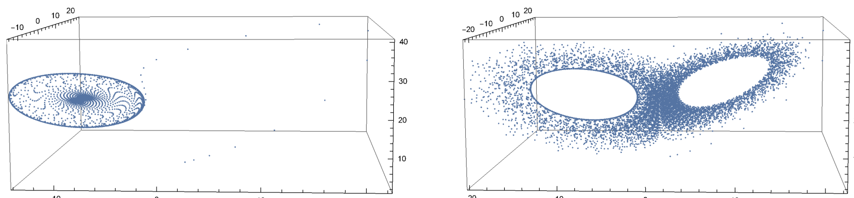
For the taken value $d = 25$, minimum fractional order on which the system shows chaotic behavior is $v = 0.8726$.

For $(x_0, y_0, z_0) = (1, 1, 1)$ and fixed values of v, $v = 0.75$, $v = 85$, $v = 0.998$, we change the value of $d \in [5, 38]$, with the step size $h = 0.01$ and time $t \in [0, 100]$. For $v = 0.75$ and $d > 13.03923$, $v = 0.85$ and $d > 12.05219$, $v = 0.998$ and $d > 9.51495$ the system shows chaotic behavior. For $v = 0.75$ and $d = 13.03923$, $v = 0.85$ and $d = 12.05219$, $v = 0.998$ and $d = 9.51495$ the system is periodic.

In Figure 1, Figure 2 and Figure 3 we plot 3D phase portrait of system (4) for parameters $a = 40, b = 8, c = 10$ according to varied d arbitrary taken in the interval $d \in [5, 38]$.

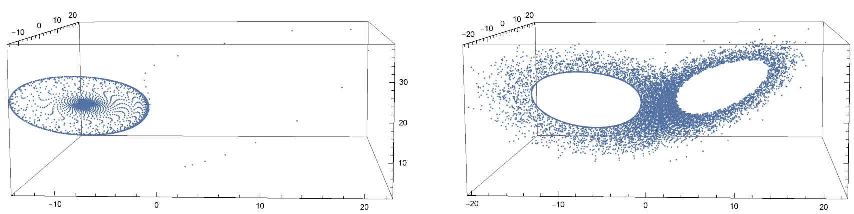
For fixed value of $d = 9.51495$, the eigenvalues of Jacobian matrix for the equilibrium point $E_0 = (0, 0, 0)$ are $(-47.0691, 16.5841, 3.0000)$ with $|\arg(\lambda_1)| = \pi > 0.998\frac{\pi}{2}$ and for $E_{1/2} = (\pm 7.65146, \pm 7.65146, 19.515)$ are $(-33.2144, -0.135325 + 11.874i, -0.135325 - 11.874i)$ with $|\arg(\lambda_{2/3})| = 3 > 0.998\frac{\pi}{2}$. According to Definition 4, E_0 is saddle point and $E_{1/2}$ are unstable focus node points.

For fixed value of $d = 9.51496$, the equilibrium points E_0, E_1, E_2 shows values with absolute difference $1.96046 * 10^{-6}$ to the ones taken for $d = 9.51495$. The difference makes different chaotic behavior of the system!



a) b)

FIGURE 1. 3D phase portrait of system (4) using MFDTM for fixed value of $v = 0.75$, $d \in [5, 38]$, initial values $(x_0, y_0, z_0) = (1, 1, 1)$, step size $h = 0.01$, time $t = 100$; a) $d = 13.03923$ the system (4) is periodic, b) $d = 13.03924$ the system (4) is chaotic.

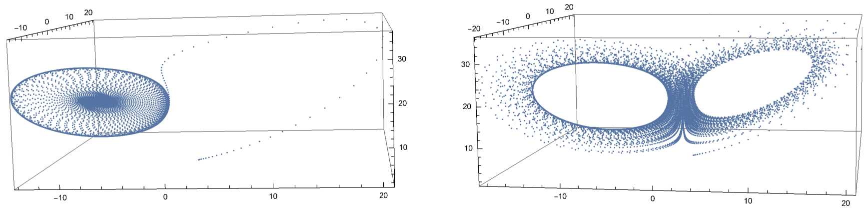


a) b)

FIGURE 2. 3D phase portrait of system (4) using MFDTM for fixed value of $v = 0.85$, $d \in [5, 38]$, initial values $(x_0, y_0, z_0) = (1, 1, 1)$, step size $h = 0.01$, time $t = 100$; a) $d = 12.05219$ the system (4) is periodic, b) $d = 12.0522$ the system (4) is chaotic.

4. **Conclusions**

There are found some points where the system shows changes on its behaviour, $d = 13.03923$ with $d = 13.03924$, $d = 12.05219$ with $d = 12.0522$ and $d = 9.51495$ with $d = 9.51496$, using 3D phase portrait and numerical experiments. In all cases the system pass from periodic stable behavior to chaotic behavior. With aim to divide the interval of varied $d \in [5, 38]$, for different values of fractional order, greater than 0.5 and closer to 1, according to many numerical experiments with FMDTM. In all simulations we use FMDTM, the time of integration is $t = 100$, which means that the method should be very fast deal with it.



a) b)

FIGURE 3. 3D phase portrait of system (4) using MFDTM for fixed value of $v = 0.998$, $d \in [5, 38]$, initial values $(x_0, y_0, z_0) = (1, 1, 1)$, step size $h = 0.01$, time $t = 100$; a) $d = 9.51495$ the system (4) is periodic, b) $d = 9.51496$ the system (4) is chaotic.

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