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COUNTEREXAMPLES FOR SOME CONJECTURES OF FINITE GROUPS

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Abstract

For a finite group G, let $\pi_e(G)$ be the set of orders of its elements. The order classes of G, denoted as $OC(G)$, are collections of pairs $[k, m]$, where $k \in \pi_e(G)$ and m represents the number of elements in G with order k. Two finite groups G and H are said to be of the same order type if $OC(G) = OC(H)$. It is unequivocal that two simple groups G_1 and G_2 with the same order classes are isomorphic. This paper presents counterexamples demonstrating that this property does not necessarily hold for non-simple groups. Furthermore, we will examine several properties of non-simple groups that derived from their group order and element orders, and address certain open problems related to these properties.

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1. Introduction

One of the central themes in group theory is the classification of groups and determining whether two given groups are isomorphic. This concept is crucial for understanding these structures since all group properties can be held through associated isomorphisms. Completing such classification requires describing a specific isomorphism map, which is not straightforward. However, some group axioms and properties can simplify this process.

The purpose of this paper is to study the effect of group order and order type on whether or not such groups are necessarily isomorphic.

Let G be a finite group, and define $\pi_e(G) = \{k \in \mathbb{N} \mid k = o(x), x \in G\}$, which represents all possible element orders within G.

In 1987, W. J. Shi [1] posed a notable conjecture:

"Let G be a group and S a finite simple group. Then $G \cong S$ if and only if (a) $\pi_e(G) = \pi_e(S)$, and (b) $|G| = |S|$."

In 2009, Vasilev et al. [2] proved that this conjecture is true only for simple groups, affirming that all finite simple groups can be characterized by their two orders (the group order and the set of all element orders).

In [3], Moretó explored a related conjecture concerning the relationship between the number of elements of order p (the largest prime divisor of $|S|$) for simple groups, which we quote below:

"Let S be a finite simple group and p the largest prime divisor of $|S|$. If G is a finite group with the same number of elements of order p as S and $|G| = |S|$, then $G \cong S$."

In Section 3, we will present a counterexample to this conjecture and outline another open problem from [4], along with additional conjectures for which we will either provide proofs or counterexamples.

The Groups, Algorithms and Programming (`GAP`) software will be used to construct our examples and do the required calculations.

2. Preliminaries and notations

For a finite group G, the order of G is denoted by $|G|$, and the identity element is e. The order of $x \in G$ is the smallest positive integer n such that $x^n = e$ and denoted by $o(x)$. Let $\pi_e(G) = \{k \in \mathbb{N} \mid k = o(x)\ ,\ x \in G\}$. Then, the order classes of G are defined as $OC(G) = \{[k,\ |O_k|] \mid k \in \pi_e(G)\}$, which consists of all order pairs that assign each element order with the number of elements in G of this order. For more about the order classes of finite groups, see [5, 6, 7].

DEFINITION 2.1. Two finite groups G_1 and G_2 have the same order type if and only if $OC(G_1) = OC(G_2)$.

Certainly, any two isomorphic groups share the same order and the same order classes. Conversely, groups that share the same order classes do have the same order, but this does not guarantee that they are isomorphic.

In order to discuss the coming conjectures, some of the basic definitions are listed below.

DEFINITION 2.2. A non-trivial group G is simple group if and only if G has no non-trivial normal subgroup. Otherwise, it is called not simple group.

The commutator of x and y in a group G is $[x, y] = x\ y\ x^{-1}\ y^{-1}$. The subgroup of G that generated by all of the commutators is called the commutator subgroup or the derived subgroup of G and denoted by G'.

EXAMPLE 2.1. Let $x = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $G = \langle x,\ y \mid x^4 = (x\ y)^3 = e,\ x^2 = y^3 \rangle \cong SL(2, 3)$ the special linear group of 2×2 matrices over $\mathbb{Z}_3$ with determinant 1 of order 24. The commutator of x and y in G is:

$$[x, y] = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

The derived subgroup of G is $G' = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\} \cong Q_8$ the Quaternion group of order 8.

The derived series of a group G is a series that obtained by the derived subgroups as $G_{i+1} = G'_i$ for $i = 1, 2, \cdots$.

Consider the group G in Example 2.1. Then $G' \cong Q_8$, implies that $G_2 = G'_1 = (G')' = \langle x^2 \rangle \cong \mathbb{Z}_2$ and $G_3 = G'_2 = (G'_1)' = \{e\}$. Therefore, the derived series of G is $Q_8, \mathbb{Z}_2, \{e\}$.

Definition 2.3. A group G is said to be solvable if the derived series reaches the trivial subgroup in a finite number of steps. Otherwise, it is called non-solvable group.

Consider the group G in Example 2.1. The group G is solvable, as its derived series is $Q_8, \mathbb{Z}_2, \{e\}$, which reaches the trivial subgroup in three steps.

The exponent $E(G)$ of G is the least common multiple of the element orders.

3. **Some open conjectures**

The study of order classes provides a refined method for distinguishing between finite groups. While two finite simple groups with the same order classes are necessarily isomorphic, this paper demonstrates that such a property does not extend to non-simple groups. In this section, we provide additional insights into how order classes interact with solvability, nilpotency, and isomorphism classifications. Note that, if two finite groups have the same order classes (the same order type), then both groups have the same order, the same set of element orders (π_e) and the number of elements of any order $k \in \pi_e$ in both groups are equal.

A central question in group theory is whether two groups with the same order classes must have specific structural properties. The following conjectures address this issue:

Conjecture 3.1. If two groups G_1 and G_2 are of the same order type, is it necessarily to be isomorphic?

Conjecture 3.2. Suppose G_1 and G_2 are finite groups of the same order type. Suppose also that G_1 is Abelian. Is it true that G_2 is necessarily Abelian?

Conjecture 3.3. Suppose G_1 and G_2 are finite groups of the same order type. Suppose also that G_1 is nilpotent of class c. Is it true that G_2 is necessarily nilpotent of the same class?

The next example can be considered as a counterexample of Conjectures 3.1, 3.2 and 3.3.

EXAMPLE 3.1. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and $H = \mathbb{Z}_2 \times Q_8$, where Q_8 is the quaternion group of order 8. It is clear that, both groups are 2-group and $|G| = |H| = 16 = 2^4$. Also, $OC(G) = OC(H) = \{[1,1],[2,3],[4,12]\}$. Certainly, G is Abelian group. Thus G is nilpotent of class 1, and also it is solvable (p-group). On the other hand, H is non-Abelian solvable group, and it is nilpotent group of class 2.

From these calculations, it follows that G and H are of the same order type. But they are not isomorphic groups, not both groups are Abelian and these groups have distinct nilpotency classes. This shows that, Conjecture 3.1, Conjecture 3.2 and Conjecture 3.3 all are not mainly correct.

In [2], the author showed that two finite simple groups of the same order type are isomorphic. The following example illustrates that groups that are not simple but share the same order classifications (order type) are not necessarily isomorphic.

EXAMPLE 3.2. Let $G = \mathbb{Z}_4 \times A_5$ and $H = SL(2,5) \rtimes \mathbb{Z}_2$. Then, $|G| = |H| = 240 = 2^4 \cdot 3 \cdot 5$. Both are not simple and non-solvable groups, and

$$OC(G) = OC(H) = \{[1,1],[2,31],[3,20],[4,32],[5,24],\\ [6,20],[10,24],[12,40],[20,48]\}.$$

But, $|G'| = |A_5| = 60$ and $|H'| = |SL(2,5)| = 120$. So, $G \ncong H$.

Thus, non-simple groups, with the same order type are not need to be isomorphic.

Let p represent the largest prime divisor of $|G|$. In [3], the author showed that some families of simple groups are basically determined just by $|O_p|$ (the number of elements of order p). Also, he posted the following conjecture.

CONJECTURE 3.4. Let S be a finite simple group and p the largest prime divisor of $|S|$. If G is a finite group with the same number of elements of order p as S and $|G| = |S|$, then $G \cong S$.

This conjecture is not valid; the following serves as a counterexample.

EXAMPLE 3.3. Consider $S = PSL(3, 2)$ which is a simple group of order $|S| = 168 = 2^3 \cdot 3 \cdot 7$ and

$$OC(S) = \{[1, 1], [2, 21], [3, 56], [4, 42], [7, 48]\}.$$

The largest prime divisor of $|S|$ is $p = 7$, the number of elements in S of order 7 is 48. Let $G = (\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_3$ which is not simple group of order 168, and $OC(G) = \{[1, 1], [2, 7], [3, 56], [6, 56], [7, 48]\}$. This implies that S and G have the same order and the same number of elements of order $p = 7$ (the largest prime divisor of 168). Clearly, they have distinct exponents; $E(S) = 84 \neq 42 = E(G)$. Therefore S and G are not isomorphic groups.

CONJECTURE 3.5. Two solvable groups which have the same order classes need to be isomorphic.

Consider the groups G and H in Example 3.1, both groups have the same order classes and they are solvable groups, but they are none isomorphic groups. This shows that this conjecture is not true.

A major unresolved question concerns whether solvability is preserved under order type equivalence. In [4], we encountered the following open problem:

PROBLEM 3.1. [4] *Suppose G_1 and G_2 are finite groups of the same order and $\pi_e(G_1) = \pi_e(G_2)$. Suppose also that G_1 is solvable. Is it true that G_2 is also necessarily solvable?*

The answer of this conjecture is illustrated in the next example.

EXAMPLE 3.4. Let $G = \mathbb{Z}_5^2 \times A_4$. Then, G is solvable. In which $OC(G) = \{[1, 1], [2, 3], [3, 8], [5, 24], [10, 72], [15, 192]\}$.

Also, let $H = \mathbb{Z}_5 \times A_5$. The group H is non-solvable. In which $OC(H) = \{[1, 1], [2, 15], [3, 20], [5, 124], [10, 60], [15, 80]\}$.

Clearly, $|G| = |H| = 300 = 2^2 \cdot 3 \cdot 5^2$. Moreover $\pi_e(G) = \pi_e(H) = \{1, 2, 3, 5, 10, 15\}$.

Upon comparing the properties of the groups, we find that the solvable group G and non-solvable group H possess the same two orders (the group order and the set of element orders), which is insufficient for establishing solvability.

To support this conjecture, we implemented `GAP` calculations across groups of order at most 2^{11}. The computational results reveal that for small solvable groups, all groups with the same order type were also solvable. This suggests:

RESULT 3.6. Let G be a solvable finite group of order at most 2^{11}. Then any group of the same order classes as G is also solvable.

This raises the following open problem:

PROBLEM 3.2. *Can a counterexample be found for groups of larger order, or does solvability preservation hold universally for order type equivalence?*

These findings indicate that order type analysis provides a useful but incomplete tool for classifying finite groups. The results suggest a structured approach to studying order classes and their impact on solvability.

4. **Conclusion**

This study explores the concepts of order type and order classes of finite groups to elucidate their impact on group structure. It is shown that, while simple groups can be classified by their order classes, the order type alone is insufficient to fully categorize all finite groups.

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